ON DEFINITIONS IN FORMAL SYSTEMS (1)

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1. INTRODUCTION

The traditional conception of a definition, in relation to a mathematical system, is that it is a convention in regard to the use of language. The idea is to introduce a new symbol or symbol-combination, which we may call the definiendum, with the stipulation that it is to stand for another symbol combination, the definiens, whose meaning is already known in terms of certain given symbols. It is intended that, given any combination of the symbols, new and old, which is correctly formed, one can by successive replacements of definienda by their corresponding definientia reduce the given expression to one which is properly formed in terms of the given symbols; this latter expression, which we may call the ultimate definiens of the given one, is supposed to always exist and be unique. The definitions are considered a kind of shorthand which can, theoretically at least, be dispensed with altogether; it has no relation to the content of the theory being considered, but only to the language in which it is expressed.

On trouble with this idea is that we do not confine ourselves to making definitions in this simple fashion. We define not single symbol combinations, but whole schemes or families of them, and it is often convenient to call the whole scheme of equations a definition. Thus in a Peano arithmetic we call the schemes

(1)
$$x + 0 = x,$$
$$x + y' = (x + y)'$$

where the accent indicates the successor function and the letters x, y are intuitive variables for natural numbers, a definition of addition; and it is indeed true that the infinite set of equations formed by substituting numerals (i.e. expressions in the series $0, 0', 0'',\ldots$) for 'x' and 'y' in (1) in all possible ways constitute a definition of addition in the sense that, given any expression formed from '0', the accent, '+', and parentheses in the usual fashion, there is a unique numeral which can be obtained from the expression by a series of steps, each of which is a replacement of the left side of one of those equations by the corresponding

(1) This article presents, in somewhat abbreviated form, the content of [1], § 2E (numbers in brackets refer to the Bibliography). There is, however, enough of the background to make the present discussion self-contained, so far as explanations of ideas are concerned; but for formal proofs it will be necessary to refer to [1]. In certain matters of detail some improvements of presentation have been made.

right side. But, if we generalize the notion of definition in the way suggested by this example, it is not clear under what circumstances the series of replacements has a unique termination. Indeed it is known that the question of whether a set of equations in one or more new numerical functions satisfies these conditions is undecidable (2). Thus the problem of what constitutes a definition under these general circumstances is far from being a trivial matter.

The example (1) illustrates another fact about definitions. We cannot say that they are matters of language to be distinguished sharply from content. Indeed, there are those who maintain that formal reasoning, as such, has nothing to do with content anyway — that it is *all* a matter of language — and therefore it is impossible to make a sharp distinction between definitions and other sorts of conventions. Even if one does not subscribe to this point of view, one must agree that the conventions (1) are something more than abbreviations. In fact it is more natural to think of them as postulates for an extension of Peano arithmetic with addition as a new operation. This point of view, whereby a definition is an extension of the underlying system, is a fruitful one, which includes the notion of abbreviation as a special case. It will form the basis of the discussion given here.

2. Formal systems

Up to the present I have been rather vague as to the nature of the «mathematical system» with which we are supposed to be dealing. Before we go further it will be necessary to clarify this concept (3).

There are now in common use two sorts of deductive system which do not presuppose logic. I shall call these *syntactical systems* and *formal systems* respectively. These two sorts have much in common. I shall begin by describing their common features; I shall comment on the differences and interrelations between them.

To begin with, both sorts of system generate their theorems from certain initial ones (the axioms) by explicitly stated rules. This is done in such a way that an alleged process of derivation (proof) can be effectively checked. To this end, a class of elementary statements is first specified : this is done by postulating certain predicates, and certain objects; then an elementary statement is one ascribing one of these predicates to an ordered sequence of the proper number of objects. The predicates may be unary (4), like that symbolized

(2) It is well known (see e.g., [2]) that there is no recursive method for determining whether a set of equations of a certain form constitutes a recursive definition of a function. This may be regarded as refinements of the statement made in the text. Cf. « Church's thesis » in [2], § 62.

(3) Cf. [1] pp. 14-28; also [3].

