THE INFERENTIAL APPROACH TO LOGICAL CALCULUS (part II)

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6. *The elimination theorem.* Besides the rules stated in § 4, Gentzen stated a further rule, which he called «Cut» (*Schnitt*). This may be formulated as follows:

$$(10) \qquad \frac{\mathfrak{X}_1, A \Vdash \mathfrak{Y} \qquad \mathfrak{X}_2 \Vdash A, \mathfrak{Z}}{\mathfrak{X}_1, \mathfrak{X}_2 \Vdash \mathfrak{Y}, \mathfrak{Z}.}$$

Here we retain the conventions of § 4 according to which $\mathfrak{Y}$ is singular and $\mathfrak{Z}$ void in all singular systems.

The principle that concepts are determined by their rules of introduction indicates that (10) ought to be derivable as a theorem. Gentzen introduced it as a primitive rule, and then his principal theorem (*Hauptsatz*) was to the effect that it is redundant. This is a consequence of Gentzen's motivation; from the standpoint of foundations of ordinary logical calculus it is a quite natural assumption. In [TFD] the rule (10) was taken as a theorem and called the *elimination theorem* (abbreviated ET). Because of its role in Gentzen's work it is often called the cut-elimination theorem; but from the point of view of [TFD] it is *A*, rather than a cut, which is eliminated.

The original proof of Gentzen involved a double induction. There was a primary induction on the number of operations in *A*, and a secondary induction on the number of steps in deriving the premises. The proof in [TFD], later modified in [ETM], was similar. It applies in all the cases which satisfy certain conditions, of which the most important is that, if the general formulation of the system allows an additional parameter in the premises, inference from the premise so altered to the correspondingly altered conclusion will remain valid. This incudes all the singular systems, and those multiple ones in which none of the rules are restricted to be singular, such as LC_m, LE_m, LK_m.

The invertibility properties mentioned in § 5 can easily be derived from ET. On the other hand those properties can be derived independently [36], and in that case the proof of ET can be shortened. Indeed in cases where all the rules are invertible, such as LC_m, LE_m,

([36]) See § 8.

and LK_m the secondary induction in unnecessary ([37]). But it seems to be necessary, at least for certain cases, in all systems involving quantifiers and in all systems which are absolutely based.

The proof requires that *K* be admissible. There is some interest in systems in which the rules *K* are absent or modified. For these ET is still open.

The elimination theorem is necessary in order to prove the equivalence of the different forms of systems. That the singular L-systems can be interpreted in the T-systems can be shown directly; but the converse requires ET. That the multiple systems can be interpreted in the corresponding singular systems can be shown by the use of ET for the latter; the converse argument requires also ET for the multiple systems in case any of the rules Nx, Px is postulated. Thus ET, and the equivalence to the corresponding singular system, holds for LA_m^*, LM_m^*, LJ_m^* ([38]), LE_m^*, LK_m^*. In regard to LD_m the question is still open, although I conjecture that ET and the equivalence holds.

Another consequence of the elimination theorem is the *replacement theorem*. This says that if

$$A \Vdash B \qquad B \Vdash A \tag{1}$$

both hold, then an occurence of A can be replaced by one of B. A more general theorem, applying to cases where only one of (1) holds, can be proved by distinguishing positive and negative occurrences much as in [CLg] § 2D3.

In the following we shall be concerned with properties which, for the most part, are quite independent of ET. The significance of ET is that these properties, and the conclusions drawn from them, can be extended under obvious limitations to the more conventional formulations of logical calculus.

7. *Deducibility theorems.* Let us call a *proper L-system* one formed with only (some selection of) the structural and operational rules here considered. Then a proper algebraic ([39]) L-system has the property that, although the rules may introduce new components ([40]),

([37]) This appears from the work of Schütte and Schmidt.

([38]) The equivalence with ET added as postulate for both systems was first shown by Maehara [DIL]. I understand from Umezawa [IPL], p. 20 footnote 31 that ET for LJ* was proved by Ohnishi.

([39]) For 'algebraic system' see § 3.

