PROBLEMS ARISING IN THE FORMALIZATION OF INTENSIONAL LOGIC (1)

The possibility of a «logic of intension» has been hotly debated during the last decade, notably by QUINE and CHURCH. Quine's extensionalism is spiritually kin to his nominalism, but has no simple logical connection therewith. True, a nominalist can countenance intensions as little as he can extensions; but he may (like Leibniz) be a nominalist and still be intensional in the sense of admitting modal operators; and Quine's arguments against intensions and modalities alike are entirely independent of his arguments for nominalism.

Church's intensionalism raises a curious methodological problem. Some philosophers of science maintain that in the absence of counterindications the simpler of two empirical theories is to be preferred, but one may hesitate to extend this principle to the *a priori* sciences. To accept such an extension one must be at once sufficiently a platonist to believe in a strong parallelism between the methods applicable to empirical and a priori matters, and sufficiently an empiricist to be hypothetical rather than dogmatic in the latter sphere. Whatever one's feelings on this, however, few of those whose minds are not already made up on the issue of intensions would see anything to object to in the quest for an approach which is immune from all the criticisms which the Frege-Church theory is designed to answer, and at the same time escapes the formidable complexities of the latter.

The clue to such an approach is perhaps to be found in a paper of Church himself «A Formulation of the Simple Theory of Types» in an early issue of the Journal of Symbolic Logic. This paper was designed not to solve any problems in the explication of intensional concepts, but purely to provide the mathematician with as many extensions as he needs for his particular business, while remaining non-commital about extensionality in general.

The types in this paper are the type o of propositions, the type ι of individuals, and for any types α and β, the type $(\alpha\beta)$ of mappings of β into α. The classical propositional calculus is assumed, together with the classical functional calculus in each type. The rule of λ-conversion is assumed in the form that

$$(\lambda x_\alpha)(\ldots x_\alpha -)y_\alpha = \ldots y_\alpha - \tag{1}$$

is a theorem; here identity is defined classically and the system develops in such a way that two sides of a proved identity are intersubstitutable in all contexts. Extensionality holds in the form

$$(x_\beta)(f_{\alpha\beta}x_\beta = g_{\alpha\beta}x_\beta) \rightarrow f = g \tag{2}$$

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but in the absence of $(p_o \longleftrightarrow q_o) \rightarrow p = q$ this is an incomplete principle of extensionality, which at first sight seems to offer us a world in which intensions and extensions can coexist peacefully.

Intensions (for example predicates of individuals) are denoted by λ-expressions in the following way. If $\ldots x_\iota$ — is a propositional function in which x_ι and x_ι alone is free, then $(\lambda x_\iota)(\ldots x_\iota —)$ is the *property* asserted by $\ldots x_\iota$ — to characterize x_ι; one cannot prove, despite (2), that two coextensive properties of this kind are identical. (Of course by (1) and (2) we can assert

$$(x_\alpha)(\ldots x_\alpha — = - \, . - x_\alpha \, . - .) \rightarrow (\lambda x_\alpha)(\ldots x_\alpha —) = (\lambda x_\alpha)(- \, . - x_\alpha \, . - .); \quad (3)$$

but unless we could prove $(p_o \longleftrightarrow q_o) \rightarrow p = q$ we could not establish (3) with the first identity replaced by a material equivalence, and so we are entitled to regard λ-expressions of type $(o\alpha)$ as denoting predicates rather than sets.)

Extensions, running parallel with intensions, are introduced by the following ingenious device. A *primitive* description-operator $\iota_{\alpha(o\alpha)}$ obeys the axiom

$$(E!\, x_\alpha)(f_{o\alpha}\, x) \rightarrow f(\iota_{\alpha(o\alpha)} f) \quad (4)$$

where E! denotes unique existence. To each predicate $f_{o\alpha}$ we now assign the characteristic function

$$\text{Ext}\, f = (\lambda\, x_\alpha)\, \iota(\lambda\, n)\, [(n = 1 \,\&\, fx) \vee (n = 0 \,\& \sim fx)] \quad (5)$$

where 1 and 0 can be any two objects of the same type. Now if we define $x \,\varepsilon\, y$ as short for