(4) For certain objections to the word ‚singulary', used to replace ‚unary' by Quine, Carnap, Church and some others, see my review of [4] in Journal

by the Frege prefix '⊢', Hilbert's «ist beweisbar» or Huntington's «is in T»; binary, like the equality or partial ordering of algebraic systems or the convertibility relation of Church's λ-conversion; or of still higher degree like the betweenness relation (degree three), separation of point pairs (degree four), or the sense-class equality of Veblen and Young (degree six). The axioms must be a subclass of these elementary statements such that it is effectively decidable whether or not an elementary statement is an axiom; and the rules must be such that it is effectively decidable whether or not an alleged application of the rule is correct. All of these relationships must, of course, be expressed in language; the language used for the actual statement of the necessary conventions is called the U-language (5). The U-language must then contain means of forming names fot the objects; and it must contain verbs of the proper sort for expressing the predicates.

So much for the ways in which the two sorts of systems are alike. They differ in the nature of the objects.

In a syntactical system the objects are the *words* (or expressions) of a suitable «object language» which may or may not be part of the U-language. That is, there is specified a (finite or infinite) set («alphabet») of symbols or *letters*; the words are then the linear strings of these letters. The words are thus generated from the letters by concatenation; since this operation is associative a word can generally be constructed in more than one way. If all words are admissible the system will be called *pantactic*; if the system can be so formulated that only certain «well-formed» words (*wefs*) need be taken account of, it will be called *eutactic*; if certain modes of construction can be singled out so that for any (word or) wef there is a unique construction (6) using such modes it will be called *tectonic* (7).

In a formal system, on the other hand, it is simply specified that the objects, called *obs*, are generated from certain primitive ones (the *atoms*) by certain *operations*. In the most rigorous form of this conception (the only one considered here) there is a list of the atoms and operations, with a definite

of the Franklin Institute, vol. 264, pp. 244-246 (1957), especially p. 246. There is, unfortunately, an error in that review, in that the word at the end of the next to the last line of the left hand column of p.246 should be ,second' rather than ,first'. The arguments in that review do, I maintain, demolish the claim of ,singulary' to be « etymologically more correct ». However the real objection to ,singulary' is that it is too much like ,singular', particularly when one tries to translate it into French (cf. [6], p. 252). The word ,singular', however, might be acceptable; but to a person accustomed to English, rather than to Latin, it appears parallel to ,dual' rather than ,binary'. (An other correction is that the statement, made in the review, that the footnotes were numbered serially is incorrect, and remarks on that point should be revised).

(5) See [1] pp. 25, 36.

(6) A construction may be exhibited in the form of a tree. When it is said that the construction is unique, it is meant that this tree is unique. Cf. [1], § 2B.

(7) See [3].

positive integer assigned to each of the latter as its *degree*; it is then understood that the obs are an inductive class whose basic elements are the atoms, and such that the application of an operation of degree k to an ordered sequence of k obs is an ob. There are no restrictions on the applicability of the operations other than that the number of arguments must be as stated. It is important that obs constructed in different ways are considered as distinct obs. Nothing is said about the nature of the obs. However there is no objection to assigning a unique concrete object to each ob; if this is done in such a way that distinct objects are assigned to distinct obs, we say that we have a *representation* of the formal system in the set of objects concerned. A formal system cannot be conceived without a representation, since the names of the obs in the U-language constitute a particular representation, called a presentation. But the representation has no effect on the truth of the theorems.

Now the essential difference between these two conceptions does not lie in the fact that a syntactical system is linguistic whereas a formal system is not. In fact it has been pointed out, e.g. by Carnap and Lorenzen, that the letters of a syntactical system do not have to be letters in the ordinary sense; they may be stones or other physical objects, or even sounds. Again any formal system can be represented in terms of symbols; so represented the formal system is a tectonic syntactical system. The characteristic feature of a formal system is rather the uniqueness of construction of its obs. A tectonic system also has this property. Thus a tectonic system can always be regarded as a represented formal system. But since the representation is irrelevant, a tectonic system is a formal system as it stands. This is true of most systems of interest in logic and mathematics which, as syntactical systems, are both eutactic and tectonic. Other syntactical systems can be reduced to tectonic systems, and hence to formal systems, by formalizing the notion of concatenation.

