AXIOMATIZATION OF INFINITE VALUED LOGICS

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1. *The propositional calculus.* The infinite valued propositional calculus was first described in 1930 by Łukasiewicz and Tarski (see [1]). They used the two propositional connectives **C** and **N**, construed as generalizations of implication and negation respectively. The propositions involved were to be considered as taking any of an infinity of truth values, these being real numbers lying in the range 0 to 1 inclusive. Interpretations of such a situation are necessarily unnatural, but the suggestion was that one might conceive of 0 as the ultimate of falsehood and 1 as the ultimate of truth. With **C** and **N** are associated the respective truth functions:

(1) $\mathbf{c}(x, y) = \min(1, 1 - x + y)$

(2) $\mathbf{n}(x) = 1 - x.$

Note that **C** can be thought of as embodying the notion of $\leq$, since **C***PQ* can have the ultimate of truth if and only if the value assigned to *P* is not greater than that assigned to *Q*.

Łukasiewicz and Tarski defined

(3) **A***PQ* for **CC***PQQ*

(4) **K***PQ* for **NAN***P***N***Q*

and observed that **A** and **K** can serve fairly well as generalizations of disjunction and conjunction respectively. In particular, they satisfy many familiar laws. For instance, they satisfy the commutative laws that no matter what values are assigned to *P* and *Q*, the value of **A***PQ* is the same as that of **A***QP*, and the value of **K***PQ* is the same as that of **K***QP*. They also satisfy the associative laws that **AA***PQR* takes the same value as **A***P***A***QR* and that **KK***PQR* takes the same value as **K***P***K***QR*. We note that **N** satisfies the law of double negation, that **NN***P* takes the same value as *P*; accordingly, as we have embodied one of de Morgan's laws in the definition of **K**, we readily infer the other, namely that **A***PQ* takes the same value as **NKN***P***N***Q*. We note also the familiar results that **A***PP* and **K***PP* take the same value as *P*, and **C***PQ* takes the same value as **CN***Q***N***P*. There are also the distributive laws that **K***P***A***QR* takes the same value as **AK***PQ***K***PR*, and **A***P***K***QR* takes the same value as **KA***PQ***A***PR*.

However, **C***P***C***QR* does not always take the same value as **CK***PQR*, nor does **A***PQ* always take the same value as **CN***PQ*. This suggests defining **B** and **L** as alternative notions of disjunction and conjunction respectively by

(5) $\quad \mathbf{B}PQ$ for $\mathbf{CN}PQ$

(6) $\quad \mathbf{L}PQ$ for $\mathbf{NC}P\mathbf{N}Q$.

For these also, we have the commutative and associative laws, and de Morgan's laws. Of the six additional distributive laws that one could write by combining either of the disjunctions with either of the conjunctions, only two are valid, namely that **L***P***A***QR* takes the same value as **AL***PQ***L***PR*, and **B***P***K***QR* takes the same value as **KB***PQ***B***PR*. In addition, **C***P***C***QR* takes the same value as **CL***PQR*. Also **B***P***N***P* always takes the value unity, which **A***P***N***P* commonly does not do; dually, **L***P***N***P* always takes the value zero, which **K***P***N***P* commonly does not do. Commonly **B***PP* and **L***PP* do not take the same value as *P*.

There are truth functions associated with **A**, **K**, **B**, and **L** respectively, namely:

(7) $\quad \mathbf{a}(x, y) = \max(x, y)$

(8) $\quad \mathbf{k}(x, y) = \min(x, y)$

(9) $\quad \mathbf{b}(x, y) = \min(1, x + y)$

(10) $\quad \mathbf{l}(x, y) = \max(0, x + y - 1)$.

In view of the proposed interpretation of unity as the ultimate of truth, especial interest attaches to those propositions, such as **B***P***N***P* or **C***P***C***QP*, which always take the value unity regardless of the values assigned to their constituents. A conjecture of Łukasiewicz, recorded in [1], asserted in effect that the propositions which always take the value unity are just those derivable by Rule **C** below from the five axiom schemes displayed below.

Rule **C**. If *P* and **C***PQ*, then *Q*.

L1. **C***P***C***QP*.

L2. **CC***PQ***CC***QR***C***PR*.

L3. **CA***PQ***A***QP*.

L4. **AC***PQ***C***QP*.

L5. **CCN***P***N***Q***C***QP*.

A proof of this conjecture was announced in 1935 by Wajsberg (see [2]), but publication was delayed and the proof was lost during the dislocations of the war. A later, and probably quite different, proof was published in 1958 by Rose and Rosser (see [3]), and in 1959 still another proof was published by Chang using algebraic techniques (see [4]). Proofs of axiom scheme **L4** from the others were published in 1958 by Meredith (see [5]) and Chang (see [6]). A proof that **L4** cannot be derived from **L1**, **L2**, and **L3** alone was communicated privately to us by A. R. Turquette, and is in process of publication. This is of interest because the first four of Łukasiewicz's axiom schemes do not involve **N**.

