THE INFERENTIAL APPROACH TO LOGICAL CALCULUS
(part I)

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1. *Introduction.* In his doctoral thesis [ULS] (1) Gentzen presented a new approach to the logical calculus whose central characteristic was that it laid great emphasis on inferential rules which seemed to flow naturally from meanings as intuitively conceived. It is appropriate to call this mode of approach to logic the ***inferential approach.***

This approach has been a major interest of Canon Feys, as is shown by his publications on the subject (2). Because of its bearing on some basic methodological questions, it has also a close connection with combinatory logic. During the past academic year I gave at the Pennsylvania State University a course of lectures principally devoted to it; and in these lectures several innovations, contributed by various persons, were included. These reasons make it seem a suitable subject for a paper in this collection. The present paper contain a summary account of the inferential systems with special emphasis on features which go beyond or modify the presentations I have previously made. The detailed proofs of the results are too voluminous for presentation here. Eventually I hope to publish the lectures in book form; the reader is referred to that book, or to original sources in the literature, for more details (3).

It will be appropriate, before we begin, to make a few remarks about the historical background.

Gentzen's work was preceded by a series of papers of Paul Hertz. In a paper [EUA] written before his thesis, Gentzen refers explicitly to Hertz; and shows that a certain rule, which Hertz called «Syll», could be reduced to a simpler rule which Gentzen called «Cut» (*Schnitt*). There is also enough resemblance between Gentzen's work and that of Herbrand [RTD] to lead one to infer that there was some influence of the latter on the former.

At about the same time as Gentzen, Jaskowski presented, in his

(1) For explanation of the letters in bracklets see the Bibliography (to appear at the end of the second part of this paper).

(2) See his [MRD], [NCM], [TFD].

(3) See [FML]. For the previous treatments see [TFD] and other papers listed in the Bibliography.

[RSF] [4], a closely related system. From statements made by the author one gathers that his work was inspired by Łukasiewicz; Jaskowski states that similar ideas were presented by Łukasiewicz as far back as 1927. Jaskowski gave inferential formulations of the classical propositional algebra, of the «positive» (in the present terminology «absolute») propositional algebra, of the algebra of Kolmogorov [5], of an «extended theory of deduction» allied to protothetics, and of a form of predicate calculus which does not presuppose that the domain of individuals be nonvoid. The rules are of the type which Gentzen called N-rules (here called T-rules) with a technique for expressing them in linear form somewhat similar to that used by Fitch [SLg]; there is nothing analogous to Gentzen's L-rules. The latter are the principal concern of this paper, which thus owes but little to Jaskowski and his predecessors.

Shortly after his thesis appeared Gentzen published his famous proof of the consistency of arithmetic using a transfinite induction up to the first ε-number [6]. This sensational result should have drawn attention to his inferential methods; but it seemed to do rather the reverse. At any rate, as World War II was drawing to a close, and Gentzen went to his death in a concentration camp in Prague, the thesis was relatively little known. I understand that Kleene, who has since contributed greatly to the elaboration of Gentzen systems, only rediscovered them about 1947.

In recent years, however, there has been a considerable development of Gentzen's ideas. Space limitations do not permit me to discuss all these developments or even to mention the names of the authors. However the principal publications which have come to my attention and bear directly on the subject of this paper are listed in the Bibliography.

This paper does not attempt to handle, even summarily, all the questions connected with the inferential approach. The discussion will be limited to questions which are 1) strictly constructive, 2) related to first order predicate calculus and to a formulation rather closely allied to Gentzen's own. Even then there is some selection.

(4) A resumé of this paper is given in Feys [NCM].

(5) See Kolmogorov [PTN]. I have never seen this paper. Jaskowski treats only the part in which the only operations are implication and negation. From the statements made by him and also Feys [NCM], Church [IML] note 210, it appears that this part is exactly the same as the corresponding part of the «minimal logic» of Johansson [MKR].

(6) The original presentation was made in Gentzen [WFR], a revised presentation in his [NFW].

However, I shall go into some detail in regard to questions of motivation.

2. *Preliminary analysis.* In the inferential systems the axioms, which, for reasons shortly to be explained, are called *prime statements,* play a relatively minor role; the essential content of the system is contained in the inferential (or deductive) rules. Except for a few rather trivial rules of a special nature, these rules are associated with the separate operations; and those which are so associated with a particular operation express the meaning of that operation. In order to show in what sense this is true, I shall discuss here in some detail the motivation back of the rules with particular emphasis on implication. The motivation is quite different from that given by either Gentzen or Jaskowski, who were interested in setting up logical calculus as an end in itself; but is similar to those of Hertz and Lorenzen, and it is in some respects an improvement over that found in [TFD]. The main point is that the logical calculus is regarded as an instrument in the broader problem of methodology of formal reasoning.

