REALIZABILITY AND SHANIN'S ALGORITHM FOR THE CONSTRUCTIVE DECIPHERING OF MATHEMATICAL SENTENCES

S. C. KLEENE

1. In [2] or [5] § 82 Kleene interpreted number-theoretic statements through a notion of «realizability», which can be introduced in the following manner.

Consider a formal language like that in [5] but with symbols for some further primitive recursive functions and predicates (besides $'$, $+$, $\cdot$, $=$). We merely assume that these include all the ones appearing below with notations similar to the informal notations of [5] (indexed bottom p. 538). We also write $\langle a_0, \ldots, a_n \rangle$ for $p_0^{a_0} \cdot \ldots \cdot p_n^{a_n}$ (1).

An algorithm is given ([2] top p. 120, or § 5 below) which to each formula E of this language correlates another formula r E of the same language. A closed formula E is said to be *realizable*, if r E is true; an open formula, if r $\forall$E, where $\forall$E is the closure of E, is true. Thus a closed formula is interpreted by reading r E in place of E itself.

Readers of [5] § 82 will not find r E, but only «E is realizable» in the informal language. Thence r E is to be obtained by translation into the formal language (now more extensive than in [5]). Specifically, r E is $\exists$e[e r E] where, if E is closed, e r E expresses «*e* realizes E». If E is open, e r E expresses that *e* realizes E for given natural numbers $y_1, \ldots, y_m$ as the values of the free variables $\mathrm{y}_1, \ldots, \mathrm{y}_m$ of E; or in the terminology of [5], writing E as $\mathrm{E}(\mathrm{y}_1, \ldots, \mathrm{y}_m)$ and $\mathbf{y}_1, \ldots, \mathbf{y}_m$ for the numerals expressing $y_1, \ldots, y_m$, that *e* realizes $\mathrm{E}(\mathbf{y}_1, \ldots, \mathbf{y}_m)$.

In general, certain constructive properties are made explicit in r E which, Kleene argued, are implicit in E from the intuitionistic standpoint but not from the classical. For this reason, r E is not in general equivalent to E classically; for E may be true classically without

(1) To get by here without extending the intuitionistic number-theoretic system of [5], we could use the familiar procedures for eliminating the additional symbols ([5] § 74 Example 9 p. 415), except (for a reason to appear in § 5 below) using $\forall$ instead of $\exists$ (cf. [5] *181 p. 408). The formula in the proof of Theorem 27 middle p. 243 can be replaced by $\forall \mathrm{cd}\{\forall \mathrm{u}[\mathrm{B}(\mathrm{c},\mathrm{d},0,\mathrm{u}) \supset \mathrm{Q}(\mathrm{x}_2,\ldots,\mathrm{x}_n,\mathrm{u})] \,\&\, \forall \mathrm{i}[\mathrm{i}<\mathrm{y} \supset \forall \mathrm{uv}[\mathrm{B}(\mathrm{c},\mathrm{d},\mathrm{i}',\mathrm{u}) \,\&\, \mathrm{B}(\mathrm{c},\mathrm{d},\mathrm{i},\mathrm{v}) \supset \mathrm{R}(\mathrm{i},\mathrm{v},\mathrm{x}_2,\ldots,\mathrm{x}_n,\mathrm{u})]] \supset \mathrm{B}(\mathrm{c},\mathrm{d},\mathrm{y},\mathrm{w})\}$, $\mathrm{B}(\mathrm{c},\mathrm{d},\mathrm{i},\mathrm{w})$ by $\neg \forall \mathrm{v}[\mathrm{c} \neq (\mathrm{i}'\cdot\mathrm{d})'\cdot\mathrm{v}+\mathrm{w}] \,\&\, \mathrm{w}<(\mathrm{i}'\cdot\mathrm{d})'$ (cf. [5] *180b), and $\mathrm{i}<\mathrm{y}$ by $\forall \mathrm{a}[\mathrm{i} \neq \mathrm{y}+\mathrm{a}]$.

those properties holding, so that r E is false (example in [2] § 9 or [5] bottom p. 513). A fortiori, r E is not in general equivalent to E in the prior intuitionistic formal systems of number theory, which are subsystems of the classical (though in [2] Kleene proposed adjoining r E $\sim$ E to such systems).

Subsequently (in [6]) Kleene receded from the position that this realizability algorithm r is wholly appropriate as an intuitionistic interpretation of number-theoretic formulas; it enforces constructivity in such a drastic form as to leave no room for relativized constructivity. In [6] he introduced another realizability notion r′. The algorithm r′ takes any formula E of intuitionistic analysis (construed to include a sufficient collection of symbols for primitive recursive functionals and predicates), and hence any formula of the present language, into a formula r′ E of intuitionistic analysis. Most of what is done below for r will be applicable to r′ also (²).