There is considerable diversity of choice as to the possible sets of truth values. Each must be a set of real numbers, infinite in number, lying in the range 0 to 1 inclusive, and closed under application of the truth functions **c** and **n**. It is clear that a proposition always takes the value unity relative to one such set if and only if it does so relative to all such sets. As illustrations of such sets we might cite: all rational numbers in the range 0 to 1; all real numbers in the range 0 to 1; all rational numbers in the range 0 to 1 having a fractional representation whose denominator is a power of 2; all real numbers in the range 0 to 1 differing by an integer from an integer multiple of π.

One cannot construct all truth functions by means of **c** and **n**; indeed one clearly cannot construct a truth function whose value will be different from 0 or 1 when the value 1 is assigned to all the arguments. This inability to construct all truth functions is described by saying that the calculus in terms of **C** and **N** is functionally incomplete. A simple cardinality argument shows that any calculus with a most a denumerable number of propositional connectives must be functionally incomplete.

Instead of confining our attention to formulas whose value is always unity, we could inquire about the class of formulas each of which always takes a value $\geqslant r$. For instance, if $r=0{,}5$, then **A***P***N***P* would lie in the corresponding class. A reasonable question to ask is for which choices of r is the corresponding class of formulas recursively enumerable. As there is a non-denumerable number of possible r's, and only a denumerable number of recursively enumerable classes of formulas built from **C** and **N**, the chance of choosing an r which corresponds to a recursively enumerable class is vanishingly small in this case. Since the axiomatization of Łukasiewicz shows that the choice $r = 1$ produces a recursively enumerable class, one must conclude that this choice of r is quite special. Naturally it is also quite special in that it is the largest possible choice of r, but it is not clear that there is any relationship between these two special properties of $r = 1$.

We need hardly note that the determination of a class associated with r depends on both the set of propositional connectives and the set of truth values that one has chosen.

2. *Indication of proofs.* The proof of Łukasiewicz's conjecture by Rose and Rosser involves a frightening amount of detail, but we will attempt to give the basic idea and to make it plausible that one could expand this into a complete proof by adjoining a sufficient number of details.

As developed by Rose and Rosser there were originally two quite separate phases of the proof. In the first phase, it was established that there are finitely many axiom schemes (more than seventy were required at first) which suffice (with Rule **C**). In the second phase, these schemes were derived from Łukasiewicz's five axiom schemes. Many opportunities for ingenuity and dexterity in handling formulas occurred in the second phase, but there were no conceptual difficulties, and so little real interest attaches to this phase. In preparing [3], the historical origins of the proof were completely submerged, and what is left of the pattern of the first phase is obscured by the tremendous mass of details derived from the second phase.

Conceptually, there is heavy reliance on a result of McNaughton (see [7]). For conciseness, let us temporarily refer to a polynomial, $a + \Sigma\, b_j\, x_j$, in which a and the b_j are all integers, as a McNaughton polynomial. The relevant result by McNaughton is the following.

Let P be a proposition built by **C** and **N** from some or all of the basic propositions $P_1, \ldots, P_n$. Using $x_1, \ldots, x_n$ as variables denoting values of the respective P_m's, there are McNaughton polynomials f_{ij} and g_j in the x's such that in the j-th domain of the unit n-cube, given by

$$f_{ij} \geqslant 0 \qquad\qquad i = 1, \ldots, \alpha_j,$$

the value of P is given by g_j. Such domains cover the entire cube.

The proof goes by induction on the number of symbols in P, and it suffices to indicate one portion of the proof, namely where we assume the result for P_1 and P_2 and deduce it when P is $\mathbf{C}P_1P_2$. That is, we assume that when

$$(11) \qquad f_{ij}^m > 0 \qquad\qquad i = 1, \ldots, \alpha_j^m$$

holds, then the value of P_m is given by g_j^m ($m = 1, 2$). Then (see (1) above) if (11) holds for both $m = 1$ and 2 and if

$$g_j^2 - g_j^1 \geqslant 0,$$

take $g_j = 1$, while if (11) holds for both $m = 1$ and 2 and if

$$g_j^1 - g_j^2 \geqslant 0,$$

take $g_j = 1 - g_j^1 + g_j^2$.

Although this result is metamathematical, there is a formal equivalent. To get this, one shows that for each McNaughton polynomial f there is a proposition F such that F takes the value f when $0 \leqslant f \leqslant 1$ and otherwise F takes the value 0 or 1 according as $f \leqslant 0$ or $f \geqslant 1$. This is proved by induction on the sum of the absolute values of the coefficients appearing in f. Consequently, one has the possibility of proving results involving F and f by induction, which means that each such proof is reducible to a finite number of explicit, direct steps; by having available an appropriate formal equivalent for each of these steps, one can formalize the result in question. As the first phase of the proof was originally conceived, formalization of each step was commonly achieved by adjoining a suitably chosen axiom, but in [3] one commonly cites a result proved earlier for just this purpose.

Assuming that F_{ij} has been chosen to go with $f_{ij} + 1$ and G_j to go with g_j, one can formalize McNaughton's key result as signifying that if one assumes each of $F_{ij} (i = 1, \ldots, \alpha_j)$ in addition to Łukasiewicz's axioms, then $G_j \equiv P$ is deducible, where $Q \equiv R$ denotes **L**C*Q*R**C***RQ*.

The proof of this formal result follows the same inductive schema that was used in the proof of the metamathematical result. There are a finite number of direct steps, each of which embodies one or more results involving a McNaughton polynomial f and its corresponding proposition F; as noted above, each proof of one of these results is done by induction at the cost of adjoining a finite number of axioms or having proved a finite number of formal results from Łukasiewicz's axioms.