Before we start on this, it is necessary to explain that I am using the terms «sentence», «statement», and «proposition» in somewhat peculiar senses. A *sentence* is a string of words in the language, called the *U-language,* which is actually being used, such that the string constitutes a sentence in the sense of ordinary grammar. Each such sentence expresses a *statement,* and this is capable of being understood by the users of the U-language. When, in a particular formal context, we talk about certain objects which, when that formal context is interpreted, become statements, I shall call these objects *propositions*. Thus strings of symbols in an object language are propositions provided they correspond to sentences in the interpreted object language (7). The distinction between statements and sentences is left to the readers' philosophy, and, within limits, it is not important what sort of philosophy he adopts; but that between either of these and a proposition is quite important. Propositions are named in the U-language; statements are asserted, denied, believed, and understood; and although the U-language does possess devices for talking about its statements, that is quite different from the formal type of consideration which characterizes propositions.

(7) From this point of view the U-language is what is often called the «metalanguage». But the terms «U-language» and «metalanguage» are not synonymous. In some semantical studies the U-language is the metalanguage, in others it is the metametalanguage.

We shall be concerned with statements which relate to some unspecified underlying formal theory or system 𝔖. Here by *theory* I mean a set or class 𝔈 of statements, called the *elementary statements* of 𝔖, within which there is distinguished in some objective way the subclass of true elementary statements, or *elementary theorems* of 𝔖. We shall suppose the class 𝔈 is a definite class, in that given any statement, we can determine effectively whether or not it belongs to 𝔈. It is not, however, necessary that there be an effective process for deciding whether or not an elementary statement is true. A *deductive theory* is one in which the elementary theorems form an inductive class (8). The basis is a class 𝔄 of *axioms*; from these the elementary theorems are generated by certain *deductive* rules 𝔑. Ordinarily it is required that the class 𝔄 be definite and that the rules 𝔑 be such that an alleged proof can be effectively checked; in such a case we can say that the deductive theory is effectively generated or *formal*.

A (formal) theory becomes a (formal) system when there is a definite class of formal objects and one or more predicates such that every elementary statement is to the effect that such and such a predicate applies to such and such an ordered sequence of formal objects. The formal objects may be taken as strings of letters in an object language, or as generated from certain *atoms* by *operations*, so as to form structures like branched trees rather than strings. The predicate may be binary, like «=» or «$\leqslant$»; ternary like betweenness; or unary, in which case we have elementary statements which say that such and such a formal object belongs to a certain class.

We are now concerned with the following question. Suppose that from the elementary statements of 𝔖 we form new statements by means of the ordinary propositional connectives of the U-language. How are these new statements to be interpreted? Is it possible to explain their meaning in a strictly constructive way without doing violence to our intuitions?

As stated we shall consider this question with reference to implication. If A and B are elementary statements of a deductive theory 𝔖, what is meant by the statement

(1) If A, then B?

It is useless to say that this means that either A is false or B is true;

(8) I.e. a class defined by a fundamental inductive definition in the sense of Kleene [IMM] p. 258.

for in general there is no constructive meaning to be attached to the falsity of an elementary statement — in the present context, as we shall see, falsity is more difficult to define semantically than is implication. There are two principal ways by which a constructive meaning can be attached to (1). In the first way, we form a theory $\mathfrak{S}'$ by adjoining A to $\mathfrak{S}$ as a new axiom; then (1) is true just when B is a theorem of $\mathfrak{S}'$. I shall call this the *deducibility interpretation* of (1). A second method, which Lorenzen has emphasized, is to form a system $\mathfrak{S}''$ by adjoining a deductive rule (which has, in this case, only one instance) permitting the passage from A to B; then (1) is true if every theorem of $\mathfrak{S}''$ is also a theorem of $\mathfrak{S}$. Following Lorenzen's terminology I shall call this second interpretation the *admissibility* (*Zulässigkeit*) *interpretation*. These two interpretations are not the only ones possible; both have the property that (1) is true wherever B is itself a theorem of $\mathfrak{S}$. The first interpretation is adopted here. The second interpretation is more general than the first; but it is curious that Lorenzen, starting with it, arrives at exactly the same logical calculus as that which is described later. Relative to a vacuous system $\mathfrak{S}$ the two interpretations are equivalent.

These considerations answer the question about (1) if A and B are elementary. But what if they are not? How should one explain, for instance:

If (if A then B), then (if C, then A and E) ?

Statements formed in this way from elementary statements of $\mathfrak{S}$ I shall call the *compound statements* of (or relative to) $\mathfrak{S}$; we are concerned with finding an interpretation for them.

At this point let us proceed to a higher stage of formalization. Let us form a system L($\mathfrak{S}$) whose formal objects are called *propositions*. We shall interpret these propositions as compound statements of $\mathfrak{S}$; but of course this interpretation is to be used only as motivation for setting up the systems, and has nothing to do with the formal developments themselves. In discussing interpretations however, it will be permissible to identify propositions with the statements which interpret them. Then in L($\mathfrak{S}$) the «logical connectives» become operations; the statement (1) becomes a proposition which we write

$$A \supset B.$$

The elementary statements of L($\mathfrak{S}$) will be of the form

(2) $$A_1, \ldots, A_m \Vdash B,$$

with $m \geqslant 0$ (for $m=0$ we have simply a blank on the left). We shall interpret this as meaning that B is, in some sense to be explained later, derivable from $A_1, \ldots, A_m$.