([40]) A proposition A is said to be a component of a proposition B if A is no component once introduced is ever entirely eliminated. Consequently they have the following *composition property* (also called «*subformula property*»): every component of a statement in a formal derivation of a statement Γ is also a component of Γ itself. Since for any given Γ there are only a finite number of components, and since for some of the equivalent formulations mentioned in § 5 the number of possible statements which can be made from these components is finite also, it follows that *every proper algebraic L-system is decidable.*

If an L-system admits quantifiers or special rules, then the composition property fails, because the special rules and the rules for quantifiers allow constituents (and hence possible components) to be eliminated. But they may admit weaker forms of the composition property from which interesting results follow. Thus the rules $\vdash^*$ and Fj allow only elementary statements to be eliminated; the same is true for F* if all counteraxioms are elementary. The quantification rules allow the elimination of arbitrary components, but replace them by others which differ only in their terms. Let us say that two propositions which differ only in their terms have the same *propositional matrix.* Then any L-system (formed with the rules listed here) whose special rules eliminate only elementary propositions has the composition property with respect to compound propositional matrices.

Consider now the following properties, Γ being a statement of form (8):

Separation property: If Γ holds, then it is derivable without using any rules related to operations not appearing in Γ.

Conservation property: If Γ holds and all its constituents are elementary, then some constituent on the right is derivable in $\mathfrak{S}$ itself from the constituents on the left side.

Alternation property: If Δ is a derivation of

$$\mathfrak{X} \Vdash A \vee B,$$

one of the parts from which B may be constructed by the operations. This is here to include the case where A is B itself. For the precise definition see [CLg], p. 49. A component of a statement (8) is a component of any of its constituents.

then we can find effectively a derivation of one of

$$\mathfrak{X} \Vdash A \qquad \mathfrak{X} \Vdash B;$$

likewise if Δ is a derivation of

$$\mathfrak{X} \Vdash (\exists x)\ A(x),$$

then we can find effectively a derivation of

$$\mathfrak{X} \Vdash A(t)$$

for some term t.

We shall study the conditions for validity of the properties.

If there are no special rules other than $\vdash^*$, then the separation property follows by the composition property for compound matrices; and the conservation theorem is easily deduced from it. But if Fj or F* (with counteraxioms elementary) are assumed, then caution is necessary. For the rules Fj, F* are rules for negation, and since in the F-formulation negation is defined only in terms of F, we must think of negation as being at least implicitly present if F is. A more elaborate investigation is necessary, in which the complex relations between the F-, N-, and FN-formulation enter. The upshot is that under certain reasonable assumptions on the counteraxioms, which assumptions hold in particular if there are no counteraxioms or Fj is not assumed, the separation and conservation properties hold under the hypothesis that Γ does not contain F, the conclusion of the former being that Γ can be derived without the rules for F.

The alternation property has a different status. In the first place it is a theorem about singular systems which are absolutely based. In the second place $\mathfrak{X}$ must be formed from $\supset$, $\wedge$, and universal quantification only. In that case it is true without further restriction.

In all the foregoing it is supposed that the special rules Px and Nx do not occur. Let us now discuss the case in which they do. The cases of LC_1^*, LE_1^*, LK_1^* can be taken care of by their equivalence to LC_m^*, LE_m^*, LK_m^* respectively. There remain only the various forms of LD. In [SLD] a form of Glivenko theorem was proved which showed that LD_1 could be reduced to LM_1. Not much has been done with LD since that time. It seems likely that LD_m (when properly formulated) is equivalent to LD_1. Concerning LD* not much is known.

8. *Permutation of rules.* When we have a proof Δ of a statement Γ of form (8) containing a compound parametric consti-

tuent A, it is sometimes desirable to know whether Δ can be recast into a proof Δ^* which ends with a rule introducing this A as principal constituent. This question will be discussed here with reference to the proper multiple systems. The discussion is a modification of Kleene's [PIG] ([41]). (See also Schütte [SWK].)

In accordance with § 3 we may suppose that the prime statements of Δ have only elementary constituents. If we use Formulation III we may also suppose that *W* does not occur ([42]). Furthermore it may be supposed that A has at least one ancestor in a prime statement of Δ; for otherwise one can show that, by the omission of A and all its ancestors, together with certain superfluous branches of Δ, one can have a proof of the Γ' formed by omitting A from Γ, and from this one can have Γ by K*.

