NOTES ON A GROUP OF NEW MODAL SYSTEMS

A. N. PRIOR

There is a certain modal system Q which I have elsewhere (*Time and Modality*, Ch. V) defined as that system consisting of all, and only, those formulae in C,N,M,L and propositional variables which are verified by the following infinite matrix: The «values» of propositions are infinite sequences containing 1's, 2's, 3's or combinations of these, and beginning with 1 or 3. The designated values are all those not containing any 3's. Wherever the value of a proposition p has a 2, the value of any function of p will have a 2 also; at other places the function sequences are as follows: Cpq (if p then q) has 1 at places where p has 3 and q has 1 or 3 and where p and q both have 1, and 3 at places where p has 1 and q 3. Np (Not p) has 1 where q has 3 and 3 where q has 1. Lp (Necessarily p) has 1's throughout if and only if p has 1's throughout, otherwise has 3's except where p has 2. If p has 1 anywhere, Mp (Possibly p) has 1's except where p has 2, otherwise 3's except where p has 2.

The intuitive basis of this matrix is that the first place in a sequence gives the truth value or the given proposition in the actual world (1 for 'true' and 3 for 'false'), and the other places its truth value in other possible worlds, apart from worlds in which there could not be any such proposition as the one under consideration (e.g. because of the non-existence in that world of some object which the proposition is directly about), the number 2 being put in a place which represents a world which, with the given proposition, is of this sort. L is taken to mean 'true in all worlds', this being a stronger characterisation than 'false in none' (NMN); and M, 'true in some world' this being a stronger characterisation than 'not false-in-all' (NLN). The designation of all 3-less sequences expresses the principle that a proposition is logically true if it is true in all circumstances in which there is any such proposition. Distinctive features of the system are the complete absence of theses beginning with L (though we have $\vdash NMN\alpha$ whenever we have $\vdash \alpha$), and the impossibility of proving, when the ordinary rules for quantification theory are adjoined, such formulae as $CL\Pi x\varphi x\Pi xL\varphi x$ $C\Pi xL\varphi xL\Pi x\varphi x$ and $CM\Sigma x\varphi x\Sigma xM\varphi x$ (though we do have $\vdash C\Sigma xM\varphi xM\Sigma x\varphi x$).

I do not know of any set of postulates for which this matrix has been proved to be characteristic, but in 1956 E. J. Lemmon produced a strongly supported conjecture about this which is worth recording (and which I have his permission to communicate). He had already proved at this time (cf. JSL, 1956, pp. 347-9) that the following postulates of my own, subjoined to the propositional calculus, yield a system equivalent to Lewis's S5: —

M1: $\vdash C\alpha\beta \rightarrow \vdash CM\alpha\beta$, provided that every variable in β is modalised (i.e. falls within the scope of some M).

M2: $\vdash C\alpha\beta \rightarrow C\alpha M\beta$ (or the axiom: $\vdash CpMp$).

Df.L: L = NMN.

In Q, L is not definable in terms of M, and Lemmon conjectured that if we drop Df.L above, and add to M1 the further proviso that β contain no variables that are not in α, we will have a sufficient basis for that part of Q in which L does not occur. To obtain the rest, he further suggested, we need only add the rule

RL : $\vdash C\alpha\beta \rightarrow \vdash CL\alpha L\beta$, provided that β contains no variables not in α, and the axioms

1. CKLpLqLKpq 2. CLpp 3. CLpLLp 4. CLNLNpMp
5. CMqMCLpq

(also allowing 'modalisation' in M1 to include falling within the scope of some L).

There is now a further result, of my own, which seems to give considerable added weight to this conjecture of Lemmon's; namely that these postulates, supplemented by a further definition, are demonstrably equivalent to another set which express very directly indeed the basic intuitions underlying the Q matrix. The peculiarity of Q as a modal system is that it allows for propositions not being under all circumstances 'statable'. Suppose we write Sp for 'It is necessarily statable that p', and define the Lp of Q as 'p is necessarily statable, and cannot be false' i.e. KSpNMNp. This direct introduction of the idea of 'statability' into the formalism was suggested to me by J. L. Mackie. Beside this definition of L in terms of it, we may lay down for S the following two rules (reflecting the fact that a proposition is statable when and only when all its constituent propositions are):

RS1 : $\vdash CS\alpha Sp$, where p is any variable in α,

RS2 : $\vdash CSpCSq\ldots S\alpha$, where p, q, etc are all the variables in α.