The first explanation of this derivability is the existence of a tree constructed according to certain rules, which are here called the T-rules ([9]). I shall call such a tree a T-proof. The nodes of a T-proof for (2) are propositions; the bottom node is B, and the top nodes consist of «suppositions», which are either instances of $A_1, \ldots, A_m$, or are cancelled in the course of the construction. The rules state under what conditions on the portion of the tree above a given node a proposition may be put at that node, and, in certain cases, suppositions cancelled. There are two rules for each operation; one for introduction and one for elimination. The former allows a new proposition formed with the operation to be placed at the node; the latter allows a proposition formed by the operation and placed immediately above the node to be broken up, and, perhaps, disappear. The rules for introduction are determined by the meaning of the operations in question. Thus the rule Pi for introduction of $A \supset B$ says that a sufficient condition for putting $A \supset B$ at a given node is that the part of the T-proof above that node be a T-proof with B at the bottom; then all occurrences of A among the suppositions may be cancelled. If we think of the T-proof above B as a derivation of B from the theory generated by its uncancelled top nodes, this is evidently in accord with the meaning of (1) as presented in the third preceding paragraph.

Now it is a well recognized principle that the meaning of a notion is determined by the conditions under which it can be introduced into discourse. The consequences which can be inferred from its presence are, in fact, determined by these conditions. This, in essence, is Lorenzen's «principle of inversion». According to it the T-rules for elimination are consequences, in a certain sense, of those for introduction. For example, suppose $A \supset B$ appears at a certain node, and that that node is neither a supposition nor obtained by elimination. That means that the part of the T-proof above it must be a T-proof ending in B, with A possibly appearing as a supposition. In that case we cannot expect to draw any inference unless A is also derivable; then we can infer B by putting the derivation of A over each occurrence of A as a supposition. If we accept the principle that we can make the same inferences from $A \supset B$ no matter how it is obtained that we can when it is first introduced, then the rule Pe for elimination of $A \supset B$ is

([9]) Gentzen called these the N-rules. The change from «N» to «T» was made in [TFD] on account of possible confusion of «N» with negation. The rules of Jaskowski are essentially T-rules.

Pe $$\frac{A \supset B \qquad A}{B\,,}$$

in other words, *modus ponens*.

These rules state the cases in which (2) holds if all the operations considered are implications. However, if we stop at this point it is evident that the system L($\mathfrak{S}$) is a schematic system in the sense of [CLg] p. 37. To get a truly deductive system we must pass to the L-rules of Gentzen. This will require certain preliminary formulations, which are the same for all operations, and rules which characterize the separate operations.

For the preliminary formulation we note the following: 1°, the statement (2) is true when $m=1$ and B is the same as A_1, or $m=0$ and B an axiom of $\mathfrak{S}$; 2°, the truth of (2) is independent of the order and multiplicity of the A_i; 3°, if (2) is true, then it remains true if additional propositions are adjoined on the left; and 4°, given fixed $A_1, \ldots, A_m$, the class of propositions B for which (2) holds is closed with respect to the deductive rules of $\mathfrak{S}$. Here the property 1° can be guaranteed by taking the indicated statements as axioms of L($\mathfrak{S}$); I call these *prime statements* to distinguish them from the axioms of $\mathfrak{S}$ itself. The properties 2° and 3° are guaranteed by rules called *structural rules*. As for property 4° a special rule will guarantee that also.

This brings us to the rules, called «operational rules» ([10]), which are related to the operations. In accordance with the principle of the third preceding paragraph, it suffices to consider rules for introducing the logical operations into (2). There are two kinds of introduction, on the left and on the right. In accordance with the intended interpretation the rules on the right are the same, except for a change in notation, as for a T-rule, the cancellation of a supposition being indicated by simply omitting it in the conclusion. Thus the rule P* for introducing $A \supset B$ on the right is (here $\mathfrak{X}$ is an arbitrary sequence of propositions):

P* $$\frac{\sout{\mathfrak{X} \Vdash A \supset B} \quad \mathfrak{X}, A \Vdash B}{\mathfrak{X} \Vdash A \supset B.}$$

For the introduction of $A \supset B$ on the left we have to ask under what circumstances we can have a T-proof ending in C and starting with a supposition $A \supset B$. Inspection of the argument given above for Pe

([10]) Gentzen called them «logical rules».

shows that such a supposition cannot be eliminated unless A is already present, and in that case the supposition will be replaced by B. Thus the rule for introduction of implication on the left is

$$\text{*P} \qquad \frac{\mathfrak{X} \Vdash A \qquad \mathfrak{X}, B \Vdash C}{\mathfrak{X}, A \supset B \Vdash C.}$$

3. *Formulation of the singular systems.* The formulation of the basic «structural» rules and of the rules for implication has been discussed in some detail in § 2. We turn now to consider the rules for other operations. This will be done in rather less detail.