The modus operandi is as follows. Let Γ be represented ([43]) as

$$\mathfrak{M},\ A$$

where $\mathfrak{M}$ stands for all the constituents other than A. (I shall use letters from the middle of the alphabet for relatively fixed sets or sequences of propositions.) Let Δ be exhibited in tree form. Let Δ' be a part of Δ consisting of Γ and those nodes of Δ which contain a parametric ancestor of A. Then, since A is compound, there will be no prime statements in Δ', and there will be at least one top node of Δ' introducing A by an operational rule. Then a Δ^* can be constructed if the following conditions are fulfilled:

(a) There is a unique operational rule R_1 for introducing A (on the appropriate side, of course). This will be of the form

$$\frac{\mathfrak{X},\ \mathfrak{Q}_i \quad (i = 1, 2, \ldots, m)}{\mathfrak{X},\ A}$$

([41]) The modification is considerable. Kleene considers only the systems treated by Gentzen, viz. LJ_1 and LK_m. Further his notion of permutability is different; he is concerned more explicitly with a kind of generalized interchange of two inferences, rather than moving a certain kind of inference to the end of the proof. He also makes extensive use of structural rules, whereas here the tendency is to minimize the use of *W*.

([42]) If *W* are admitted the argument here given would be valid with certain modifications. We should have to re-define 'parametric ancestor' so as to allow passage back through *W*.

([43]) It is not necessary to indicate on which side of Γ the constituents occur; but constituents on different sides are still to be regarded as unlike.

where «$\mathfrak{X}$» stands for the parameters which vary according to the application, and «$\mathfrak{Q}_i$» for the subalterns in the i'th premise, which are uniquely determined by A.

(b) If an inference in Δ_1 is made by a rule R_2, then the inference remains valid if A is removed as parameter and parameters $\mathfrak{Q}_i$ are added.

(c) If A is introduced by *K* (or Fj), then the $\mathfrak{Q}_i$ can be introduced by *K* (perhaps with Fj).

(d) The position of A in Γ is such that it can be the principal constituent of an inference by R_1.

The proof that Δ^* can be constructed if (a)—(d) holds is quite simple (44). We proceed to discuss the conditions under which these hypotheses hold.

The condition (a) fails for non-Ketonen forms of *Λ and V*, and for quantification rules. However in Formulation IK and III it holds for all algebraic operational rules.

As for condition (b), arbitrary changes of parameters can be made for all rules except those restricted to be singular on the right and quantification rules with a characteristic variable (which must not appear in the new parameters). If R_2 is of the former kind then A, since it is a parameter, must be on the left, and there is conflict only if there is some $\mathfrak{Q}_i$ on the right; this occurs only if R_1 is *P. The case of quantifiers will concern us later.

The condition (c) causes no difficulty, since *K* is assumed with full generality, if we do not have Fj. If A is introduced by Fj, then it is on the right; and in that case there is a subaltern on the right in every premise of every R_1 (45). Thus Fj causes no difficulty.

The condition (d) causes difficulty only in the case R_1 is restricted to be singular on the right and $\mathfrak{M}$ has a constituent on the right. This can be seen from inspection of Γ without knowledge of Δ.

Thus for algebraic systems the situation is as follows: There is no restriction whatever for LC_m, LE_m, or LK_m. For LA_m, LM_m, and LJ_m

(44) For each i we simply make the transformation indicated under (b) throughout, forming a proof tree Δ_i'. If a top node of Δ' is an introduction of A by R_1, then the corresponding top node of Δ_i' will be the i'th premise of that inference; we then put over that node the proof of that premise from Δ. If a top node of Δ' is introduced by *K*, then by (c) we can derive the corresponding node of Δ_i' from the same premise in Δ. Finally from the conclusions of $\Delta_1', \ldots, \Delta_m'$ one can obtain Γ by R_1.

(45) Except N* in the N-formulation. But then we have no F.

the condition (d) must be fulfilled; then the construction is possible except in the case where R_1 is *P [46].

Let us turn now to quantified systems. There is no further restriction if R_1 is algebraic and no change in regard to (c) or (d). If R_1 is Π^* or $^*\Sigma$, then it can be shown that the bound variables can be so adjusted that there is no conflict with either of the conditions (a) [47] or (b). If R_1 is Σ^* or $^*\Pi$ there may be an unavoidable conflict with condition (b), and examples show that these cases are exceptional [48]. If no such conflict exists [49] there may still be conflict with condition (a). This can be avoided by generalizing Σ^* to [50] $\Sigma^{*\prime}$

$$\Sigma^{*\prime} \qquad \frac{\mathfrak{X} \mid \alpha \vdash A(t_1),\ A(t_2),\ \ldots,\ A(t_r),\ \mathfrak{Z}}{\mathfrak{X} \mid \alpha \vdash (\exists x)A(x),\ \mathfrak{Z}}$$

and then taking $t_1, \ldots, t_r$ to be all the terms used in the various applications of Σ^*; similarly for $^*\Pi$. Thus in the quantified systems we have Σ^* and $^*\Pi$ as additional exceptions for R_1; it does not make any difference whether the system is classically or absolutely based.