2. In [17] (³) Shanin writes «The principles of constructive understanding of mathematical sentences presented by Kleene have, in our opinion, real shortcomings of a fundamental character. These shortcomings can be clearly seen in the simplest examples.» Let F be $\forall xyz(x=y \mathbin{\&} y=z \supset x=z)$. Should we write out r F, «we would obtain an altogether cumbersome sentence F_1, more cumbersome and complex than the original sentence F. Furthermore, in the new sentence we do not reveal any features that would testify to the fact that, in some respect, the new sentence is simpler than the original and may be taken as an interpretation of the original one. ... If now we should apply the deciphering rules to F_1, we would get a third sentence F_2 much more involved than F_1, and so on. The deciphering rules for sentences presented by Kleene are not idempotent.» (F_1 is r F, F_2 is r r F, etc.)

Shanin proposes a different algorithm, which we call s, which «connects the constructive problems not with all sentences but only with some. Altering in this direction the basic theoretical apparatus of Kleene makes for greater intelligibility of the problem of constructive understanding of mathematical sentences than has hitherto been attained in other theories.»

3. Kleene aimed at simplicity in the statement of the algorithm, while Shanin aims at simplicity in the results of its application. Each

(²) A monograph [7] dealing with intuitionistic analysis along the lines indicated in [4] and [6] is in preparation.

(³) I thank A. Alexander Khoury for his translation of this paper.

kind of simplicity has its uses, and the two kinds are mutually opposed.

Kleene's r E is $\exists e[e \text{ r } E]$, where e r E is defined recursively by seven simple clauses, each applying uniformly to all formulas that have a given logical symbol (or none) outermost.

Shanin writes «The algorithm of exposition of constructive problems and the algorithm of interpretation of elementary formulas [which together constitute s] are rather unwieldy.» In [17] he only illustrates the action of s, for the detailed description of which the reader is referred to the longer paper [16].

4. Kleene's objectives did not relate to the form of the formulas r E. Therefore it was permissible in his context to simplify the formulas r E by use of any accepted constructive principles of inference.

In fact, r *is* idempotent, not «immediately» in the sense that r r E is (the same formula as) E, but «essentially» in the sense that r r E is equivalent (intuitionistically) to r E. Take the case (of primary interest for the interpretation) of a closed E. By Nelson's [14] Corollary Theorem 2 p. 323, r E $\supset$ E is realizable ([4]). So by [14] Theorem 1 p. 313 (since r E, r E $\supset$ E $\vdash$ E), if r E is realizable, so is E; i.e. r r E implies r E. Similarly, r E implies r r E ([5]). (For open E, [14] Theorem 1 is applied instead to the result of an arbitrary substitution of numerals for the free variables of E.) Another proof will be given in § 6 below.

The proofs of such equivalences might be carried out in stated formal systems, as a control that they use only principles of inference which are accepted as constructive, and also to define any differences in the extensions of constructiveness.

It is known that a formal system *S* consisting of the intuitionistic formal number theory (as in [5]) with the recursion equations for a suitable finite list of additional primitive recursive functions will suffice for formalizing the usual intuitionistically developed theory of general and partial recursive functions ([1]). The original investigation in this direction was Nelson's [14]. From work in a preliminary stage, we believe a recent new approach to recursive functions may offer some advantages in the formalization.

([4]) Nelson's [14] Theorem 1, Corollary Theorem 2, Corollary 4.1, and Corollary Theorem 5 are stated in [2] as (I), (II), (III), and (IV), respectively (for the notation, cf. p. 120). Theorem 1 is also given with a simpler proof in [5] Theorem 62 (a) p. 504.

([5]) E $\supset$ r′ E is not always realizable′, though r′ E $\supset$ E is. Nevertheless, r′ is (essentially) idempotent. A proof, in our notes since September 1952, was intended for publication in [7].

In this short communication we must content ourselves with proofs presented informally, although the results may be stated in the formal language. But it should be clear that the informal proofs can be formalized in an *S* of the sort described, or (where stated) in *S* with Markov's principle (§ 5 below) adjoined.

However for the (essential) idempotence of r, the formalization of the above proof is already available in Nelson's version S_3 of *S*. By [14] Corollary Theorem 5 p. 365, $\vdash$ r (r E $\supset$ E) (4). By [14] Corollary 4.1 p. 361, r r E, r (r E $\supset$ E) $\vdash$ r E. Thence by the deduction theorem, $\vdash$ r r E $\supset$ r E. Similarly, $\vdash$ r E $\supset$ r r E. Thus $\vdash$ r r E $\sim$ r E.