$$(\exists f)(fx \;\&\; y = \text{Ext}\, f)$$

or alternatively as short for $(yx) = 1$, we can use (2) to establish

$$(x)(x \,\varepsilon\, (\text{Ext} f) \longleftrightarrow x \,\varepsilon\, (\text{Ext}\, g)) \rightarrow (\text{Ext} f) = (\text{Ext}\, g) \quad (6)$$

and (1) (4) and (5) to establish

$$\ldots x — \longleftrightarrow x \,\varepsilon\, (\text{Ext}\, (\lambda\, y)(\ldots y —)) \quad (7)$$

so that expressions of the form $\text{Ext} f_{o\alpha}$ can function in all ways as (extensional) class-names. (We have been increasingly sparing of typesubscripts of late; this practice will be continued as long as it favors legibility.)

This concludes an account, sketchy but I hope adequate for our purposes, of the system of Church's early paper. Let us recall; it contains as theorems all formulae (1) (2) and (4), and in addition (like all the systems to be considered in this paper) the classical propositional calculus and the classical functional calculus in each type. It suggests the idea of a combined intensional-extensional system which admits both sets and predicates and distinguishes between them, thereby (we hope) escaping all the criticisms which have been made by the extensionalists without involving us in the complexities of the sense-and-denotation solution.

As it stands, the system (call it System I) is liable to a form of the number-of-the-planets paradox. For one can easily prove in it (actually without using either its intensional features or the modicum of extensionality provided by (2)) the harmless-looking formula

$$fx = y \rightarrow ((fx = y) = (fx = fx)) \tag{8}$$

from which, in the presence of the weakest possible assumptions concerning necessity, follows the fact that an identity $fx = y$, if true at all, is necessary. Taking f as a mapping which yields, when applied to any set (or predicate) the cardinality of that set (or predicate), x as the set of planets (or the property of being a planet) and y as the number nine, we obtain the familiar paradox.

The objectionable theorem (8), which in no way seems to depend on any particularly outre features of System I, and which one is equally uncomfortable denying or (in the presence of modalities) asserting, might at first sight strike us as a conclusive argument against intensions in general, or at any rate against modal distinctions. The way out of this difficulty however, has already been pointed out to us by Fitch and by Smullyan in their papers on the related Evening-Star paradox. Namely, it is claimed that the use of the notation « fx » in (8), i. e., the admission of functions other than propositional functions, is at the root of the trouble. Once discourse concerning such functions is paraphrased by the use of descriptions, and once these descriptions in turn are eliminated by Russell's device, the paradox is dissolved, or so it is claimed, without any diminution of the expressive powers of the system.

A modification of System I, designed to embody this suggestion, will now be proposed. It too will be found wanting, but it will unquestionably represent an advance.

The types will be firstly o, ι (as before), but instead of the type $(\alpha\beta)$ for arbitrary α, we are to restrict ourselves, on the present suggestion, to types $(o\beta)$ with *propositional* functions (with arguments of type β) as their members. We shall abbreviate this as (β); thus (β) is the type of predicates of subjects of type β. In addition we shall have types whose members are relations; thus if $\alpha_1, \dots \alpha_n$ are any types, $(\alpha_1, \dots ,\alpha_n)$ is the type of n-termed propositional functions whose arguments are of types $\alpha_1, \dots, \alpha_n$ respectively. (This was not necessary in System I, since relations were construed as certain mappings whose values were themselves mapping; this course is naturally closed to us. Probably we could use the Wiener-Kuratowski device, but the present alternative is more transparent.)

Again the classical propositional and functional calculi are assumed; the rule of λ-conversion appears in the form

$$(\lambda\, x_1 \dots, x_n)\,(=x_1 -.- \dots .-.\ x_n \dots)\,(y_1, \dots, y_n) = -\, y_1 -.- \dots .-.\ y_n \dots \tag{9}$$

on account of our acceptance of relations as ultimate; here of course

$-x_1-.-\ldots.-.\ x_n\ldots$ must be a *formula* rather than an arbitrary term as in System I.