A formal system has a certain advantage of emphasis. In such a system we do not specify a representation because it is irrelevant. Although a formal system cannot be communicated without a presentation (and it may be necessary to prove that the presentation is tectonic), it is natural to conceive of the presentation as accidental, and a formal system as invariant with respect to changes in (re)presentation. We can thus say that two tectonic systems of quite different character represent the same formal system. For example the Łukasiewicz prefixed-operator form of the classical propositional calculus is quite different from the one with binary infixes and parentheses; but from the standpoint of formal systems no change has been made. For this and other, more subjective, reasons the notion of formal system is preferred here. However, in the study of definitions it may help understanding to think of a formal system as symbolically represented. The reader may want to invent an object language, and to replace the word ob by the word 'wef'.

The above considerations require some modification when deductive rules are taken into account. This is irrelevant here, because definitions depend only on the obs, not what is said about them.

3. Definitional extensions

We turn now to the formulation of definitions in the generalized sense of § 1. We suppose that we have a formal system $\mathfrak{S}_0$. To form a definition over $\mathfrak{S}_0$ we form an extension $\mathfrak{S}_1$ of $\mathfrak{S}_0$ by adding new operations (and atoms), and formulating a relation of definitional reduction such that, under certain circumstances, an ob of $\mathfrak{S}_1$ reduces to an ob of $\mathfrak{S}_0$ called its ultimate definiens. If there are predicates in $\mathfrak{S}_0$ the corresponding elementary statements are to be true in $\mathfrak{S}_1$ if and only if the statements formed by replacing all obs appearing as major arguments by their ultimate definientia are true in $\mathfrak{S}_0$. This last possibility, however, causes no particular trouble; and it is here ignored.

Before making this more precise, we shall make some preliminary conventions. We shall extend the term «operation» to include the atoms as operations of degree zero. Obs and operations of $\mathfrak{S}_0$ will be described as *basic*, while those of $\mathfrak{S}_1$ which are not in $\mathfrak{S}_0$ will be called *new*. Basic obs will be denoted by letters at the beginning of the alphabet; while letters toward the end will designate obs of $\mathfrak{S}_1$ which may be new or basic. An ob of the form. $\Phi\,(A_1, \ldots, A_n)$ where Φ is a new operation of degree $n > 0$ will be called *simple*

A *definitional extension* over $\mathfrak{S}_0$ is now a formal system constituted as follows :

1. The operations of $\mathfrak{S}_1$ consist of those of $\mathfrak{S}_0$ together with certain new operations. The obs of $\mathfrak{S}_1$ are then generated by these operations as explained in the definition of a formal system in § 2.

2. The elementary statements of $\mathfrak{S}_1$ are of the form

$$X \; \boldsymbol{D} \; Y \tag{2}$$

Thus '$\boldsymbol{D}$' is a binary infix for expressing the new predicate of $\mathfrak{S}_1$. In (2), X will be called the *definiendum* and Y the *definiens*.

3. The axioms of $\mathfrak{S}_1$ consist of all instances of the reflexive law (ϱ) (i.e. (2) where Y is the same as X), together with a set $\mathfrak{E}$ of defining axioms each of which is of the form

$$\Phi\,(A_1, \ldots , A_n) \; \boldsymbol{D} \; Y, \tag{3}$$

where Φ is a new operation. Thus the definiendum of a defining axiom is always simple. The axioms may be given by means of axiom schemes, as in (1); but then the defining axioms are the individual instances of the schemes, not the schemes themselves. The definitional extension will be said to be *proper* if no two defining axioms have the same definiendum.

4. The single rule of $\mathfrak{S}_1$, called Rd, is

$$\frac{X \; \boldsymbol{D} \; Y \qquad \Phi\,(A_1, \ldots , A_n) \; \boldsymbol{D} \; B}{X \; \boldsymbol{D} \; Y',} \tag{4}$$

where Y' is the result of replacing a component $\Phi\,(A_1, \ldots. A_n)$ of Y by B. We call an application of Rd a *contraction*. If (4) is such an application, we call the left premise the *major premise*, the right premise the *minor premise*, and $X = Y'$ the *conclusion*; also we call (4) a *contraction* on Φ and the particular instance of Φ which is eliminated the *contracted operation*. Note that the contracted operation must be new, and the minor premise must have a simple definiendum and a basic definiens.

4. Definitional reductions

A dervivation from (ϱ) and $\mathfrak{E}$ by means of Rd alone will be called a *(definitional) reduction*; if it terminates in (2) it will be called a reduction from X to Y. An ob Y for which (2) holds will be called a *definiens* of X; an *ultimate definiens* is a basic ob A such that

(5) $$X \boldsymbol{D} A$$

A reduction terminating in such a statement will be called *complete*.