5. Shanin does have an objective relating to the form of the formulas s E. He finds in Lorenzen [12] ideas by which in his view formulas containing only the logical symbols &, $\supset$, $\neg$, $\forall$ can be constructively understood. Under Brouwer's interpretation of V and $\exists$, formulas composed by applying only these two symbols to components already constructively understood can be constructively understood (6). This gives Shanin the incentive to decipher arbitrary formulas by formulas of this sort (which, when only $\exists$ is applied, he calls *completely regular*; when V is applied only after $\exists$, *regular*). Acceptance of «Markov's principle» ([13], and lectures in 1952-53), which we can formalize by the formula

M_1: $\forall e x[\neg\forall y \neg T_1(e, x, y) \supset \exists y T_1(e, x, y)]$,

gives him the means for doing so (7). This suggests to us to verify that the same essentially can be done using r.

(6) We see no reason why & should not be included with the symbols $\vee$, $\exists$ applicable to the components.

(7) Replacing $T_1(e, x, y)$ by $T_n(e, x_1, ..., x_n, y)$, we have formulas M_n (n = 0, 1, 2, ...). All these formulas M_n are implied by M_1, as is $\neg\forall y \neg P \supset \exists y P$ for any prime formula P (expressing a primitive recursive predicate *P*). An equivalent formulation in different symbolism is: $\neg(x)A(x) \rightarrow (Ex)\neg A(x)$, *A* primitive recursive (Kreisel).

It was found from the beginning of the investigations of realizability in 1941 that there are cases in which the realizability of a formula was proved only classically, including cases where what is lacking intuitionistically is this principle to infer that a value of a partial recursive function $\varphi(x) = \mu y P(x, y)$ is defined ($\exists y P(x, y)$) when only the absurdity of its indefinition ($\neg\neg \exists y P(x, y)$ or $\neg \forall y \neg P(x, y)$) is given. One class of examples dating from February 1951 is in G. F. Rose's [15] p. 11. Cf. Remark 2.5 below.

Especially in connection with the author's investigations of r′, it became apparent to him (before he learned of Markov's [13]) that considerable inter-

Let us write the definition of e r E in what seems now the most convenient form, changing it from [2] p. 120 or the direct formalization of [5] § 82 in respects immaterial for the definition of r E as $\exists e[e \text{ r } E]$ ([8]). The bound variables are to be chosen to avoid collisions (similarly later).

1. e r P is P (for P a prime formula).
2. e r (A & B) is $(e)_0$ r A & $(e)_1$ r B.
3. e r (A V B) is $[(e)_0 = 0$ & $(e)_1$ r A] V $[(e)_0 \neq 0$ & $(e)_1$ r B].
4. e r (A ⊃ B) is $\forall a[a \text{ r } A \supset \exists y[T_1(e, a, y) \;\&\; U(y) \text{ r } B]]$.
5. e r ¬A is $\forall a[\neg a \text{ r } A]$.
6. e r $\forall xA(x)$ is $\forall x \exists y[T_1(e, x, y) \;\&\; U(y) \text{ r } A(x)]$.
7. e r $\exists xA(x)$ is $(e)_1$ r $A((e)_0)$.

THEOREM 1. *For each formula* E, *the formula* e r E *is equivalent to a formula* e r_1 E *in which* ∃ *occurs only in parts of the form* $\exists yT_1(t, x, y)$ (x *and* y *distinct variables,* t *a term not containing* x *or* y) *and* V *does not occur.* (Then r E is equivalent to $\exists e[e \text{ r}_1 E]$.)

PROOF. We define e r_1 E like e r E, changing three clauses (cf. [5] *158 and § 63 (61a)).

3. e r_1 (A V B) is $[(e)_0 = 0 \supset (e)_1 \text{ r}_1 A]$ & $[(e)_0 \neq 0 \supset (e)_1 \text{ r}_1 B]$.

est attaches to intuitionistic systems with this principle adjoined, because of a variety of results which it then (or only then) becomes possible to obtain. An example recently come to light is Gödel's result that, if strong completeness of the intuitionistic predicate calculus is provable intuitionistically, so is $\neg(x)A(x) \rightarrow (Ex)\neg A(x)$ for each primitive recursive A (cf. Kreisel [9] Remark 2.1 and [10]).

Kreisel [10] shows that the principle is not provable in the existing intuitionistic formal systems.

A different but related extension of intuitionistic systems, expressible by $\neg\neg(\neg\forall xA(x) \supset \exists x\neg A(x))$ for x ranging over the natural numbers but A(x) not necessarily prime, was proposed by Kuroda in [11], known to us only through Ohnishi's review.