The result is independent of ET. It includes the results on invertibility mentioned in § 5 and § 6 [51]. But although it may shorten the proof of ET, the exceptional cases make it seem unlikely that it can completely replace ET.

By the aid of this theorem one can prove the theorem which Gentzen called his «*erweiterter Hauptsatz*», but which, from its close connection with the Herbrand theorem, is better called the *Herbrand-Gentzen* theorem [52]. This is to the effect that if Γ is a theorem of LK_m of form (8) such that all the constituents are in prenex normal form, then there is a proof of Γ in which no quantificational inference is above an algebraic one. This is not a direct consequence of the present result, because it has to be shown that conflict between the quantificational inferences does not occur.

[46] Since *P is an exception we do not have to worry about the quasi-singular constituent in Formulation III.

[47] I.e., one can make the $\mathfrak{Q}_1$ the same in all the applications of R_1 in Δ.

[48] See Kleene [PIG], p. 25.

[49] This will occur, in particular if there are no instances of Π^* or $^*\Sigma$ in Δ.

[50] This inference can be obtained by Σ^* and W*.

[51] Indeed it is only another way of formulating them.

[52] See Craig [LRN]. The theorem is not a direct consequence of ET nor is ET directly deducible from it; so that the term 'extended ET' seems inapt.

9. *Proof tableaux.* A little experience with inferential methods will soon convince one that there is a great deal of rewriting (53); and that although the systems are decidable, yet an actual decision may require the exploration of a large number of alternatives which are difficult to keep track of. There is need of an algorithmic procedure for carrying out the analysis. Such an algorithm should be capable of giving a decision in the case of the decidable systems; but for the quantified systems the most that can be expected is that in all cases where a proof exists the algorithm, if pushed far enough, will produce a proof (54), whereas in other cases the algorithm may produce a negative decision or continue indefinitely. When this is so the algorithm will be said to be *complete* (55).

Algorithms of the sort described were proposed by Beth in his theory of «semantic tableaux». This is presented for classical calculus in his [SEF], a revision for intuitionistic calculus in his [SCI]. In the latter publication it is explicitly tied to a Gentzen-like system given on p. 362. Both systems are presented and discussed at some length in his [FMt]. Briefer or less formal explanations appear in his other papers cited in the Bibliography. These formulations do not agree exactly with one another. Moreover, as inferential formulations of intuitionistic predicate calculus the formulations of his [SCI] and [FMt] are not complete; for Kripke (56) has remarked that the valid statement

$$A, A \supset B \parallel\!\!- B$$

is not obtainable by Beth's algorithm (57).

In this section there will be proposed modifications of Beth's algorithms which are complete and have some other advantages. Because the interpretation in terms of «model» construction, which is Beth's

(53) Various devices, which will not be discussed here, can be used to reduce the amount of rewriting.

(54) Not necessarily the same proof as that which was known to begin with.

(55) This completeness of the algorithm is not to be confused with semantical completeness of the system.

(56) In correspondence. This correspondence contains many suggestions in regard to inferential methods.

(57) Beth's methods should be compared with those of Hintikka. See also Rasiowa and Sikorski [GTh]. The latter shows that the Gödel completeness theorem can be derived quite elegantly by similar methods. Cf. Schütte [SVS].

principal interest, is irrelevant here, the arrays constructed according to the algorithm are called simply *proof tableaux*.

The principle of the algorithm is that we attempt to construct a proof tree upside down. The process is defined by a sequence of rules each of which, when applicable, assigns to a statement called the *datum* another statement called the *result* ([58]). The datum is always an elementary statement (i.e., of form (8)); the result may be an elementary statement, or a conjunction or alternation of such statements ([59]). The rules are ordered, so that, given a datum Γ, the rule to be applied to Γ is uniquely determined. The algorithm begins with a certain initial datum Γ_0, which we call the *head* of the tableau. If Γ_0 is of form (8) with some A_i like some B_j, then no rule is applicable and the algorithm is said to *close* at Γ_0. If this is not the case and still no rule is applicable, then we say the tableau is *open* at Γ_0. If neither of these cases occurs, there is a rule R applicable to Γ_0. If the result of R is a single elementary statement Γ_1, then we proceed with Γ_1 as new head, and the original tableau is closed or open according as the new one is. If the result of R is a conjunction Γ_1 & Γ_2 & ... & Γ_n, then we say the tableau splits conjunctively into the subtableaux headed respectively by $\Gamma_1, \ldots, \Gamma_n$; the original tableau closes just when all of these subtableaux close, and is open when at least one of them is open. If R is an alternation Γ_1 or Γ_2 or ... or Γ_n, then we say the tableau splits alternatively into the subtableaux headed by $\Gamma_1, \ldots, \Gamma_n$ respectively; it is closed when at least one of the subtableaux closes, and is open when they are all open. It is further agreed that if in this process we have a Γ_q which is the same ([60]) as some Γ_p which has already appeared above it, then the rule applied to Γ_p cannot be applied again with the same principal constituent to Γ_q; also that we do not have to repeat constituents.