For M, we introduce the usual M2 and the following modification of M1 :

SM1 : $\vdash C\alpha\beta \leftrightarrow \vdash CSpCSq \ldots CM\alpha\beta$, where every variable in β is modalised (i.e. falls within the scope of some M or S) and p, q, etc. are all the variables in β that are not in α.

(If there are no variables in β that are not in α, the consequent of the last rule becomes the simple $\vdash CM\alpha\beta$; i.e. we have Lemmon's modification of M1 as a special case of SM1).

In a system where L rather than S is undefined, Sp is easily definable as LCpp (giving it, in the Q matrix, the value 1's-all-through if p has no 2's, and if it has, 3's except where p has 2's); and with this definition, Lemmon's postulates in M and L can be shown to yield exactly the same asserted formulae as the above set in M and S. Here, for exemple, is the proof of SM1 from Lemmon's postulates and Df.S (the key step is to prove the lemma 12):

6. CCLpqCLpKLpq (p.c.)
7. CMCLpqMCLpKLpq (6, M2, mod. M1).
8. CMqMCLpKLpq (5, 7, Syll).
9. CCLpKLpqCLpMKLpq (M2, Syll.
10. CMCLpKLpqCLpMKLpq (9, mod. M1).
11. CLpCMCLpKLpqMKLpq (10, Comm).
12. CKLpMqMKLpq (8, 11, CCqrCCpCrsCKpqs).
13. Cαβ (hypothesis; β fully modalised).
14. CKLCppKLCqq...αβ (CKpqp, 13, Syll; p, q etc. all the variables in β but not in α).
15. CLKpqLp (CKpqp, RL).
16. CLKpqLq (CKpqq, RL).
17. CLKpqKLpLq (15, 16, CCpqCCprCpKqr).
18. CKLKCppKCqq...αβ (14, repeated use of 17, Syll).
19. CMKLKCppKCqq......αβ (18, mod. M1).
20. CKLKCppKCqq.....Mαβ (12, 19, Syll).
21. CKLCppKLCqq... Mαβ (17, repeated use of 1, Syll).
22. CLCppCLCqq......CMαβ (21, exportation).
23. CSpCSq......CMαβ (22, Df.S).

Conversely, in the M-S system we prove Lemmon's 5 (CMqMCLpq) by applying CCpqCCNpqq to CSpCMqMCLpq (from SM1 and CqMCLpq, from Simp and M2) and CNSpCMqMCLpq (i.e. CNSpCMqMCKSpNMNpq ; proved by a series of simple steps from CNSpCSpq). The other parts of the proof of equivalence are fairly obvious.

In 1956 it was also shown by Lemmon that a number of interesting *extensions* may be made to Q without falling into inconsistency or completely destroying its character as a modal system, namely

(i) Q + ⊢CLpp, equivalent to Lewis's S5, in which L and M remain genuinely modal functors, but are not both needed as primitives (Lp = NMNp, Mp = NLNp).

(ii) Q + ⊢NLp, in which L is destroyed as a modal functor (Lp becoming contradictory) but M is not.

(iii) Q + ⊢CMpp, in which M is destroyed as a modal functor (Mp = p) but not L.

(iv) Q + ⊢ CLMpp.

(iv) is contained in both (ii) and (iii), and it (and so *a fortiori* (ii) and (iii))) cannot be combined with (i) without destroying both L and M as modal functors (Q + ⊢ LCpp + ⊢ CLMpp → ⊢ CpLp, ⊢ CMpp). System (i) is verified by O's matrix with every value containing 2 removed; (ii) by Q's matrix with every value *not* containing 2 removed; (iii) by Q's matrix with every value which contains both 1's and 3's removed; (iv) by Q's matrix with every value which contains both 1's and 3's but not 2's removed. (In each of these cases, the application of C, N, L or M to any value in the restricted set will not yield a value outside it).