([8]) The reader who prefers may work directly from the definition of e r E in [2] p. 120; e.g. then e r (A & B) is equivalent to e = $\langle (e)_0, (e)_1 \rangle$ & $(e)_0$ r A & $(e)_1$ r B.

Some of the differences between the present definition and [2] p. 120 are due to Nelson [14] p. 356. The equivalence of the present and former r E (call them r E and r_0 E, respectively) is proved by defining for each E a pair of primitive recursive functions $\zeta_E(e)$ and $\eta_E(e)$ such that: when e realizes the result E(Y) of substituting any numerals Y for the free variables Y of E, $\zeta_E(e)$ realizes$_0$ E(Y); and when e realizes$_0$ E(Y), $\eta_E(e)$ realizes E(Y). For example, $\zeta_{A \vee B}(e) = \overline{sg}((e)_0)\cdot\langle 0, \zeta_A((e)_1)\rangle + sg((e)_0)\cdot\langle 1, \zeta_B((e)_1)\rangle$, $\zeta_{A \supset B}(e) = \Lambda a\ \zeta_B(\{e\}(\eta_A(e)))$, etc. (Some similar proofs are given in the text in more detail.)

4. e r_1 (A ⊃ B) is ∀a[a r_1 A ⊃ ∃yT_1(e, a, y) &

∀y[T_1(e, a, y) ⊃ U(y) r_1 B]].

6. e r_1 ∀xA(x) is ∀x[∃yT_1(e, x, y) &

∀y[T_1(e, x, y) ⊃ U(y) r_1 A(x)]].

COROLLARY 1.1. *Using Markov's principle* M_1 (and [5] *83a), e r E *is equivalent to a formula* e r_2 E *in which* ∨ *and* ∃ *do not occur.* (Then r E is equivalent to ∃e[e r_2 E], which is completely regular.)

6. Shanin's objection to r that it alters, without clarifying, some simple formulas suggests to us to verify that, for a certain class of E's (including all regular formulas), r E is equivalent to E. The (essential) idempotence of r will become an application of this result.

LEMMA 2.1a. *To each formula* E(Y) *containing free only the variables* Y *and not containing* ∨ *or* ∃, *there is a number* e_E *such that, for each choice of natural numbers* Y *(with corresponding numerals* **Y**)*:*

(i) *If* E(**Y**) *is realizable, then* E(**Y**) *is true.*

(ii)a *If* E(**Y**) *is true, then* e_E *realizes* E(**Y**).

PROOF, by induction on the number of logical symbols in E(Y) (briefly, E). According as E is of the form P (a prime formula), A & B, A ⊃ B, ¬A, or ∀xA(x), let $e_E = 0, \langle e_A, e_B \rangle, \Lambda a\, e_B, 0$, or $\Lambda x\, e_{A(x)}$, respectively. One case will suffice for illustration.

CASE 4: E is A ⊃ B. Then $e_E = \Lambda a\, e_B$. (i) Suppose A(**Y**) ⊃ B(**Y**) is realizable; say e realizes it. Suppose A(**Y**) is true. By the hypothesis of the induction (ii)a, e_A realizes A(**Y**), so $\{e\}(e_A)$ realizes B(**Y**) (i.e. in the formal language, ∃y[T_1(**e**, $\mathbf{e}_A$, y) & U(y) r B(Y)]), and by hyp. ind. (i), B(**Y**) is true. Thus A(**Y**) ⊃ B(**Y**) is true. (ii)a Suppose A(**Y**) ⊃ B(**Y**) is true. Suppose a realizes A(**Y**); then by hyp. ind. (i), A(**Y**) is true, so B(**Y**) is true, and by hyp. ind. (ii)a, e_B realizes B(**Y**). Thus $\Lambda a\, e_B$ realizes A(**Y**) ⊃ B(**Y**).

LEMMA 2.1b. *To each formula* E(Y) *containing free only the variables* Y, *and containing no* ∃, *other than in parts of the form* ∃xP(x) *with* P(x) *prime, and no* ∨, *there is a partial recursive function* ε_E(Y) *such that, for each* Y [9] :

(i) *If* E(**Y**) *is realizable, then* E(**Y**) *is true.*

(ii)b *If* E(**Y**) *is true, then* ε_E(Y) *(is defined and) realizes* E(**Y**).

[9] In the corresponding lemma for r′, ш is admitted also in parts of the form ∃αP(α) with α a function variable and P(α) prime.