We wish to have extensions as well as intensions in the system, but see no reason to accept (2) as it stands. Rather we shall introduce new types $[\alpha_1, \ldots, \alpha_n]$ of propositional functions in extension whose arguments are of the types $\alpha_1, \ldots, \alpha_n$ respectively; in particular $[\alpha]$ is the type of sets whose members are of type α. For these we postulate extensionality in the form

$$(x_1)\ldots(x_n)\,(r_{[\alpha_1\ldots\alpha_n]}(x_1, \ldots, x_n) \longleftrightarrow s_{[\alpha_1\ldots\alpha_n]}(x_1, \ldots, x_n)) \rightarrow r = s \quad (10)$$

and abstraction in the form

$$(\lambda^* x_1 \ldots x_n)\,(-x_1-.-\ldots.-.\ x_n\ldots)\,(y_1, \ldots, y_n) = -y_1-.-\ldots.-.\ y_n\ldots \quad (11)$$

Thus λ^* is an extensional abstraction operator as λ is an intensional abstraction operator.

The description-operator ι and with it (4), is unavailable in either an extensional or an intensional form, on account of our restriction to propositional functions; this is no loss since descriptions can be introduced contextually.

This system shall be called System II. It escapes the number-of-the-planets paradox, but falls prey to another.

For suppose $a_{(\iota)}$ and $b_{(\iota)}$ are two coextensive predicates. We have

$$(x)\,(ax \longleftrightarrow bx) \quad (12)$$

By (10) and (11)

$$(\lambda^* x)\,(ax) = (\lambda^* x)\,(bx)$$

and then by (11) again

$$\begin{aligned} ax &= (\lambda^* x)\,(ax)\,x \\ &= (\lambda^* x)\,(bx)\,x \\ &= bx \end{aligned} \quad (13)$$

for every x. The logical implication of (13) by (12) is however utterly counterintuitive, and bids fair to abolishing the distinction between coextensive predicates.

This suggests the weakening of System II by replacing (11) by the weaker statement

$$(\lambda^* x_1 \ldots x_n)\,(-x_1-.-\ldots.-.\ x_n\ldots)\,(y_1, \ldots, y_n) \longleftrightarrow -y_1-.-\ldots.-.\ y_n\ldots \quad (14)$$

and (by analogy, though we know of no paradox that is thereby avoided) replacing (9) by

$$(\lambda\, x_1 \ldots, x_n)(-x_1-.-\ldots.-.\ x_n\ldots)\,(y_1, \ldots, y_n) \longleftrightarrow -y_1-.-\ldots.-.\ y_n\ldots \quad (15)$$

The resulting system shall be called System III. It escapes the paradoxical implication of (12) by (13) as well as the number-of-the-planets paradox, but, at least if it is to support a non-trivial modal logic, is still in need of amendment.

For observe first that

$$a = b \rightarrow [\ldots a - \longleftrightarrow \ldots b -]$$

is a theorem of System III. Hence if that system is extended by the introduction of a modal operator $\Box$ for which

$$\Box\, [a = a] \tag{16}$$

is a theorem, we have that

$$a = b \rightarrow [\Box\, [a = a] \longleftrightarrow \Box\, [a = b]]$$

and hence

$$a = b \rightarrow \Box\, [a = b] \tag{17}$$

are theorems. Also by (10) and (14)

$$p \longleftrightarrow [(\lambda^* x)\, [x = x \,\&\, p] = (\lambda^* x)\, [x = x]] \tag{18}$$

By (17) and (18)

$$p \rightarrow \Box\, [(\lambda^* x))\, [x = x \,\&\, p] = (\lambda^* x)\, [x = x]] \tag{19}$$

By (18), if $\Box\, \boldsymbol{A}$ is a theorem whenever $\boldsymbol{A}$ is

$$\Box\, [[(\lambda^* x)\, [x = x \,\&\, p] = (\lambda^* x)\, [x = x]] \rightarrow p]$$

is a theorem; consequently, if

$$[q \rightarrow r] \rightarrow [\Box\, q \rightarrow \Box\, r] \tag{20}$$

is a theorem, so by (19) is

$$p \rightarrow \Box\, p \tag{21}$$

the wished-for paradox. (21) will hold in every extension of System III in which (20) is a theorem and in which also a «rule of necessitation» holds (so that in particular (16) is a theorem).