If P happens to take only the value unity, then G_j is **C***RR* for each j. Then the formal conclusion is the same in each region of the unit n-cube. As the regions cover the cube, the corresponding sets of F_{ij} are exhaustive, which can be proved formally by an induction based on a finite number of formal results. Thus one can infer **C***RR* $\equiv P$, from which P itself readily follows.

We shall now summarize Chang's algebraic proof of the completeness of Łukasiewicz's axioms. This makes extensive use of a very interesting algebraic structure, which Chang refers to as an MV-algebra (see [8]) but which other authors generally refer to as a Chang algebra.

A Chang algebra is a generalization of a Boolean algebra. It is a system $\langle A, +, \cdot, -, 0, 1 \rangle$ depending on a set A which contains at least the elements 0 and 1 and which is closed under the binary operations + and · and the unary operation —. In the classically careless algebraic tradition, A is commonly used alone to denote the entire structure, which after making the definitions

Def. $x \vee y = (x \cdot \bar{y}) + y \qquad x \wedge y = (x + \bar{y}) \cdot y$

is required to satisfy the axioms:

A 1.	$x + y = y + x$.	**A 1′.**	$x \cdot y = y \cdot x$.
A 2.	$x + (y + z) = (x + y) + z$.	**A 2′.**	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
A 3.	$x + \bar{x} = 1$.	**A 3′.**	$x \cdot \bar{x} = 0$.
A 4.	$x + 1 = 1$.	**A 4′.**	$x \cdot 0 = 0$.
A 5.	$x + 0 = x$.	**A 5′.**	$x \cdot 1 = x$.
A 6.	$\overline{x + y} = \bar{x} \cdot \bar{y}$.	**A 6′.**	$\overline{x \cdot y} = \bar{x} + \bar{y}$.
A 7.	$x = \bar{\bar{x}}$.	**A 8.**	$\bar{0} = 1$.
A 9.	$x \vee y = y \vee x$.	**A 9′.**	$x \wedge y = y \wedge x$.
A 10.	$x \vee (y \vee z) = (x \vee y) \vee z$.	**A10′.**	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$.
A 11.	$x + (y \wedge z) = (x + y) \wedge (x + z)$.	**A 11′.**	$x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$.
A 12.	$0 \neq 1$.		

We have $x \vee x = x = x \wedge x$. The following conditions are equivalent, and usually fail: $x + x = x$, $x \cdot x = x$, $\bar{x} + \bar{x} = \bar{x}$, $\bar{x} \cdot \bar{x} = x$, $\bar{x} \vee \bar{x} = 1$, $x \wedge \bar{x} = 0$.

Each Boolean algebra is a Chang algebra. Specifically, Boolean algebras are just those Chang algebras in which the additional axiom $x + x = x$ holds.

Before proceeding further with properties of Chang algebras, we shall note the connection between the Łukasiewicz-Tarski infinite valued propositional calculus and a Chang algebra. If on the basis of Łukasiewicz's axioms, one divides propositions built out of **C** and **N** into equivalence classes with respect to the $\equiv$ defined above, the resulting set of classes, A_L, is a Chang algebra on the basis that the class containing **C***RR* is 1, the class containing **NC***RR* is 0, + parallels **B**, · parallels **L**, and — parallels **N**. Incidentally, V parallels **A**, and Λ parallels **K**.

The fact that A_L is a Chang algebra is proved in [8] by making use of the consequences of the Łukasiewicz axioms given in the early part of [3]. That A_L is a Chang algebra embodies a relationship which parallels the relationship between the two valued propositional calculus and Boolean algebra. This suggests that one might seek an algebraic proof of te completeness of the Łukasiewicz axioms by paralleling the algebraic proof of completeness for the two valued propositional calculus. In synopsis, this latter proof goes as follows. By the Stone decomposition theorem, any Boolean algebra is isomorphic to a subalgebra of a direct product of simple Boolean algebras; a simple Boolean algebra contains only the elements 0 and 1. Let us now consider a proposition P, built up out of $P_1, \ldots, P_n$, which is not deducible in the two valued propositional calculus. The equivalence class to which it belongs is not the unity for the Boolean algebra of equivalence classes, and hence it must have a 0 assigned to it in one of the simple Boolean factors of the direct product decomposition. In this factor, each of the equivalence classes containing $P_1, \ldots, P_n$ is assigned a value $x_1, \ldots, x_n$ which is 0 or 1. Upon assigning these same values to the $P_1, \ldots, P_n$ themselves, the value 0 will be assigned to P. By contraposition, any P which takes only the value 1 must be provable.

We first look for the generalization of the Stone decomposition theorem to the infinite valued case. Let us define an ideal as a subset I of A with the following properties:

(i) $0 \varepsilon \mathrm{I}$.

(ii) If $x, y \varepsilon \mathrm{I}$, then $x + y \varepsilon \mathrm{I}$.

(iii) If $x \varepsilon \mathrm{I}$ and $y \varepsilon \mathrm{A}$, then $x \wedge y \varepsilon \mathrm{I}$.