PROOF, by ind. According as E(Y) is of the form P, A & B, A ⊃ B, ¬A, ∀xA(x), or ∃xP(x), let $\varepsilon_E(Y) \simeq 0, \langle \varepsilon_A(Y), \varepsilon_B(Y) \rangle, \Lambda a\ \varepsilon_B(Y)$, $0, \Lambda x\ \varepsilon_{A(x)}(x, Y)$, or $\langle \mu x P(x, Y), 0 \rangle$ where $P(x, Y)$ is the predicate expressed by the prime formula P(x).

LEMMA 2.2. *For each formula* E *composed without using* ⊃ *or* ∀ *from components* $C_1, \ldots, C_l$ *(not necessarily prime),* r E *is equivalent to the formula composed correspondingly from* r C_1, ..., r C_l (10).

PROOF. By induction, it will suffice to verify that r (A & B), r (A ∨ B), r ¬A, and r ∃xA(x) are equivalent to r A & r B, r A ∨ r B, ¬ r A, and ∃x r A(x), respectively. For example:

CASE 3: E is A ∨ B. PART 1. Assume r (A ∨ B), i.e., for some e, $[(e)_0 = 0$ & $(e)_1$ r A] ∨ $[(e)_0 \neq 0$ & $(e)_1$ r B]. CASE A: $(e)_0 = 0$ & $(e)_1$ r A. Then $(e)_1$ r A, whence r A, whence r A ∨ r B. CASE B: $(e)_0 \neq 0$ & $(e)_1$ r B. Similarly. PART 2. Assume r A ∨ r B. CASE A: r A, i.e., for some a, a r A. Then $\langle 0, a \rangle$ r (A ∨ B), so r (A ∨ B). CASE B: r B. Similarly.

THEOREM 2. *For each formula* E *in which no* ∃, *other than in parts of the form* ∃xP(x) *with* P(x) *prime, and no* ∨ *lies in the scope of any* ⊃ *or* ∀:

(iii) r E *is equivalent to* E, *i.e.* E(**Y**) *is realizable if and only if* E(**Y**) *is true.*

PROOF. Consider the minimal components C_1, ..., C_l of E which stand inside no ⊃ or ∀. These components contain no ∃, other than in parts of the form ∃xP(x), and no ∨. By Lemma 2.1a (or Lemma 2.1b if C_i contains parts ∃xP(x)), r C_i is equivalent to C_i ($i = 1, \ldots, l$). But E is composed from C_1, ..., C_l without using ⊃ or ∀. By Lemma 2.2, (iii) holds.

REMARK 2.3. The condition on E in this theorem does not cover all cases in which (iii) holds. For, if E_1 is equivalent intuitionistically to E by inferences formalizable in *S*, then r E_1 is equivalent to r E, using Nelson's [14] Theorem 1 or formally [14] Corollary 4.2 p. 362 (4); and hence, if E satisfies the condition, r E_1 is equivalent to E_1. (Conversely, by Theorem 1 above, if r E_1 is equivalent to E_1, then E_1 is equivalent to an E satisfying the condition.) Any ∃-prenex part $\exists x_1 \ldots x_n A(x_1, \ldots, x_n)$ is equivalent in *S* to a part ∃xP(x). An E_1 might have ∨'s and ∃'s which could by intuitionistic equivalences

(10) In the corresponding lemma for r′, ¬ and ∃α for a function variable α must also be excluded.

be brought into $\exists$-prenex parts, or be removed by advance across $\neg$'s ([5] *63, *86).

COROLLARY 2.4. *For each formula* E:
(iv) r r E *is equivalent to* r E.

REMARK 2.5. We have stated the theorem so that applications do not depend on accepting Markov's principle M_1. By the theorem, M_1 is equivalent to r M_1, i.e., Markov's principle implies, and is implied by, its own realizability (by reasoning formalizable in *S*).

7. Shanin gives no indication in [17] that he has considered the question whether s is «essentially» distinct from r in the sense that s E is not always equivalent to r E. The current situation, in which several pairwise essentially distinct interpretations have appeared (the identical interpretation, r Kleene [2], r′ Kleene [6], Gödel's interpretation [1], [8], and others in Kreisel [8]), surely makes it of interest to know whether Shanin's s is still another. In § 8 we shall show that, using Markov's principle, s is «essentially» equivalent to r, i.e. s E is always equivalent to r E. Two preliminary difficulties face us.

First, we must arrange that s operate in the same language as r. Shanin illustrates s starting in another language in which the terms are built from numerals and variables by repeated use of (essentially) Kleene's universal partial recursive functions $\{z\}(x_1, ..., x_n)$ ($\simeq U(\mu y T_n(z, x_1, ..., x_n, y))$) [5] § 65. His prime (or «elementary») formulas are !t («t is defined») and $t_1 \simeq t_2$ (complete equality [5] § 63) where t, t_1, t_2 are any terms. He first applies «the algorithm of exposition of constructive problems» within this language, and then applies «the algorithm of interpretation of elementary formulas» to express !t and $t_1 \simeq t_2$ in a language like ours.