To avoid this last paradox, we introduce the last of our modifications of Church's System I. This we describe in somewhat fuller detail than the others, since in our estimation it and its modal extension discussed below are the simplest and most natural intensional systems available and the principal contributions of this paper.

Type-subscripts are o, ι and $(\alpha_i, \ldots, \alpha_n)$ and $[\alpha_i, \ldots, \alpha_n]$ whenever $\alpha_i, \ldots, \alpha_n$ are such. The variables $x, y, z, \ldots$ affected with subscript α are terms of type α. The unique primitives of the system are quantifiers and $\rightarrow$ of type (oo). If $\boldsymbol{A}$ is of type o and $\varkappa_{\alpha_i}$ is a variable of type α_i $(i = 1, \ldots, n)$ then $(\lambda \varkappa_{\alpha_1} \ldots \varkappa_{\alpha_n} \boldsymbol{A})$

is of type $(\alpha_1, \ldots, \alpha_n)$. If $\boldsymbol{A}$ is of type $(\alpha_1, \ldots, \alpha_n)$ or $[\alpha_1, \ldots, \alpha_n]$ and $\boldsymbol{B}_i$ is of type α_i for $i = 1, \ldots, n$, then $\boldsymbol{A}(\boldsymbol{B}_1, \ldots, \boldsymbol{B}_n)$ is of type $o. \rightarrow(\boldsymbol{A}, \boldsymbol{B})$ is abbreviated $(\boldsymbol{A} \rightarrow \boldsymbol{B})$. If $\boldsymbol{x}$ is a variable and $\boldsymbol{A}$ is of type o, so is $(\boldsymbol{x})\boldsymbol{A}$. $=_{(\alpha\alpha)}$ is short for $(\lambda x_\alpha\, y_\alpha)\,(f_{(\alpha)})\,(fx \longleftrightarrow fy)$, where $\longleftrightarrow$ and the other truth-functional connectives are defined in familiar fashion. The definition of the existential quantifier is obvious.

The usual axioms and rules of the propositional calculus are assumed, as are the axioms and rules of the classical functional calculus for each type. In addition we retain axiom (15) of System III. Axiom (14) gives way to

$$(\exists\, x_{[\alpha_1 \ldots \alpha_n]})\,(y_{\alpha_1}) \ldots (y_{\alpha_n})\,(x(y_{\alpha_1}, \ldots, y_{\alpha_n}) \longleftrightarrow w_{(\alpha_1 \ldots \alpha_n)}\,(y_{\alpha_1} \ldots y_{\alpha_n}) \qquad (22)$$

where $x_{[\alpha_1 \ldots \alpha_n]}, y_{\alpha_1}, \ldots, y_{\alpha_n}, w_{(\alpha_1, \ldots, \alpha_n)})$ are variables of the types indicated.

Finally axiom (10) of Systems II-III is retained to secure uniqueness of the x asserted to exist by (22).

Thus the new system (call it System IV) differs from System III only in the absence of the extensional abstractor λ^*; so that the extensions asserted by (22) to exist have no names in the system. This appears to preclude the derivation of the paradox (21), though unfortunately we have no formal proof of this.

Of course sets and extensional relations can be introduced contextually by Russell's theory of descriptions and their usual properties proved, so the absence of the λ^* operator does not impoverish the expressive powers of the system.

A somewhat distantly related but entirely natural question to ask at this point is what axioms we should add for necessity. In addition to (10), (15) and (22) the modal extension of System IV which we are now proposing (call it System V) has as axioms the axioms of Lewis's system S5 for the modal propositional calculus, and as a rule the «rule of necessitation»; if $\boldsymbol{A}$ is a theorem, so is $\Box\, \boldsymbol{A}$. The case for S5 as against the weaker Lewis systems has already been eloquently pleaded by Carnap and others, and there seems to be no need to repeat their arguments in this place. Indeed I know of only one considerable argument *against* S5, due essentially to Church in his article on sense and denotation in the Sheffer memorial volume. It runs as follows: Let f be the property of being a proposition entertained by a particular person x at a particular time t, and let there be exactly one proposition that has that property. Further let this proposition be necessary. Then it would seem that we have

$$\Box(\imath\, p) f(p) \qquad (23)$$

but not

$$\Box\Box(\imath\, p) f(p) \qquad (24)$$

For to assert the latter proposition would be to assert that it is *necessary* that the proposition entertained by x at t be a necessary one, which is clearly not the case. This argument, which seems to be valid against attempts to combine S5 with systems in which descriptions are primitive, carries no weight against systems, like our own System V, in which they are contextually defined à la Russell. One has only to write the formulae (23)-(24) in primitive notation, careful attention being given to the matter of scope, to be convinced of the immunity of our system to this particular objection.