This parallels the definition of an ideal in Boolean algebras. We continue the parallel by defining a maximal ideal of A as an ideal different from A which is not a proper subclass of any ideal different from A. We say that A is simple if $\{0\}$ is the sole maximal ideal in A. We say that A is representable if it is isomorphic to a subalgebra of a direct product of simple Chang algebras. By Thm. 4.9 of [8], A is representable if and only if $\{0\}$ is the intersection of all maximal ideals.

Here the parallel with Boolean algebras ends. It is not true that every Chang algebra is representable; an example of one which is not representable is cited by Chang on p. 486 of [8].

There are of course non-simple Chang algebras which are representable, for instance any direct product of simple Chang algebras. Moreover, by appealing to the known fact of the completeness of the infinite valued propositional calculus, and using Thm. 5.3 of [8], one can infer that $\mathrm{A_L}$ is representable. If one could find a different proof that $\mathrm{A_L}$ is representable, one could conversely infer the completeness of the infinite valued propositional calculus without circularity, and the analogy with the two valued case would be maintained. Some measure of the difficulties which might be expected to attend the search for an alternate proof that A_L is representable can be inferred from the result (communicated privately by Chang) that the class of representable Chang algebras is not an arithmetic class.

The procedure used in [4] by Chang to get an algebraic proof of completeness starts with a deeper analysis of ideals. Let us say that an ideal I is a prime ideal if for each x and y at least one of $x \cdot \overline{y}$ or $\overline{x} \cdot y$ is in I. For Boolean algebras, prime ideals coincide with maximal ideals. This is not the case for Chang algebras, in which maximal ideals are prime ideals, but prime ideals need not be maximal ideals. Indeed one of the key results of [4] is that for each Chang algebra A, $\{0\}$ is the intersection of all prime ideals; we earlier noted that this can fail for maximal ideals.

For any ideal I, the relation R defined by

$$xRy \quad \text{iff} \quad (\overline{x} \cdot y) + (\overline{y} \cdot x) \ \varepsilon \ I$$

is an equivalence relation. The set of equivalence classes, A/I, is again a Chang algebra.

Def. $$x \leqslant y \quad \text{iff} \quad x \vee y = y.$$

$\leqslant$ is a partial ordering. Each of $+$, $\cdot$, $\vee$, and $\wedge$ is monotone with respect to $\leqslant$. Also $x \leqslant y$ iff $\overline{y} \leqslant \overline{x}$.

$$0 \leqslant x \cdot y \leqslant x \wedge y \leqslant x \leqslant x \vee y \geqslant x + y \leqslant 1.$$

Usually, a Chang algebra A is not linearly ordered by $\leqslant$. However, the quotient algebra A/I is linearly ordered if and only if I is a prime ideal. Consider the direct product of all factors of the form A/I, where I is a prime ideal. Since $\{0\}$ is the intersection of all prime ideals, it follows that A is isomorphic to a subalgebra of the direct product.

Thus we have another key result of [4], that each Chang algebra is isomorphic to a subalgebra of a direct product of linearly ordered Chang algebras. Thus, let P be a proposition built from $P_1, \ldots, P_n$ by **C** and **N** which is not derivable from the Łukasiewicz axioms. Then the equivalence class to which it belongs in A_L is not unity. So, looking at the isomorphism of A_L to a subalgebra of a direct product of linearly ordered Chang algebras, we must find a factor A^* in which the equivalence class of P is not represented by unity. Thus in A^* the representative p^* of the equivalence class of P is such that the structure of p^* is exactly similar to that of P and $p^* \neq 1$ holds in A^*.

If A is a linearly ordered Chang algebra, we can define a closely related linearly ordered Abelian group as follows. Let the elements of G be ordered pairs, $\langle x, m \rangle$, in which x is some element of A different from unity and m is an integer.

We define

$$\langle x, m \rangle + \langle y, n \rangle = \begin{cases} \langle x + y, m + n \rangle & \text{if } x + y \neq 1 \\ \langle x \cdot y, m + n + 1 \rangle & \text{if } x + y = 1 \end{cases}$$

$$- \langle x, m \rangle = \begin{cases} \langle \bar{x}, -(m+1) \rangle & \text{if } x \neq 0 \\ \langle 0, -m \rangle & \text{if } x = 0 \end{cases}$$

$$\langle x, m \rangle \leqslant \langle y, n \rangle \text{ if } \begin{cases} m = n \text{ and } x \leqslant y \\ \text{or } m < n. \end{cases}$$

Then G is a linearly ordered group, with $\langle 0, 0 \rangle$ serving as the zero of the group.

For the A* above, we form the corresponding G*, writing c for $\langle 0, 1 \rangle$. Then we construct an element q^* of G* analogous to p^* by replacing

$$\begin{aligned} x + y &\text{ by } \min(\mathrm{c}, x + y) \\ \bar{x} &\text{ by } \mathrm{c} - x \\ x \cdot y &\text{ by } \max(\mathrm{c}, x + y - \mathrm{c}) \end{aligned}$$

where the operations on the right are group operations. Then because $p^* \neq 1$ holds in A*, we will have

$$(0 < \mathrm{c}) \;\&\; (0 \leqslant x_1 \leqslant \mathrm{c}) \;\&\; \ldots \;\&\; (0 \leqslant x_n \leqslant \mathrm{c}) \;\&\; (q^* \neq \mathrm{c})$$

holding in G*, where the x's are the ultimate constituents of q^* corresponding to the constituents $P_1, \ldots, P_n$ of P. So

$$(12) \qquad (0 < \mathrm{c}) \;\&\; (0 \leqslant x_1 \leqslant c) \;\&\; \ldots \;\&\; (0 \leqslant x_n \leqslant \mathrm{c}) \rightarrow (q^* = \mathrm{c})$$

is not a universal theorem of ordered Abelian groups. Since the rational numbers constitute a universal model for linearly ordered groups (see [9]), a counter-example for (12) can be found in the rationals. Then we can assign the rational values x_i/c to the constituents of P and it will take a value different from unity.