In Shanin's first language of course he can express all primitive recursive functions and predicates. The fact that he *must* in his first algorithm so express even such simple predicates as = and < requires in general more steps in his second algorithm. There can be no objection from the standpoint of the interpretation to adding to Shanin's first language all our symbols for primitive recursive functions and predicates.

Now we have a language with both his symbols and ours. But his symbols { }( ), !, $\simeq$ which ours lacks can be expressed in terms of ours. In the case of no nesting of { }( ), equivalents in ours of !t and $t_1 \simeq t_2$ are given by [5] § 63 Theorem XIX (a) fourth line and Example 2 next page. The $\exists$'s which these equivalents contain can be removed as in the proof of Theorem 1 and Corollary above, granting Markov's principle. Nested uses of { }( ) can be eliminated progressively from inside, similarly to [5] § 74, but using ∀ instead of ∃ (cf. [5] *181). This essentially is Shanin's second algorithm (11). We can thus interpret each formula in the language having both his symbols and ours as standing for a formula in ours.

In some steps his first algorithm introduces new instances of { }( ), !, ≃. We can instead pass directly to te result of their elimination (as indeed is done in defining e r E, e r_1 E or e r_2 E compared to defining «*e* realizes E(**Y**)» verbally).

In brief, he applies his two algorithms successively. Instead, we apply his second algorithm intermittently, so as to remain in our language throughout the application of his first algorithm.

Our second preliminary difficulty is that his two algorithms are described in detail only in [16], of which we have seen only Heyting's review. But the illustrations of their action in [17] are sufficient to enable us, we believe, to set up an algorithm s which must agree with his to within inessential details when the languages are brought into correspondence in the manner just indicated. We use the slight simplifications which our symbolism affords within the spirit of his procedure.

8. Now we find four general differences between Shanin's procedure and our r. (1) His utilizes Markov's principle to end up in [16], according to Heyting's review, with completely regular formulas (in [17] with regular formulas). (2) His in [16] leaves completely regular formulas (in [17] regular formulas) unaltered, and alters regular components of irregular formulas less than r does. (3) His handles consecutive occurrences of a given one of &, V, ∀, ∃ in one operation (12). (4) His does not contract pairs (or *n*-tuples) of natural numbers into single natural numbers (13).

In §§ 5 and 6 we have seen the way to bridge the differences (1) and (2), respectively. We arrange our treatment to allow the choice (A or B) of following [16] or [17], and also the choice (a or b) of assuming Markov's principle or not. Shanin calls a formula *normal*, if it contains no V or ∃; we call a formula *seminormal*, if it contains no ∃, except in parts of the form ∃xP(x) (P(x) prime), and

(11) He requires an extra step because he doesn't assume for his ϱ_n the property $\varrho_n(z, x_1, ..., x_n, y)=0$ & $\varrho_n(z, x_1, ..., x_n, w)=0 \supset y=w$, which is already built into the representing function of our predicate $T_n(z, x_1, ..., x_n, y)$.

(12) For *prenex* ∀'s, this was introduced into the handling of the realizability notion in [3] (cf. [5] pp. 503, 508).

(13) Is $\exists x_0 ... x_n A(x_0, ..., x_n)$ or $\exists x A((x)_0, ..., (x)_n)$ simpler?

no V. Allowing seminormal instead of only normal components, we obtain *completely semiregular* (*semiregular*) formulas instead of completely regular (regular) ones.

Under Choice b, to each formula E we now correlate a list $\mathfrak{d}$ of variables $d_1, \ldots, d_l$ ($l \geqq 0$) and a seminormal formula $\mathfrak{d}$ s E. Just as e r E expresses «*e* realizes E(**Y**)» and r E expresses «E(**Y**) is realizable», we can read $\mathfrak{d}$ s E as «$\mathfrak{d}$ solve E(**Y**)» and s E (below) as «E(**Y**) is solvable».

1. If E is seminormal, $\mathfrak{d}$ is empty ($l = 0$) and $\mathfrak{d}$ s E is E.

In the remaining cases E shall not be seminormal. We write $\mathfrak{a}$ for the list of variables $a_1, \ldots, a_n$ correlated to A, and $\mathfrak{b}$ for the list $b_1, \ldots, b_m$ correlated to B.

2. If E is A_1 & ... & A_k where $k > 1$ and none of $A_1, \ldots, A_k$ is a conjunction, then $\mathfrak{d}$ is $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ and $\mathfrak{d}$ s E is $\mathfrak{a}_1$ s A_1 & ... & $\mathfrak{a}_k$ s A_k.