M-S systems equivalent to these may be obtained by enlarging Q thus: For (i) add ⊢ Sp (or define Sp as Cpp); for (ii) add ⊢ NSp (or define Sp as NCpp); for (iii) add ⊢ CMpp (or define Mp as p); for (iv) add ⊢ CSpCMpp. Whether or not the M-S postulates given above successfully define Q, the addition of ⊢ Sp to these postulates definitely yields Q + ⊢ Sp, i.e. S5 (SM1 collapses by a series of detachments to M1, and KSpNMNp, i.e. Lp, to NMNp, giving the postulates of S5 very neatly). And if we equate a proposition's being necessarily statable with the necessary existence of all objects directly named in it, the above postulates (i.e. those for Q plus ⊢ Sp) exhibit S5 as the result of assuming that all beings are necessary beings. (ii) results similarly from the contrary assumption that none are. In (iii) the addition (of ⊢ CMpp) makes Lp (KSpNMNp) equivalent to KSpp, 'p is necessarily statable and actually true', the laws of this queer 'necessity' being deducible from this equivalence and RS1 and 2 (SM1 being now redundant). And now we see what (iv) is — it is a modal system which, unlike (i) and (ii), allows for *both* necessary and contingent beings, but it assumes (for it is a simple matter to deduce CSpCpLp in this system) that no proposition which is about necessary beings only, can be contingent; all such propositions are either necessary or impossible (though ones which are about *both* necessary and contingent beings — e.g., perhaps, '9 is the number of planets in this solar system' — may be contingent.)

In System (iii) the collapse of M and transformation of L is of such a character that the system may be characterised not only by Lemmon's infinite matrix but by the following finite one:

K	1	2	3	4	N	S	L	M
*1	1	2	3	4	4	1	1	1
*2	2	2	3	4	3	3	3	2
3	3	3	3	3	2	3	3	3
4	4	3	3	4	1	1	4	4

Proof that RS1-2 + Dff. L,M + propositional calculus exactly fits this matrix is easy. Read 'p = 1' as KSpp, 'p = 2' as KNSpp, 'p = 3' as KNSpNp, 'p = 4' as KSpNp, and prove the implications corresponding to the use of the table (e.g. prove CKKSppKNSqqKNSKpq-Kpq for 'If p = 1 and q = 2, Kpq = 2'). A simple equivalent set of postulates in L (with Sp defined as LCpp and Mp as p) would be Lemmon's RL, 1 (CKLpLqLKpq) and 2 (CLpp), plus 3′: CLCppCpLp (subjoined to propositional calculus). As these are all in the weaker calculus (iv) also, it is clear that this has the same M-less fragment as (iii). (3′ is in (iv) since RL, 2, CLMpp give CLMpLp, and CLCppCpLMp is in Q.)

The system (iii), or rather its M-less portion, has at least one interesting extension. Add ⊢ CLqLCpp to RL, 1,2,3′, or alternatively replace 3′ by 3″. CLqCpLp, and RL and 1 become superfluous. The resulting calculus (but with Mp, if used at all, for NLNp rather than for p) is equivalent to that part of the L-modal system of Lukasiewicz (*Aristotle's Syllogistic*, 2nd ed., ch. VII) which is expressible without variable functors. The distinctive feature of the L-modal is that, if we write Fp for the contradictory function NCpp and Sp for the plain p, then ⊢ f(L) if and only if both ⊢ f(F) and ⊢ f(S), and in fact on my view this L is simply a variable functor with its value restricted to F and S. The use of 2 and 3″ to derive ⊢ f(L) given ⊢ f(F) and ⊢ f(S) may be illustrated by the following example, where f is C′pN′Np (the apostrophe for F,S, or L):

24. CFpNFNp (from CFpq, p.c.)
25. CSpNSNp (i.e. CpNNp, p.c.)
26. ANLqCpLp (3″, C = AN)
27. ANLqELpp (26,2)
28. AELqFqELpp (27, CNpEpFq)
29. AELqFqELpSp (28, Df.S)
30. AELNqFNqELpSp (29 q/Nq)
31. AKELqFqELNqFNqELpSp (29, 30, CKAprAqrAKpqr)

32. AKELqFqELNqFNqELNpSNp (31 p/Np)
33. AKELqFqELNqFNqKELpSpELNpSNp (31, 32)
34. CCpNqCKErpEsqCrNs (p.c.)
35. CKELpFpELNpFNpCLpNLNp (24,34)
36. CKELpSpELNpSNpCLpNLNp (25, 34)
37. CLpNLNp (35, 36, 33q/p, CCprCCqrCApqr).

The same procedure, with slight variations, will take us from CFNFpFNp and CSNSpSNp to CLNLpLNp (one of this system's odder laws): and similarly in all other cases.

(University of Manchester)