This shows that if P is not provable, then one can find rational values for its constituents which will make P different from unity. That the values of the constituents can be taken rational rather than real seems at first a slightly stronger result that was obtained before. However, since the value of P is a continuous function of the values of its constituents, and since each acceptable set of truth values is dense in the interval 0 to 1, it follows that if in one set of truth values one can find values for the constituents which make P different from unity, one can do so in each set of truth values.

The algebraic completeness proof given by Chang in [4] is non-constructive, whereas that given by Rose and Rosser in [3] is constructive, in the sense that if P always takes the value unity then from the structure of P one can proceed in a strictly determinate way to write out a proof for P. This constructiveness is mainly theoretical, since if P is of reasonable length or complexity, the proof supplied for it by the methods of Rose and Rosser will be impracticably long.

3. *Preliminaries for the predicate calculus.* There are some other results in the propositional calculus which are of use for the predicate calculus. One such is referred to as *Lemma B* in [10], and we shall adhere to this nomenclature.

Lemma B. Let $P_1, \ldots, P_n$ be a set of propositions, and let $P, S_1, \ldots, S_m$ be propositions built out of the P_i by means of **C** and **N**. Suppose that no contradiction is deducible from the S_j by means of Rule **C** and the Łukasiewicz axiom schemes. Then P is deducible from the S_j by means of Rule **C** and the Łukasiewicz axiom schemes if and only every assignment of rational truth values to the P_i which makes each S_j equal to unity also makes P equal to unity.

We first note that if a class of propositions all have the value unity, then any proposition which is deducible from the class by Rule **C** also has the value unity. Thus it is immediate that if P is deducible from the S_j, then it must take the value unity whenever they do. Now suppose that P is not deducible from the S_j. If on the basis of Łukasiewicz's axioms plus the S_j one divides the propositions into equivalence classes with respect to $\equiv$, one gets a Chang algebra A_S; the hypothesis that no contradiction is deducible from the S_j is needed to verify **A12**. As in Chang's algebraic proof of completeness, we find a linearly ordered Chang algebra A^* in which a representative p^* of P is different from unity while the representatives s^*_j of S_j are equal to unity. We generate the ordered Abelian group G^* corresponding to A^* in which there is q^* corresponding to p^* and t^*_j corresponding to s^*_j. Then the relation

$$(0<c) \,\&\, (0\leqslant x_1\leqslant c) \,\&\, \ldots \,\&\, (0\leqslant x_n\leqslant c) \,\&\, (t^*_1 = c) \,\&\, \ldots \,\&\, (t^*_m = c) \rightarrow (q^* = c)$$

is violated in G*, and so must be violated in the rationals. This gives us rational values of the P_i such that each S_j is unity but P is not.

The proof sketched above is similar to that given in [10]. It will be noted that the proof depends on the fact that the set of S_j was finite in number. Indeed, *Lemma B* fails to hold if the set of S_j is allowed to be infinite, as is shown in [10]. This was disappointing at the time it was noted, since a proof of completeness for the predicate calculus would have been forthcoming if *Lemma B* should have held for infinite classes of S_j as well as for finite. This is shown in [10], and sketched below.

Another useful result from [10] is:

Lemma A. Let P_1, ... be a set of propositions, and let S_1, ... be propositions built out of the P_i by means of **C** and **N**. A set $\mathscr{S}$ of the S_j can be added to Łukasiewicz's axioms without contradiction if and only if values can be assigned to the P_i such that the members of $\mathscr{S}$ are all simultaneously unity.

As before, if all members of $\mathscr{S}$ are unity, then all propositions derivable from $\mathscr{S}$ must be unity; thus no contradiction can be derived from $\mathscr{S}$. To prove the converse, we first restrict attention to the case where $\mathscr{S}$ is finite. Suppose no contradiction is derivable from $\mathscr{S}$. In *Lemma B,* take P to be $\mathbf{L}P_1\mathbf{N}P_1$. Then P is not derivable from $\mathscr{S}$ and so there must be rational values of the P_i which give the value unity to every member of $\mathscr{S}$. A limiting procedure can now be used to cover the case where $\mathscr{S}$ is (denumerably) infinite.

In [10] a proof of *Lemma A* is given directly from the fact that the Łukasiewicz axioms are complete, without having to go through a generalization of Chang's algebraic proof of completeness.