The other positive operations, $\wedge$ (conjunction) and $\vee$ (alternation) cause no particular difficulty. By methods analogous to those of § 2 we have the rules

$$^*\Lambda \quad \frac{\mathfrak{X}, A \Vdash C}{\mathfrak{X}, A \wedge B \Vdash C}, \quad \frac{\mathfrak{X}, B \Vdash C}{\mathfrak{X}, A \wedge B \Vdash C} \qquad \Lambda^* \frac{\mathfrak{X} \Vdash A \quad \mathfrak{X} \Vdash B}{\mathfrak{X} \Vdash A \wedge B}$$

$$^*\text{V} \quad \frac{\mathfrak{X}, A \Vdash C \quad \mathfrak{X}, B \Vdash C}{\mathfrak{X}, A \vee B \Vdash C} \qquad \text{V}^* \frac{\mathfrak{X} \Vdash A}{\mathfrak{X} \Vdash A \vee B}, \quad \frac{\mathfrak{X} \Vdash B}{\mathfrak{X} \Vdash A \vee B.}$$

Note that $^*\Lambda$ and V^* each consist of two separate rules; Λ^* and $^*\text{V}$ are single rules with two premises each.

If we adjoin these rules to those considered in § 2 we have a system which has been called LA_1, or the *singular absolute system*. The T-rules for it form the system TA. Let us call a proposition A such that

(4) $\qquad \Vdash A$

is true in LA_1, a *thesis* of LA; this term will also be used for other systems. The totality of these theses of LA constitute the *absolute propositional algebra*. This turns out to be identical with the «*positive Logik*» of Hilbert and Bernays. The formulation given by them in [GLM.I] as well as suitable modifications of it will be referred to as HA. We thus have an interpretation of HA in terms of formal deducibility.

The absolute algebra HA does not contain all classically valid propositions which involve only the operations $\supset$, $\wedge$, $\vee$. Let us call the latter HC. Then it is well known that Peirce's law, viz.

(5) $\qquad \Vdash A \supset B. \supset A. \supset A,$

is not valid in HA; and that the adjunction of (5) to HA gives a formulation of HC. If we adjoin to LA_1 the rule

$$\text{Px} \qquad \frac{\mathfrak{X},\ A \supset B \ \Vdash A}{\mathfrak{X} \ \Vdash A\ ,}$$

we have a system LC_1 in which, it turns out, the theses are precisely those for which the corresponding statement is derivable in HC. These theses constitute *the classical propositional algebra.* It has a corresponding T-form which will be called TC.

We turn now to consider the adjunction of negation and quantification. These may be adjoined either to LA or LC. In the former case we call the system *absolutely based,* in the later *classically based.*

In regard to negation, it has been remarked already that it is not in general a constructive concept (11). Proving constructively the falsity of a single elementary statement would demonstrate the consistency of $\mathfrak{S}$, and this is not in general possible. There are however, at least two ways of introducing a constructive notion analogous to negation. The first of these is to define a statement as false for $\mathfrak{S}$ in case its adjunction to the axioms for $\mathfrak{S}$ will make the latter inconsistent (in the sense that every elementary statement is derivable); this kind of negation is called *absurdity* (12). Again we may consider theories $\mathfrak{S}$ in which, in addition to the axioms there are *counteraxioms,* and we say that a statement is false if a counteraxiom can be derived from it. This kind of negation is called *refutability* (13). With each kind of negation there is associated a definition of consistency and completeness; those associated with absurdity are Post consistency and completeness. Completeness with respect to any sort of negation is equivalent to a law of excluded middle; if we postulate that law we say we have a *complete negation.*

Returning now to higher formalization, we express the negation operation, following the practice of the intuitionists, by the prefix «$\neg$». We may also introduce a fixed, perhaps fictitious, counter-

(11) At least not in the strictest sense of constructiveness.

(12) The term was introduced by the intuitionists; indeed their system, LJ, is precisely that obtained by basing absurdity on LA.

(13) The term is taken from Carnap [ISm]. (In his [LSL] the word had a different sense.) Łukasiewicz [ASS] uses the term «rejection».

axiom F. Such an F can be defined as the negation of any absolute thesis; conversely negation can be defined in terms of F thus:

$$(6) \qquad \neg A \equiv A \supset F.$$

We may take $F, \neg$, or both together (with (6) as a theorem) as primitives, giving F, N, and FN-formulations respectively. The relations between these are rather complex ([14]); and I shall consider here only the F-formulation. Then refutability requires only (6) and, if there are additional counteraxioms $F_1, F_2, \ldots$, the rule

$$\text{F*} \qquad \frac{\mathfrak{X} \Vdash F_i}{\mathfrak{X} \Vdash F.}$$

This will not occur if $\mathfrak{S}$ is void. Absurdity can be expressed by the rule

$$\text{Fj} \qquad \frac{\mathfrak{X} \Vdash F}{\mathfrak{X} \Vdash A,}$$

and completeness by the rule

$$\text{Nx} \qquad \frac{\mathfrak{X}, \neg A \Vdash A}{\mathfrak{X} \Vdash A,}$$

which, in view of (6), is a special case of Px. This gives us five kinds of negation (the subscript 1 being understood in each case), as follows:

- LM, minimal negation ([15]), or absolute refutability. Formed as above stated with (6), and, possibly, F*.
- LJ, intuitionistic negation, or absolute absurdity. Formed by adjoining Fj to LM.
- LD, strict negation ([16]), or complete refutability. Formed by adjoining Nx to LM.