Under these circumstances, if the algorithm rules are such that the inference from result to datum can always be made by an L-rule, then a closed algorithm will give a proof if it is turned upside down and superfluous alternatives are discarded. But it may not be a decidable question whether the tableau closes or not for a given Γ_0.

([58]) The terms 'premise' and 'conclusion' would be misleading here, since the rules work in the opposite direction from the inferential rules themselves.

([59]) The conjunction and alternation are connections here (i.e. they form statements from other statements), not operations.

([60]) I. e., has the same constituents except possibly for order. Without this convention we might have infinite cycles.

The ordering of the rules is governed by the following heuristic principles, the earliest stated having the greatest priority: 1°. The rules must be equivalences; more precisely, while the inference from result to datum is valid in the sense of deducibility, that from datum to result is valid in the sense of admissibility (cf. § 2). 2°. Rules which can be moved to the end of the proof in § 8 precede the others. 3°. Rules which split the tableau follow those which do not. 4°. Rules on the right should precede those on the left.

With this understanding the following algorithm is proposed. The rule to be applied in any case is the first one that can be applied subject to the above conventions. The sequences $\mathfrak{Z}_1$, $\mathfrak{Z}_2$ are to be void in systems which are absolutely based, otherwise they are not so restricted. The sequences $\mathfrak{X}_1$, $\mathfrak{Y}_1$, $\mathfrak{Z}_1$ must not contain any constituent with the same main operation as the principal constituent, which immediately follows them ([61]); whereas $\mathfrak{X}_2$, $\mathfrak{Y}_2$, $\mathfrak{Z}_2$, $\mathfrak{X}$, $\mathfrak{Y}$ are not so restricted ([62]). The range α_r consists of $a_1, a_2, \ldots, a_r$; a_{r+1} is distinct from any of these. The L-rule permitting the inference from result to datum is indicated on the right, it being understood that *C* can be used wherever needed. In the classically based systems XI is to be omitted and XII is modified; in those without F_j one must omit X.

I $$\frac{\mathfrak{X} \Vdash \mathfrak{Z}_1, A \supset B, \mathfrak{Z}_2}{\mathfrak{X}, A \Vdash \mathfrak{Z}_1, B, \mathfrak{Z}_2} \qquad (P^*)$$

II $$\frac{\mathfrak{X} \mid \alpha_r \vdash \mathfrak{Z}_1, (\forall x)A(x), \mathfrak{Z}_2}{\mathfrak{X} \mid \alpha_{r+1} \vdash \mathfrak{Z}_1, A(a_{r+1}), \mathfrak{Z}_2} \qquad (\Pi^*)$$

III $$\frac{\mathfrak{X}_1, (\exists x)A(x), \mathfrak{X}_2 \mid \alpha_r \vdash \mathfrak{Y}}{\mathfrak{X}_1, A(a_{r+1}), \mathfrak{X}_2 \mid \alpha_{r+1} \vdash \mathfrak{Y}} \qquad ({}^*\Sigma)$$

IV $$\frac{\mathfrak{X} \Vdash \mathfrak{Y}_1, A \vee B, \mathfrak{Y}_2}{\mathfrak{X} \Vdash \mathfrak{Y}_1, A, B, \mathfrak{Y}_2} \qquad (V^*)$$

([61]) E.g., in IV the $\mathfrak{Y}_1$ must not contain any constituent of the form $C \vee D$.