3. If E is $A_1 \vee \ldots \vee A_k$ where $k > 1$ and none of $A_1, \ldots, A_k$ is a disjunction, then $\mathfrak{d}$ is u, $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ and $\mathfrak{d}$ s E is
$[u = 0 \supset \mathfrak{a}_1 \text{ s } A_1]$ & ... & $[u = \mathbf{k}-2 \supset \mathfrak{a}_{k-1} \text{ s } A_{k-1}]$ & $[u > \mathbf{k}-2 \supset \mathfrak{a}_k \text{ s } A_k]$.

4. If E is $A \supset B$, then $\mathfrak{d}$ is $\mathfrak{b}$ and, if $n > 0$, $\mathfrak{d}$ **s** E is $\forall\mathfrak{a}[\mathfrak{a} \text{ s } A \supset \exists y_1 T_n(b_1, \mathfrak{a}, y_1)$ & ... & $\exists y_m T_n(b_m, \mathfrak{a}, y_m)$ & $\forall y_1 \ldots y_m [T_n(b_1, \mathfrak{a}, y_1)$ & ... & $T_n(b_m, \mathfrak{a}, y_m) \supset U(y_1), \ldots, U(y_m) \text{ s } B]]$ (which for $m = 0$ reduces to $\forall\mathfrak{a}[\mathfrak{a} \text{ s } A \supset \text{s } B]$); but if $n = 0$, $\mathfrak{d}$ s E is s $A \supset \mathfrak{b}$ s B.

5. If E is $\neg A$, then $\mathfrak{d}$ is empty and $\mathfrak{d}$ s E is $\forall\mathfrak{a}[\neg\, \mathfrak{a} \text{ s } A]$.

6. If E is $\forall\mathfrak{x}A$ where $\mathfrak{x}$ consists of $k > 0$ variables and A does not begin with $\forall$, then $\mathfrak{d}$ is $\mathfrak{a}$ and $\mathfrak{d}$ s E is $\forall\mathfrak{x}[\exists y_1 T_k(a_1, \mathfrak{x}, y_1)$ & ... & $\exists y_n T_k(a_n, \mathfrak{x}, y_n)$ & $\forall y_1 \ldots y_n [T_k(a_1, \mathfrak{x}, y_1)$ & ... & $T_k(a_n, \mathfrak{x}, y_n) \supset U(y_1), \ldots, U(y_n)$ **s** A]] (which for $n = 0$ reduces to $\forall\mathfrak{x}[\text{s } A]$).

7. If E is $\exists\mathfrak{x}A$ where $\mathfrak{x}$ consists of $k > 0$ variables and A does not begin with $\exists$, then $\mathfrak{d}$ is $\mathfrak{x}, \mathfrak{a}$ and $\mathfrak{d}$ s E is $\mathfrak{a}$ s A.

Lemma 3.1b. *To each formula* E(Y) *containing only the variables* Y, *there are partial recursive functions* γ_E *and* δ_E *such that:*

(v) *If* $\mathfrak{d}$ *solve* E(**Y**), *then* $\gamma_E(\mathfrak{d}, Y)$ (*is defined and*) *realizes* E(**Y**).

(vi) *If e realizes* E(**Y**), *then* $\delta_E(e, Y)$ (*is defined and*) $= \langle d_1, \ldots, d_l \rangle$ *where* $d_1, \ldots, d_l$ *solve* E(**Y**).

Proof, by ind. on the number of logical symbols in E, with cases corresponding to those in the definition of $\mathfrak{d}$ s E.

Case 1: E is seminormal. Then $\mathfrak{d}$ is empty and $\mathfrak{d}$ s E is E. (v) Let $\gamma_E(Y) \simeq \varepsilon_E(Y)$ (Lemma 2.1b). If $\mathfrak{d}$ solve E(**Y**), then E(**Y**) is true, and by (ii)b $\varepsilon_E(Y)$ realizes E(**Y**). (vi) Let $\delta_E(e, Y) = 1$. Suppose *e* realizes E(**Y**). By Lemma 2.1b (i), E(**Y**) is true, i.e. $d_1, \ldots, d_l$ for $l = 0$ solve E(**Y**); and (for $l = 0$) $\langle d_1, \ldots, d_l \rangle = 1$.