([14]) An account of the relations between the F and N formulations was attempted in [DNF]. Essential changes will be made in [FML].

([15]) The name comes from Johansson [MKR]. For relation to Kolmogorov [PTN] see footnote 5.

([16]) A term introduced in [TFD], I am not too happy about it, but I can think of no better one.

LE, classical refutability ([17]). Formed by adjoining Px to LM; or (6) and, possibly, F*, to LC. It includes LD.

LK, classical negation, or complete absurdity. Formed by adjoining Nx and Fj to LM or Fj to LD; in either case Px is redundant.

Of all these systems Gentzen considered explicitly only LJ and LK.

We turn now to quantification. This will only arise when the underlying theory $\mathfrak{S}$ is itself a system. The system L($\mathfrak{S}$) will then contain two kinds of objects: terms, which represent the formal objects of $\mathfrak{S}$ itself, and propositions, which represent its (compound or elementary) statements. If we think of (2) as stating that B is a thesis of some extension, then we have now to think of extensions formed by adjoining terms as well as propositions. Let α be the set of terms adjoined, then in order to make α explicit we can write (2) in the form

$$(7) \qquad A_1, \ldots, A_m \mid \alpha \vdash B.$$

The set α will be called the range of (7). A lot of fuss has to be made over collision of bound variables, substitution etc. But in the long run we end up with rules as follows:

$$^*\Pi \; \frac{\mathfrak{X},\, A(t)\ \alpha \vdash B}{\mathfrak{X},\, (\forall x)A(x) \mid \alpha \vdash B} \qquad \Pi^* \; \frac{\mathfrak{X} \mid \alpha, b \vdash A(b)}{\mathfrak{X} \mid \alpha \vdash (\forall x)A(x)}$$

$$^*\Sigma \; \frac{\mathfrak{X},\, A(b) \mid \alpha, b \vdash B}{\mathfrak{X},\, (\exists x)A(x) \mid \alpha \vdash B} \qquad \Sigma^* \; \frac{\mathfrak{X} \mid \alpha \vdash A(t)}{\mathfrak{X} \mid \alpha \vdash (\exists x)A(x)}$$

Here t is any term, and b is a variable, called the *characteristic variable,* which does not otherwise occur. These rules may be adjoined to any of the systems previously discussed; the new system is indicated by affixing an asterisk, so that, for example, from LA we have LA*. In such systems the rules for operations other than quantification — these will be called the *algebraic rules* — hold for an arbitrary range which is the same in all premises and conclusion. Systems with only algebraic rules will be called *algebraic* systems; other systems, in contradistinction, will be called *quantified* systems.

4. *Multiple formulations.* Gentzen presented, besides the system LJ_1, a system LK_m ([18]) which differed from LJ_1 solely in the fact that

([17]) Due to Kripke [SLE].

([18]) Gentzen called these systems LJ and LK respectively.

sequences of more than one constituent were allowed on the right. The elementary statements of the new system are those of the form

$$(8) \qquad A_1, \ldots, A_m \mid \alpha \mid\!\!-B_1, \ldots, B_n.$$

No commitment is made to a specific interpretation for (8); but if we interpret it in terms of the usual two-valued truth tables with 1 for truth and 0 for falsity, regarding it as true when some A_i has the value 0 or some B_j has the value 1, and as false if all A_i have the value 1 and all B_j the value 0, then the rules of LK_m preserve truth. The system LK_m is called the *multiple form* of LK. It is developed by a formal analogy. The rules allow arbitrary changes in the order and multiplicity, and also the adjunction of new propositions, on the right as well as on the left. The special rules for the operation admit the adjunction of arbitrary elements on the right; these are carried down unchanged from the premises to the conclusion. The theses of LK_m, viz. those propositions A for which (4) is derivable in LK_m, turn out to be the same as those for LK_1.

If this transformation is applied to LA_1 we get a system in which Px is redundant. The theses of the system are the same as those of HC. It is therefore appropriate to call it LC_m, or the multiple form of LC.

This raises the question of whether or not there is a system LA_m, constituting a multiple formulation of LA, whereas LA_1 is a singular formulation of LA. This question was, so far as I know, first raised in [NRG]. As shown there, if we interpret (8) as

$$A_1, \ldots, A_m \mid \alpha \mid\!\!- B_1 \vee B_2 \vee \ldots \vee B_n$$

(with association, preferably, to the right), then we have an interpretation of the multiple system in the singular. The only rule of LC_m which is not valid when so translated into $LA_1{}^*$ is P*. Thus if we restrict P* to be singular, allowing all other rules to be multiple, we have a formulation LA_m.