A definitional *reduction* can be exhibited in the form of a tree in the usual fashion. We agree that above any junction (*node*) of such a tree the major premise shall be on the left and the minor premise on the right. The branch furthest to the left will then be called the *principal branch*, and the node at its head the *leading node*. Now the statements corresponding to nodes on the principal branch will all have the same definiendum, viz. that of the leading node; we call this the *definiendum* of the reduction. We can modify the tree by omitting the definienda throughout, provided that at the top of each branch where there was originally an axiom we write an additional node giving the definiendum of that axiom. In such a case the leading node will always be the definiendum of some axiom; the nodes corresponding to minor premises will always be basic, and each will be the terminus of a partial reduction having a simple definiendum.

Next we observe that it is unnecessary to repeat the definiendum in case the axiom used is an instance of (ϱ). For if the X in such an axiom is basic the axiom cannot appear as either a major or minor premise; and if the X is simple it can appear only as major premise for which the conclusion is the same as the minor premise. In the latter case the major premise is superfluous. We can therefore suppose, without explicit indication, that where the definiendum of a reduction is simple the axiom is a defining axiom with that definiendum and the next node below as definiens; in any other case the axiom is an instance of (ϱ), and the definiens of that axiom can be omitted.

With this understanding we conclude the (2) holds if and only if there is a tree of one or more nodes with X as definiendum and Y as terminating node. It follows at once that $\boldsymbol{D}$ is reflexive and transitive. It further has following replacement property: Suppose (2) holds, and let V by obtained from U by replacing a component X of U by Y. Then it is clear that the same replacements which reduce X to Y will also reduce U to V. Thus $\boldsymbol{D}$ is the monotone quasi-ordering generated by $\mathfrak{E}$.

We now consider a standard form to which any reduction can be transformed. Let $\mathfrak{R}$ be a reduction from X to Y, and let Y be $\Phi(Y_1, \ldots, Y_n)$, $n \geqslant 0$, Φ an operation not necessarily new. If Φ is basic, or if $n > 0$ and some argument Y_k is new, the next step on the principal branch cannot be a contraction on Φ. It must therefore be a replacement inside some argument Y_k. Now the replacements inside the different Y_k are clearly permutable with one another. We can therefore require that reductions inside the various Y_k be made in order of increasing k. A reduction satisfying this condition throughout (i.e. for all partial reductions), and such that no new operation is left uncontracted when its turn comes, will be called a *standard reduction*. In such a reduction the contracted operation is always the first operation (in the above ordering) which satisfies the restrictions of Rd. Then it is clear that there will be a complete reduction of X if and only if there is a standard complete reduction.

Suppose we return to the situation of the preceding paragraph with all Y_k basic, so that Y is $\Phi(A_1, \ldots, A_n)$. If Φ is basic so is Y, $\mathfrak{R}$ is complete, and Y is an ultimate definiens of X. If Φ is new, the next step on the principal branch must be an application of Rd in which the minor premise will be that stated in (4). We must then find a partial reduction establishing that minor premise. Such a partial reduction must have $\Phi(A_1, \ldots, A_n)$ as definiendum. This must correspond to a defining axiom. If there is no such defining axiom we say the reduction is *blocked*. If there is more than one there is a choice at this point. Otherwise the axiom is unique.

It follows from this that in a proper definitional extension a standard reduction is an algorithm in the sense of Markov (revised for a formal system rather than a syntactical one). It may terminate in a basic ob, which is then the unique ultimate definiens of X; it may be blocked; or it may continue indefinitely.

5. Complete definitions

If Φ is a new operation of a definitional extension $\mathfrak{S}_1$, we shall say that $\mathfrak{S}_1$ defines Φ *completely* just when there is a unique ultimate definiens for every $\Phi(A_1, \ldots, A_n)$; it defines Φ *univalently* just when there is at most one ultimate definiens for every $\Phi(A_1, \ldots, A_m)$. It defines Φ *explicitly* (9) in terms of certain $\psi_1, \ldots, \psi_m$ if Φ does not appear in the definiens of any defining axiom and the axioms with Φ in the definiendum contain no new operations other than $\psi_1, \ldots, \psi_m$.