([62]) An alternative procedure, which would give an ordering of the rules more akin to those of Beth and Rasiowa-Sikorski, would be to require that $\mathfrak{X}_1$, $\mathfrak{Y}_1$, $\mathfrak{Z}_1$ be elementary. My proof of completeness would not work in that case, but presumably it could be altered so as to do so. It may be necessary to provide for alternation from side to side, and perhaps to make other changes.

$$\text{V} \quad \frac{\mathfrak{X}_1, A \wedge B, \mathfrak{X}_2 \Vdash \mathfrak{Y}}{\mathfrak{X}_1, A, B, \mathfrak{X}_2 \Vdash \mathfrak{Y}} \quad (^*\wedge)$$

$$\text{VI} \quad \frac{\mathfrak{X} \Vdash \mathfrak{Y}_1, A \wedge B, \mathfrak{Y}_2}{\mathfrak{X} \Vdash \mathfrak{Y}_1, A, \mathfrak{Y}_2 \quad \& \quad \mathfrak{X} \Vdash \mathfrak{Y}_1, B, \mathfrak{Y}_2} \quad (\wedge^*)$$

$$\text{VII} \quad \frac{\mathfrak{X}_1, A \vee B, \mathfrak{X}_2 \Vdash \mathfrak{Y}}{\mathfrak{X}_1, A, \mathfrak{X}_2 \Vdash \mathfrak{Y} \quad \& \quad \mathfrak{X}_1, B, \mathfrak{X}_2 \Vdash \mathfrak{Y}} \quad (^*\vee)$$

VIII (63) $$\frac{\mathfrak{X} \mid \alpha_r \vdash \mathfrak{Y}_1, (\exists x)A(x), \mathfrak{Y}_2}{\mathfrak{X} \mid \alpha_r \vdash \mathfrak{Y}_1, A(a_1), A(a_2), \ldots, A(a_r), \mathfrak{Y}_2, (\exists x)A(x)} \quad (\Sigma'^*)$$

IX (63) $$\frac{\mathfrak{X}_1 (\forall x)A(x), \mathfrak{X}_2 \mid \alpha_r \vdash \mathfrak{Y}}{\mathfrak{X}_1, A(a_1), \ldots, A(a_r), \mathfrak{X}_2, (\forall x)A(x) \mid \alpha_r \vdash \mathfrak{Y}} \quad (^*\Pi')$$

X (64) $$\frac{\mathfrak{X} \Vdash \mathfrak{Y}}{\mathfrak{X} \Vdash \mathfrak{Y}, F} \quad (\text{Fj})$$ (63)

XI $$\frac{\mathfrak{X} \Vdash A \text{ (65)}, \mathfrak{Y}}{\mathfrak{X} \Vdash A \quad \text{or} \quad \mathfrak{X} \Vdash \mathfrak{Y}, A} \quad (\text{K}^*)$$

$$\text{XII} \quad \frac{\mathfrak{X}_1, A \supset B, \mathfrak{X}_2 \Vdash \mathfrak{Y}}{\mathfrak{X}_2, A \supset B, \mathfrak{X}_1 \Vdash \mathfrak{Y}, A \quad \& \quad \mathfrak{X}_1, B, \mathfrak{X}_2 \Vdash \mathfrak{Y}} \quad (^*\text{P})$$

This algorithm is complete in the sense indicated. Even in the undecidable cases the tableau will eventually close if Γ_0 is derivable at all, provided one develops all alternatives in some systematic manner. In the decidable cases the tableau will eventually be either closed or open. The proof is difficult. Whether or not some other arrangement of the rules would be more advantageous for practical purposes I do not know.

(63) The $a_1, \ldots, a_r$ must contain all atoms which appear.
(64) The F_j as stated can be obtained by combining this with K*.
(65) One could restrict A to be one of the forms $B \supset C$, $(\forall x)B(x)$.

Bibliography

This bibliography lists items directly referred to in the text, and also the principal publications known to me which make a significant contribution to the subject matter of this paper and were published since 1934. Works are cited by abbreviated titles in brackets as listed below. When these citations are made without author's name (explicitly or in the context) it is understood the author is Curry, or Curry and Feys. Journals are abbreviated according to the practice of Mathematical Reviews (Providence, R.I.). The bibliography may be supplemented by those found in [TFD], in Feys [MRD] and [NCM], in the bibliographic and review sections of the Journal of Symbolic Logic, and the reviews in Mathematical Reviews.

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ERRATUM

A serious misprint has rendered unintelligible a passage in the first part of this paper. The editor apologizes for this error. The text should read as follows:

Fasc. 11-12 (1960), page 125, lines 5 and 4 from bottom, the rule P* must be:

$$\text{P}^* \qquad \frac{\mathfrak{X}\,,\, A \;||\!\!- B}{\mathfrak{X} \;||\!\!- A \supset B.}$$