In a similar way we can form multiple forms of all the systems considered in § 3. For those systems which are classically based there is no restriction on the multiple rules; but for those which are absolutely based the rules P* and Π* must be restricted to be singular ([19]). All these systems are equivalent to the corresponding singular systems ([20]).

([19]) In the case of what are called below the special rules the situation is a little complex. If N* is taken as an independent rule (see below) it must also be singular.

([20]) These multiple systems were, so far as I know, first seriously studied

Thus for all the systems we have singular and multiple L-forms. One can formulate also T-forms and H-forms ([21]). All of the different forms of the same system appear to be equivalent ([22]).

The whole set of rules, for both singular and multiple cases, will now be exhibited in a table. In this $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ represent sequences of propositions; $\mathfrak{X}$ is arbitrary, $\mathfrak{Y}$ is restricted to be singular in all singular cases, and $\mathfrak{Z}$ is void in all singular cases, but $\mathfrak{Y}$ and $\mathfrak{Z}$ are both arbitrary in the multiple cases. The notation

(9) $\qquad A_1, \ldots, A_m \vdash B$

indicates that $A_1, \ldots, A_m$, B are elementary and that there is a deductive rule in $\mathfrak{S}$ allowing us to infer B whenever $A_1, \ldots, A_m$ are verified.

PRIME STATEMENTS:

(p1) $A \Vdash A$

(p2) $\Vdash A$ where A is an axiom of $\mathfrak{S}$.

STRUCTURAL RULES.

*C If $\mathfrak{X}'$ is a permutation of $\mathfrak{X}$.

$$\frac{\mathfrak{X}' \Vdash \mathfrak{Y}}{\mathfrak{X} \Vdash \mathfrak{Y}}$$

C* If $\mathfrak{Y}'$ is a permutation of $\mathfrak{Y}$.

$$\frac{\mathfrak{X} \Vdash \mathfrak{Y}'}{\mathfrak{X} \Vdash \mathfrak{Y}}$$

*W $$\frac{\mathfrak{X}, A, A \Vdash \mathfrak{Y}}{\mathfrak{X}, A \Vdash \mathfrak{Y}}$$

W* $$\frac{\mathfrak{X} \Vdash A, A, \mathfrak{Z}}{\mathfrak{X} \Vdash A, \mathfrak{Z}}$$

*K $$\frac{\mathfrak{X} \Vdash \mathfrak{Y}}{\mathfrak{X}, A \Vdash \mathfrak{Y}}$$

K* $$\frac{\mathfrak{X} \Vdash \mathfrak{Z}}{\mathfrak{X} \Vdash A, \mathfrak{Z}}$$

by Maehara. In his [DIL] he gave the essentials of a proof that LJ_m is equivalent to LJ_1. See note (22).

([21]) There are also lattice forms. When quantifiers are present we are then lead to the cylindric algebras of Tarski, the polyadic algebras of Halmos, etc. These go beyond the scope of the present article.

([22]) This has been shown for all the cases mentioned here except LD (and, of course, LD*), for which the question is not quite settled. See § 6.

Operational Rules.

*P
$$\frac{\mathfrak{X} \Vdash A;\ \mathfrak{X}, B \Vdash \mathfrak{Y}}{\mathfrak{X}, A \supset B \Vdash \mathfrak{Y}}$$

P*
$$\frac{\mathfrak{X}, A \Vdash B, \mathfrak{Z}}{\mathfrak{X}, \Vdash A \supset B, \mathfrak{Z}}$$

*Λ
$$\frac{\mathfrak{X}, A \Vdash \mathfrak{Y}}{\mathfrak{X}, A \wedge B \Vdash \mathfrak{Y}} \qquad \frac{\mathfrak{X}, B \Vdash \mathfrak{Y}}{\mathfrak{X}, A \wedge B \Vdash \mathfrak{Y}}$$

Λ*
$$\frac{\mathfrak{X} \Vdash A, \mathfrak{Z};\ \mathfrak{X} \Vdash B, \mathfrak{Z}}{\mathfrak{X} \Vdash A \wedge B, \mathfrak{Z}}$$

*V
$$\frac{\mathfrak{X}, A \Vdash \mathfrak{Y};\ \mathfrak{X}, B \Vdash \mathfrak{Y}}{\mathfrak{X}, A \vee B \Vdash \mathfrak{Y}}$$

V*
$$\frac{\mathfrak{X} \Vdash A, \mathfrak{Z}}{\mathfrak{X} \Vdash A \vee B, \mathfrak{Z}} \qquad \frac{\mathfrak{X} \Vdash B, \mathfrak{Z}}{\mathfrak{X} \Vdash A \vee B, \mathfrak{Z}}$$

*N (23)
$$\frac{\mathfrak{X} \Vdash A, \mathfrak{Z}}{\mathfrak{X}, \neg A \Vdash F, \mathfrak{Z}}$$

N* (23)
$$\frac{\mathfrak{X}, A \Vdash F, \mathfrak{Z}}{\mathfrak{X}, \neg A \Vdash \mathfrak{Z}}$$