Because the transition from finite $\mathscr{S}$ to infinite $\mathscr{S}$ is made by a limiting process, the proof of *Lemma A* only suffices to show the existence of real (rather than rational) values of the P_i that make all members of $\mathscr{S}$ equal to unity. Indeed, in a private communication, Chang has given an illustration of an infinite $\mathscr{S}$ for which there exist irrational values of the P_i which make all the members of $\mathscr{S}$ equal to unity but for which there is no set of rational values with the desired property. Thus the limiting procedure seems an unavoidable feature of the proof.

One can paraphrase *Lemma A* by saying that a set $\mathscr{S}$ of propositions is satisfiable if and only if it is consistent.

4. *The predicate calculus.* In order to conform with the familiar notation when quantification is present, we shall write $\sim P$ for $\mathbf{N}P$,

$P \supset Q$ for **C**PQ, $P \vee Q$ for **A**PQ, $P \wedge Q$ for **K**PQ, $P+Q$ for **B**PQ, and $P \cdot Q$ for **L**PQ. We introduce individual variables x, y, ..., predicates $F(x)$, $G(x, y)$, etc., and the existential quantifier $(\mathrm{E}x)$. We define $(x)P$ as $\sim(\mathrm{E}x)\sim P$. Thus, except for the additional combinations $P+Q$ and $P \cdot Q$, the formalism is precisely that of the classical two valued predicate calculus. We introduce the abbreviations mP and P^m respectively for $P+P+ \ldots +P$ (m summands) and $P \cdot P \cdot \ldots \cdot P$ (m factors).

In terms of some universe U and some linearly ordered Chang algebra A, we say that a set of predicate functions $f(x)$, $g(x, y)$ etc., from U to A constitute an A-assignment for a set $\mathcal{S}$ of formulas if to each predicate $F(x)$, $G(x, y)$, etc., which appears in one of the $\mathcal{S}$, one of the predicate functions $f(x)$, $g(x, y)$, etc., is assigned; we further assign members of A as values to each member of $\mathcal{S}$ by interpreting the $+$, $\cdot$, $\vee$, and $\wedge$ appearing as the same relations of A, and the $\sim$ as the $-$ of A, and whenever a part $(\mathrm{E}x)\, H(x)$ occurs in one of the members of $\mathcal{S}$, its value is taken to be the least upper bound in A of the values of $H(x)$ as x runs over the members of U. It is assumed that the least upper bounds required for parts of members of $\mathcal{S}$ are available, but not necessarily that least upper bounds exist for arbitrary sets of members of A.

If we define $\bar{x}$ as $\mathbf{n}(x)$, $x+y$ as $\mathbf{b}(x, y)$, and $x \cdot y$ as $\mathbf{l}(x, y)$, as in (2), (9), and (10), then the rationals from 0 to 1 inclusive form a linearly ordered Chang algebra; so also do the reals. If A is the Chang algebra of the rationals, then rather few sets of formulas $\mathcal{S}$ can have an A-assignment (unless U is finite) because the required rational least upper bounds will commonly fail to exist. If A is the Chang algebra of the reals, then least upper bounds exist for all sets; thus one can take arbitrary U and arbitrary predicate functions $f(x)$, $g(x, y)$, etc., from U to A and they will be an A-assignment for an arbitrary set $\mathcal{S}$ provided only that each predicate appearing in $\mathcal{S}$ has a predicate function assigned to it.

We say that a formula is valid if it takes the value unity for each A-assignment for which A is the set of reals from 0 to 1. We say that a formula is strongly valid if it takes the value unity for each A-assignment over each linearly ordered Chang algebra A with respect to which it has an A-assignment.

The concepts of A-assignment and strong validity are due to Chang, and are introduced in [14].

Because of the great convenience of the reals for forming A-assignments, it is natural to make a study of valid formulas. Another consideration which justifies giving primary attention to the valid formulas is the fact, shown by Rutledge in Sect. II. 4 of [12], that the set of valid formulas is the intersection of the sets of formulas which are valid in each finite valued predicate calculus, as defined in [11].

By analogy with the two-valued case, we seek an axiomatization which characterizes the set of valid formulas. A consistent set of axiom schemes and rules would be called complete if one could derive from them exactly the set of valid formulas. It has just been discovered that there is no complete set with a finite or recursively enumerable set of rules and a finite or recursively enumerable set of axioms (see Sect. 6 below) but in the meantime considerable work of interest was done in the search for such a set. A set of axioms was proposed by Mrs. Hay in [10], where a weak completeness proof was given to the effect that if a formula P is valid then there is an m such that mP can be proved. Mr. L. P. Belluce, a pupil of Chang, has issued an abstract (see [13]) announcing a stronger completeness result for Mrs. Hay's axioms, namely that if P is valid, then $P+P$ is provable. In both [10] and [13] are given extensions of *Lemma* A to the predicate calculus, to the effect that a set $\mathscr{S}$ can be added to Mrs. Hay's set of axioms without a contradiction if and only if there is a universe U and an A-assignment from U to the set of reals from 0 to 1 such that each member of $\mathscr{S}$ takes the value unity. This is just the generalization of the Lowenheim-Skolem theorem to the infinite valued case. Whereas one could easily infer completeness from this in the two valued case, there is no way to do this in the infinite valued case. In [10], it is inferred that if P never takes the value 0, then there is an m such that mP is provable. In [13], it is inferred that if the value of P is always greater than one half, then $P+P$ is provable.