These terms apply no matter how the axioms of $\mathfrak{E}$ may be expressed.

(8) See [6], pp. 4ff., and [7] p. 4.
(9) The definition given here is a correction of that given in [1].

If the axioms are given by a finite number of axiom schemes containing intuitive variables only for basic obs, then we replace 'complete' ('univalent') by 'recursive' ('partial recursive'). For the case where $\mathfrak{S}_0$ is a formulation of arithmetic with the successor function as sole operation, these terms are in agreement with those given by Kleene ([10]). The restriction to finite axiom schemes is important for his work, and affects his arithmetization. A generalization of his whole theory of recursive definitions to the case of a general $\mathfrak{S}_0$ is not immediately available, because there is no simple analogue of the u-fonction. However one can, presumably, reduce the general case to the numerical one by the device of arithmetization.

The argument of § 4 shows that a proper definitional extension defines every one of its new operations univalently. The converse is not true. In fact let $\mathfrak{S}_0$ be the above system of arithmetic, and let $\mathfrak{E}$ be the axiom and axiom scheme

$$\Phi(0)\ \boldsymbol{D}\ 0 \qquad (6)$$

$$\Phi(x)\ \boldsymbol{D}\ \Phi(x)$$

These give a partial recursive definition of Φ; but it is not proper since the first axiom and the axiom scheme when x is 0 are two different axioms for $\Phi(0)$.

6. Definitional identity

We have seen in § 4 that the relation $\boldsymbol{D}$ is a monotone quasi-ordering generated by $\mathfrak{E}$. We now consider the monotone equivalence generated by $\mathfrak{E}$. For this relation we use the customary binary infix ,='. Its postulates are (ϱ), $\mathfrak{E}$, Rd, transitivity (τ) and symmetry (σ) it then clearly has the replacement property Rp. It seems proper to call this relation *definitional identity.*

Let us first look at the example (6). It is clear that the only value of x for which $\Phi(x)$ has an ultimate definiens is $x = 0$. On the other hand the statement

$$\Phi(x) \equiv 0$$

is true for every numeral x.

This strange result is due to the fact that (6) are defining axioms for an improper definitional extension. For a proper definitional extension it may be shown, by reasonably elementary means, that if

$$X \equiv Y \qquad (7)$$

holds and either X or Y have an ultimate definiens, then they both have the same ultimate definiens; by slightly more advanced methods, similar to those

([10]) See [2].

used by Church and Rosser in proving a famous theorem of combinatory logic, it can be shown that whenever (7) holds X and Y will have a common definiens.

The failure of these properties in the case of improper definitional extensions raises the question of whether such systems are entitled to be called definitions. At least one must be on one's guard in connection with them. It is noteworthy that Church has always been careful to use a notation, viz. the infix ‚$\rightarrow$' in making definitions (i.e. defining axioms or axiom schemes).

All this naturally raises the question of whether an arbitrary definitional extension can be made proper by omitting axioms in a constructively determined way. This is trivial in the case of (6). In the recursive case one can imagine a machine which will carry out all alternatives simultaneously and accepts only the alternative which first leads to a result. In the partial recursive or univalent case I do not know the answer, and I suspect the question is undecidable.

7. Relative definitions

In practice one makes definitions on top of previous definitions. Having given an extension $\mathfrak{S}_1$ with defining axioms $\mathfrak{E}_1$, one may wish to introduce additional new operations with additional defining axioms $\mathfrak{E}_2$. The extension $\mathfrak{S}_2$ defined in this way is not, however, a definitional extension of $\mathfrak{S}_1$ as basic system. The basic obs of the new system must be the same as for $\mathfrak{S}_1$, viz. obs of $\mathfrak{S}_0$.

Nevertheless we may compare the extension $\mathfrak{S}_2$ with another extension $\mathfrak{S}_2'$ of infinitary character in which the axioms consist of these of $\mathfrak{E}_2$ together with those assigning an ultimate definiens to every simple ob of $\mathfrak{S}_1$. This leads to theorems similar to the lemmas given by Kleene in support of his Theorem II. For the formulation and proof of these theorems, as well as for other developments of the theory of definition, the reader is referred elsewhere ([11]).

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([11]) See [1], § 2E.

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