*Π If t is a term

$$\frac{\mathfrak{X}, A(t) \mid \alpha \vdash \mathfrak{Y}}{\mathfrak{X}, (\forall x)A(x) \mid \alpha \vdash \mathfrak{Y}}$$

Π* If b is a variable which does not occur in $\mathfrak{X}, \mathfrak{Z}$

$$\frac{\mathfrak{X} \mid \alpha, b \vdash A(b), \mathfrak{Z}}{\mathfrak{X} \mid \alpha \vdash (\forall x)A(x), \mathfrak{Z}}$$

*Σ If b is a variable which does not occur in $\mathfrak{X}, \mathfrak{Y}$

$$\frac{\mathfrak{X}, A(b) \mid \alpha, b \vdash \mathfrak{Y}}{\mathfrak{X}, (\exists x)A(x) \mid \alpha \vdash \mathfrak{Y}}$$

Σ* If t is a term

$$\frac{\mathfrak{X} \mid \alpha, b \vdash A(t), \mathfrak{Z}}{\mathfrak{X} \mid \alpha \vdash (\exists x)A(x), \mathfrak{Z}}$$

Special rules

$\vdash$* If $A_1, \ldots, A_m \vdash B$, then

$$\frac{\mathfrak{X} \parallel A_i, \mathfrak{Z} \qquad i = 1, 2, \ldots, m}{\mathfrak{X} \Vdash B, \mathfrak{Z}}$$

(23) For the F-formulation, which is the only one here considered, the rules *N* are special cases of *P*. In the FN-formulation they would have to be stated as separate rules; in the N-formulation the indicated «F»s would be omitted on the right. However, for these formulations some changes would have to be made in the text.

$$\text{Px}\quad \frac{\mathfrak{X},\ A \supset B \ \|\!- \ A,\ \mathfrak{Z}}{\mathfrak{X} \ \|\!- \ A,\ \mathfrak{Z}}$$

$$\text{Nx}\quad \frac{\mathfrak{X},\ \neg A \ \|\!- \ A,\ \mathfrak{Z}}{\mathfrak{X} \ \|\!- \ A,\ \mathfrak{Z}}$$

$$\text{F*}\quad \frac{\mathfrak{X} \ \|\!- \ F_i,\ \mathfrak{Z}}{\mathfrak{X} \ \|\!- \ F,\ \mathfrak{Z}}$$

$$\text{Fj}\quad \frac{\mathfrak{X} \ \|\!- \ F,\ \mathfrak{Z}}{\mathfrak{X} \ \|\!- \ A,\ \mathfrak{Z}}$$

In connection with these rules the following terminology is useful. Let us call the individual occurrences of the propositions $A_1, \ldots, A_m, B$ in (7) or of $A_1, \ldots, A_m, B_1, \ldots, B_n$ in (8) the *constituents* (24) of that statement. Constituents which are instances of the same proposition will be said to be *alike*. We may speak also of the constituents of an inference or a proof. Among the constituents of an inference there are these kinds. Those in the sequences $\mathfrak{X}$, $\mathfrak{Y}$, or $\mathfrak{Z}$ occur in matching sets, one in each of the premises and one in the conclusion, so that these constituents pass unchanged, so to speak, through the inference; these will be called *parametric constituents,* or parameters (25). Each operational rule introduces a new constituent, formed by the operation in question, into the conclusion; this constituent will be called the *principal constituent,* and the constituent(s) in the premises representing the components from which it is formed will be called the *subaltern constituents* or subalterns. It il also convenient to define a relation of *ancestor* and *descendant* (26) among the constituents in a proof; each parametric constituent in the conclusion is descendant of all those that match it in the premises, and the principal constituent is descendant of each of the subalterns; the relation of descendant is then extended so as to be transitive, and that of ancestor is defined to be its converse. We may also speak of parametric ancestors or descendants in an obvious sense (27).

(24) In an article of this kind it is not possible to be absolutely precise in regard to the distinction between a constituent and the propositions of which it is an occurrence.

(25) The term «parameter», properly speaking, refers to the set of matching parametric constituents.

(26) Due to Kleene [PIG].

(27) These terms can be extended to the structural rules *K* and *W*, and also to the special rules, in an obvious sense. However in *K* there

5. *Modified formulations.* The formulation given in § 4 will be called Formulation I. We shall consider here a number of variants of this formulation.

In Formulation I it is required that the parameters be alike in all the premises. If we modify the rules with more than one premise so that each may have its own $\mathfrak{X}$ and $\mathfrak{Y}$ (or $\mathfrak{Z}$), all of these having mates in the conclusion, we get a formulation called Formulation II. Thus the rule Λ^* would become

$$\frac{\mathfrak{X}_1 \Vdash A, \mathfrak{Z}_1 \qquad \mathfrak{X}_2 \Vdash B, \mathfrak{Z}_2}{\mathfrak{X}_1, \mathfrak{X}_2 \Vdash A \wedge B, \mathfrak{Z}_1, \mathfrak{Z}_2.}$$

In Formulation II there is a one-one correspondence, such that corresponding constituents are alike, between the parametric constituents of the conclusion and those of the premises collectively.