Case 3: E is $A_1 \vee \ldots \vee A_k$, etc. Say e.g. E is $(A_1 \vee A_2) \vee A_3$. Then $\mathfrak{d}$ is u, $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$, etc. (v) Using [5] Theorem XX (c) p. 337, let $\gamma_E(\mathfrak{d}, Y) \simeq \langle 0, \langle 0, \gamma_{A_1}(\mathfrak{a}_1, Y)\rangle\rangle$ if $u = 0$ (Case A), $\simeq \langle 0, \langle 1, \gamma_{A_2}(\mathfrak{a}_2, Y)\rangle\rangle$ if $u = 1$ (Case B), $\simeq \langle 1, \ {}_{A_3}(\mathfrak{a}_3, Y)\rangle$ if $u > 1$ (Case C). Suppose $\mathfrak{d}$ solve $E(\mathbf{Y})$. In Case A, then $\mathfrak{a}_1$ solve $A_1(\mathbf{Y})$, so by hyp. ind. (v), $\gamma_{A_1}(\mathfrak{a}_1, Y)$ realizes $A_1(\mathbf{Y})$, so $\langle 0, \langle 0, \gamma_{A_1}(\mathfrak{a}_1, Y)\rangle\rangle$ $(= \gamma_E(\mathfrak{d}, Y))$ realizes $E(\mathbf{Y})$. The other two cases are similar. (vi) Say e.g. $n_1 = 2$, $n_2 = 0$, $n_3 = 1$. Let $\delta_E(e, Y) \simeq \langle 0, (\mathfrak{d}_{A_1}((e)_{1,1}, Y))_0, (\delta_{A_1}((e)_{1,1}, Y))_1, 0\rangle$ if $(e)_0 = (e)_{1,0} = 0$ (Case A), $\simeq \langle 1, 0, 0, 0\rangle$ if $(e)_0 = 0$ & $(e)_{1,0} > 0$ (Case B), $\simeq \langle 2, 0, 0, (\delta_{A_3}((e)_1, Y))_0\rangle$ if $(e)_0 > 0$ (Case C). Suppose e realizes $E(\mathbf{Y})$. In Case A, $(e)_{1,1}$ realizes $A_1(\mathbf{Y})$ so $\mathfrak{d}_{A_1}((e_{1,1}, Y) = \langle a_1, a_2\rangle$ where a_1, a_2 solve $A_1(\mathbf{Y})$, i.e. $(\delta_{A_1}((e)_{1,1}, Y))_0$, $(\delta_{A_1}((e)_{1,1}, Y))_1$ solve $A_1(\mathbf{Y})$, so 0, $(\delta_{A_1}((e)_{1,1}, Y))_0$, $(\delta_{A_1}((e)_{1,1}, Y))_1$, 0 solve $E(\mathbf{Y})$.

Under Choice a, we modify the construction of $\mathfrak{d}$ s E to make the result always normal: first replace each part of E of the form $\exists xP(x)$ ($P(x)$ prime) by $\neg\forall x \neg P(x)$, then apply the clauses used under Choice b, and finally replace each resulting $\exists yT$ by $\neg\forall y \neg T$. To distinguish this from the former $\mathfrak{d}$ s E, we may write them $\mathfrak{d}$ s_a E and $\mathfrak{d}$ s_b E, respectively. In Lemma 3.1a, the parameters Y are omitted from the functions γ_E and δ_E, but Markov's principle is assumed.

We give four definitions of s E. Say E is $E_1 \vee \ldots \vee E_j$ where $j \geqq 1$ and none of $E_1, \ldots, E_j$ is a disjunction. Under Definition Aa, Ab, Ba, or Bb, s E is $\exists\mathfrak{d}[\mathfrak{d}\ s_a\ E]$, $\exists\mathfrak{d}[\mathfrak{d}\ s_b\ E]$, $\exists\mathfrak{d}_1[\mathfrak{d}_1\ s_a\ E_1] \vee \ldots \vee \exists\mathfrak{d}_j[\mathfrak{d}_j\ s_a\ E_j]$, or $\exists\mathfrak{d}_1[\mathfrak{d}_1\ s_b\ E_1] \vee \ldots \vee \exists\mathfrak{d}_j[\mathfrak{d}_j\ s_b\ E_j]$, respectively. Under each definition, when $\mathfrak{d}$ is empty, $\mathfrak{d}$ s E as defined earlier and s E as defined now coincide; so the notation is consistent. Respectively, s E is (and is E when E is) completely regular, completely semiregular, regular, semiregular.

Theorem 3. *Assuming Markov's principle, if Definition* Aa *or* Ba *is chosen, for each formula* E:

(vii) s E *is equivalent to* r E.

Proof. Under Definition Aa (Ab), the theorem is immediate from Lemma 3.1a (3.1b). Under Definition Ba (Bb), it follows thence by Lemma 2.2.

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The University of Wisconsin. S. C. KLEENE
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