In [12], a study is made of the monadic predicate calculus, in which only a single individual variable occurs. For this case a set of axiom schemes is presented and a proof of completeness is given. The methods are algebraic in character, and considerably generalize the methods of [4].

In [14], it is announced that every strongly valid formula is provable from the set of axioms given by Mrs. Hay in [10].

5. *Indication of techniques.* The complications of [12] are so severe that it seems futile to say more about this development than we have already. However, the developments of [10] are sufficiently similar to the Henkin-Hasenjaeger proof of the Lowenheim-Skolem theorem for the two valued case that it seems worthwhile indicating where the differences lie for the infinite valued case.

Mrs. Hay uses Rule **C** and the rule, «If P, then $(x)P$». For her axiom schemes she uses first the Łukasiewicz schemes (with **L4** deleted, since it is dependent) and the following quantification axiom schemes:

H1. $(((Ex)P) \cdot (Ex)P) \supset (Ex)(P \cdot P)$

H2. $F(y, y) \supset (Ex)F(x, y)$, with suitable prohibitions to prevent confusion of bound variables.

H3. $(Ex)\ F(x) \supset (Ey)\ F(y)$.

H4. $(x)(P \supset Q) \supset ((Ex)P \supset Q)$ if there are no free occurrences of x in Q.

H5. $(P \supset (Ex)Q) \supset (Ex)(P \supset Q)$ if there are no free occurrences of x in P.

Of these axiom schemes, all but **H1** are familiar from two valued quantification theory. Indeed, with some aid at crucial spots from the Łukasiewicz axioms, they are just the axioms needed to carry through the initial steps of the Henkin-Hasenjaeger proof.

Specifically, let $\mathcal{S}$ be a consistent set of closed formulas. We adjoin «constants» a_1, a_2, ... without sacrificing consistency. Then we enumerate the formulas with a single free variable, $F_i(x_i)$, and adjoin axioms

$$(13) \qquad (Ex_i)\ F_i\ (x_i) \supset F_i(a_j).$$

Again, we do not lose consistency.

Because we now have $(Ex_i)\ F_i(x_i)$ equivalent to $F_i(a_j)$, we can for each formula with no free variables find an equivalent formula with no variables at all. Let $\mathcal{T}$ be the class of all provable formulas with neither free or bound variables. These have as constituents $(F(a_r)$, $G(a_r, b_s)$, etc., and by *Lemma A* we can find real values for the constituents such that each member of $\mathcal{T}$ takes the value unity. If we take U to consist of a_1, a_2, ..., then the determinations given for $F(a_r)$, $G(a_r, b_s)$, etc., define predicate functions from U to A. Again, because of the equivalence of $(E_i)\ (F_i(x_i)$ and $F_i(a_j)$, we can conclude that this choice of U and predicate functions is an A-assignment which gives the value unity to all provable closed formulas, and hence to each member of $\mathcal{S}$.

Analogously to the two valued case, one can conclude that if a formula P never takes the value 0, then a contradiction will ensue if $\sim P$ be added to Mrs. Hay's axioms. Unlike the two valued case, the inconsistency of $\sim P$ merely lets one infer mP for some m.

If *Lemma B* had been available for infinite classes of S_j, then the completeness of Mrs. Hay's axiom schemes could have been inferred by some modification of the argument given above. Suppose that R is a closed formula which is not provable. Then the addition of the constants and the axioms (13) would leave R unprovable. Consequently, R^*, the formula with no bound or free variables which is equivalent to R, is not deducible from $\mathcal{T}$. Then *Lemma B* (if the necessary strong form were available) would provide a determination which gives the value unity to all members of $\mathcal{T}$ and a smaller value to R^*, and hence to R.

The methods used in [13] and [14] are generalizations of the completeness proof of Rasiowa and Sikorski (see [15]). A key result of [15] is the following lemma, which is valid for the two valued case.

Lemma R-S. In a Boolean algebra B, let $x \neq 1$. Let $\mathrm{b}_i = \Sigma_j \mathrm{b}_{ij}$ be a denumerable set of infinite sums which exist in B. Then there is a proper maximal ideal J containing x which preserves each of the given sums, in the sense that for each i, b_i is in J if and only if b_{ij} is in J for each j.

In attempting to generalize this to a Chang algebra, there is first of all the question if we should require J to be a maximal ideal or a prime ideal, since these are not the same for Chang algebras. There seems no good reason for believing that the lemma holds generally in either case.

For the intended application, we do not need the lemma in its full strenght. Because Mrs. Hay includes Rule **C** and the Łukasiewicz axioms, the equivalence classes for the predicate calculus with respect to $\equiv$ form a Chang algebra, $\mathrm{A_H}$. We require *Lemma R-S* only for this special Chang algebra. Moreover, the set of sums which are to be preserved by J are those in which b_{ij} is the equivalence class of $F_i(y_j)$ and b_i is the equivalence class of $(\mathrm{E}y)\, F_i(y)$.

For these, by an argument not unlike that used to show that one can add all the axioms (13) without introducing a contradiction (we let y_j play the role of the a_j of (13)), it is possible to prove useful special cases of *Lemma R-S*. In [14], it is announced that if $\mathrm{a} \neq 0$, then one can find a prime ideal J in $\mathrm{A_H}$, preserving sums of the sort noted above, such that a is not in J. Because one has here a prime ideal rather than a maximal ideal, one does not conclude that every valid formula is provable, but only that every strongly valid formula is provable.