Ketonen, in 1944, introduced an important modification. We have seen that $^*\Lambda$ consists of two separate rules. Ketonen proposed to replace these by the single rule

$$\frac{\mathfrak{X}, A, B \Vdash \mathfrak{Y}}{\mathfrak{X}, A \wedge B \Vdash \mathfrak{Y}.}$$

In the multiple cases one can make an analogous change in V*. Ketonen also proposed to modify *P so as to read

$$\frac{\mathfrak{X} \Vdash A, \mathfrak{Y} \qquad \mathfrak{X}, B \Vdash \mathfrak{Y}}{\mathfrak{X}, A \supset B \Vdash \mathfrak{Y}.}$$

This again is only possible in the multiple systems. I shall indicate that Ketonen modifications have been made to the extent compatible with the singularity restrictions by adding a «K», e.g. Formulation IK.

The advantage of the Ketonen modifications is that they make certain of the rules invertible, in the sense that the inference from the conclusion to any premise is admissible [28]. Then Ketonen

are no subalterns; in *W* the subalterns are like the principal constituent, and in Px one subaltern is like the principal constituent and the other contains a replica of it as component, etc.

(28) Cf. § 2. This does not imply that one can derive the premises from the conclusion by application of the rules.

showed that all operational rules of Formulation IK are invertible in LK_m; and from this fact he was able to prove the completeness of LK with extraordinary elegance [29]. In LA_m the invertibility holds with respect to the right premise but not with respect to the left. The invertibility can be restored by introducing into the left premise a constituent, called the *quasi-principal constituent* [30], which is like the principal constituent, so that the rule becomes

$$\frac{\mathfrak{X}, A \supset B \Vdash A, \mathfrak{Y} \qquad \mathfrak{X}, B \Vdash \mathfrak{Y}}{\mathfrak{X}, A \supset B \Vdash \mathfrak{Y}.}$$

The rules Π^* and $^*\Sigma$ are invertible in all systems, but not the rules Σ^* and $^*\Pi$; these two can be made invertible by adjoining a quasi-principal constituent (which is necessary even in LK_m). The same is true for certain special rules. The formulation obtained from Formulation IK by introducing quasi-principal constituents as just indicated will be called Formulation III. In it all operational rules are invertible [31].

Certain restrictions can be made, without loss of generality, in regard to the prime statements (pl) and the rules *K*. The former can be restricted to the case that A is elementary. The latter can be restricted in all singular systems, and also in all cassically based systems, to be made initially [32]. In the absolutely based multiple systems this is true on the left; but on the right one must admit also the possibility of an application immediately following a rule which is restricted to be singular on the right (e.g. P*, Π^*). Likewise the principal constituent can be restricted to one which is elementary, or cannot introduced by other means.

The rules *W* can be shown to be redundant if in all cases where a rule other than *K* is not invertible with respect to a given premise

(29) These results hold also for LC_m, LE_m in virtue of the discussion in § 6.

(30) This is a special kind of subaltern.

(31) I have purposely not treated the special rules at length. The rules Px and Nx are invertible since the premise can be obtained from the conclusion by *K. In the systems with K*, we can replace Fj by a rule for dropping F when there is some other constituent on the right; such a rule is invertible by K*. The rules $\vdash^*$, F* are not invertible, and need to be modified as stated in Formulation III.

(32) This is equivalent to admitting as prime statements all those of form (7) or (8) in which a constituent on the left is like one on the right, and omitting *K* altogether.

a quasi-principal constituent ([33]) occurs in that premise ([34]). Thus in formulation III the rules *W* are redundant, and can be omitted altogether.

Even if *W* are redundant, it may be necessary to have two or more occurrences of the same proposition on the same side of a statement. In order to exclude this possibility, and thus to allow the sides of a sentence (8) to be interpreted as classes of propositions, it is sufficient to adjoin rules obtained from those already present by allowing the same constituent to serve two or more times as a subaltern, or as a subaltern and a parameter. The following are examples ([35]):

From Ketonen *Λ $$\frac{\mathfrak{X}, A \Vdash \mathfrak{Y}}{\mathfrak{X}, A \wedge A \Vdash \mathfrak{Y}}$$

From P* $$\frac{\mathfrak{X}, A \Vdash B}{\mathfrak{X}, A \Vdash A \supset B.}$$

The formulation so obtained from Formulation III will be called Formulation IV.

The formulations are all equivalent to one another in all cases where *K* and *W* are admissible (whether explicitly assumed or not); but if this is not the case there may be essential differences between them.

([33]) The presence of such a quasi-principal constituent does not always make the rule invertible; e.g. in the original form of *P.

([34]) In the case of singular systems this has to be done only on the left.

([35]) The examples given in [TFD] (Remark 5 on p. 37) were trivial in that the conclusion was obtained by *C* from one premise. This is true in most cases. Of those given here, the second is such that the conclusion can be obtained by the ordinary P* and *K. This is typical. It requires uses of *K* transcending the restrictions discussed above.

(to be continued)

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