In [13] it is announced that if a is of infinite order in $\mathrm{A_H}$ (that is, there is no finite integer m for which $m\mathrm{a} = 1$), then one can find a proper maximal ideal J in $\mathrm{A_H}$, preserving sums of the sort noted above, such that a is in J. From this, the general line of argument is as follows. Suppose $P+P$ not provable. Then, if p is the equivalence class of P in A_H, $p+p\neq 1$. So by Thm. 3.9 of [8], $p{\cdot}p$ is of infinite order. Then by the lemma noted above, we find a proper maximal ideal J which preserves sums as required and contains $p{\cdot}p$. Then, by the results of [8], A_H/J is simple and locally finite. So the corresponding linearly ordered Abelian group is Archimedean, and thus is isomorphic to a subset of the reals. This enables us to define an A-assignment to the reals in which $P{\cdot}P$ has the value 0. So P must have a value $\leqslant 0.5$.

By contraposition, if P always has a value greater than one half (and so, a fortiori, if P is valid), then $P+P$ is provable.

From this, one can infer the result implied by Belluce in [13] that P is valid if and only if $P+P^n$ is provable for each n. For if each of $P+P^n$ is provable, then each can take only the value unity, whence we conclude that P can take only the value unity. Conversely suppose $P+P^n$ not provable for some n. Then P^n+P^n is not provable, and so there is an A-assignment in which P^n takes a value $\leqslant 0.5$. Then $P\neq 1$, and so P is not valid.

6. *The non-axiomatizability of the infinite valued predicate calculus.* Just recently we received a private communication from Specker announcing that a student of his, B. Scarpellini, has proved that the set of valid formulas is not recursively enumerable. We have no details as to the method of proof. This shows that Mrs. Hay's set of axiom schemes is not complete, and indeed that no recursively enumerable set of axioms with the same rules can be complete.

One can obtain completeness by use of an infinitary type of rule, as noted by Belluce in [13], specifically the rule:

If each of $P+P^n$, then P.

We noted above why this would enable one to prove every valid formula, and only such.

This indicates that while one cannot characterize the set of valid formulas by means of a recursive predicate preceded merely by an existential quantifier, one can do so if one precedes the existential quantifier by a universal quantifier.

In view of the incompleteness of the polyadic predicate calculus, the result of Rutledge in [12] showing the completeness of the monadic predicate calculus takes on exceptional interest.

One can also conclude that there must be valid formulas which are not strongly valid. One waits with interest to see whether Scarpellini's proof will provide a means of exhibiting such a formula.

REFERENCES

[1] Jan Łukasiewicz and Alfred Tarski, *Untersuchungen über den Aussagenkalkül,* in *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie,* Classe III, vol. 23 (1930), pp. 30-50.

[2] M. Wajsberb, *Beiträge zum Metaaussagenkalkül I,* in *Monatshefte für Mathematik und Physik,* vol. 42 (1935), pp. 221-242.

[3] Alan Rose and J. B. Rosser, *Fragments of Many-valued Statement Calculi,* in *Transactions of the American Mathematical Society,* vol. 87 (1958), pp. 1-53.

[4] C. C. Chang, *A New Proof of the Completeness of the Łukasiewicz Axioms,* in *Transactions of the American Mathematical Society,* vol. 93 (1959), pp. 74-80.

[5] C. A. Meredith, *The Dependence of an Axiom of Łukasiewicz,* in *Transactions of the American Mathematical Society,* vol. 87 (1958), p. 54.

[6] C. C. Chang, *Proof of an Axiom of Łukasiewicz,* in *Transactions of the American Mathematical Society,* vol. 87 (1958), pp. 55-56.

[7] Robert McNaughton, *A Theorem about infinite-valued Sentential Logic,* in *Journal of Symbolic Logic,* vol. 16 (1951), pp. 1-13.

[8] C. C. Chang, *Algebraic Analysis of many valued Logics,* in *Transactions of the American Mathematical Society,* vol. 88 (1958), pp. 467-490.

[9] Abraham Robinson, *Complete Theories,* North-Holland Publishing Co., Amsterdam, 1956.

[10] Louise S. Hay, *An Axiomatization of the infinitely many-valued predicate calculus,* M. S. Thesis at Cornell University, 1958.

[11] J. B. Rosser and A. R. Turquette, *Many-Valued Logics,* North-Holland Publishing Co., Amsterdam, 1952.

[12] Joseph D. Rutledge, *A preliminary Investigation of the infinitely many-valued predicate calculus,* Ph. D. Thesis at Cornell University, 1959.

[13] L. P. Belluce, *Some Remarks on the Completeness of infinite valued predicate Logic,* abstract, in *Notices of the American Mathematical Sosiety,* vol. 7 (1960), p. 633.

[14] L. P. Belluce and C. C. Chang, *A weak completeness Theorem for infinite valued predicate Logic,* abstract, in *Notices of the Amer. Math. Soc.,* vol. 7 (1960), pp. 632-633.

[15] H. Rasiowa and R. Sikorski, *A proof of the completeness theorem of Gödel,* in *Fundamenta Mathematicae,* vol. 37 (1950), pp. 193-200.

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