The axioms of the calculus S5, together with the rule of necessitation, constitute the entire additions required to transform System IV into System V. It may come as a surprise to some that the formulae

$$(x_\alpha) \Box (\ldots x —) \rightarrow \Box (x) (\ldots x —) \tag{25}$$

supported by most adherents of S5(including Carnap) and also supported by Church who does *not* accept S5, are not amongst our axioms. My objection to (25), which is probably new, is as follows. It depends on the assumption that no proposition concerning the cardinality of the universe (except the one asserting its non-emptiness) is necessary. This assumption has seemed false to some, but to me it seems self-evidently true. I suppose we are here down to the bedrock of conflicting intuitions. In any case, there will be agreement as to the validity of the following deduction, granting the truth of the premise.

Take α as ι in (25); the disproof of (25) for this type seems to leave no particular ground for believing it in other cases. Suppose the universe of individuals is finite; suppose for example it has the five members a, b, c, d, e. Take for $\ldots x —$ in (25) the formula $x = a \vee x = b \vee \ldots \vee x = e$. Then the antecedent of (25) is true. But the conclusion implies that the universe necessarily has at most five members, which is false on our assumption that no such cardinality statement is necessary. Thus if (25) is true, the universe does *not* consist of exactly five members. By the same argument, the universe does not consist of exactly n members, for each finite number n. But then it contains infinitely many members. But this result was established by an *a priori* argument. Here it is *necessary* that the universe contains infinitely many members. But now for the second time our assumption, of the contingency of the size of the universe, is contradicted, q.e.d.

This completes our justification of the axioms and rules of System V, apart from the remark that it seems to be immune to all but two of the arguments, both against intensions and against modalities, that have been offered by Quine. The first argument which we do not claim to have answered is the argument from nominalism. This is valid, if it is valid at all, against extensions as well as intensions. I believe that at this most crucial point we are again confronted with basically conflicting intuitions between which no *reasoning* known at present either to me or to Quine is capable if deciding.

The second criticism which Quine has made of the whole notion of intension is to me far more disturbing, and I confess frankly that I am not happy with it. It runs as follows: Even assuming platonism to be true, the theory of intensions is not an acceptable theory in the same way as the theory of extensions (sets). For we have in the latter case a clear-cut *principium individuationis*, namely: Two sets are identical if and only if they have the same members. For intensions, however, there is no such *principium* to hand and no hope at present, even in principle, of deciding when two coextensive intensions are the same and when they are different. In the face of this difficulty of deciding just where identity of intensions leaves off and diversity begins, it can reasonably be maintained that no one has a clear idea of just what an intension is, and that the would-be theory of intensions has no well-defined subject-matter.

I share Quine's apprehensions on this point. I am pretty certain that intensions exist and that they satisfy (10), (15), (22) and the axioms of the Lewis calculus S5. Much more I do not know, and I shall be haunted by the fear that my constructions are a theory without a subject-matter until Quine's point has been squarely answered in the spirit of the most rigorous mathematics.

Four attempts have been made, to my knowledge, to furnish the precise lines of demarcation between intensions which Quine requires. The first is the one put forward by Carnap in *Meaning and Necessity*. Concerning this I have only two things to say: Firstly, it is in my opinion quite certainly wrong. My reasons for this opinion depend on a rather intricate argument which is given in my article «An Alternative to the Method of Extension and Intension» forthcoming in the Schilpp Carnap volume, and is too long for inclusion here. Secondly, I have a strong feeling that something very much like Carnap's account is correct. His general approach to the problem, in terms of «possible worlds» and state-descriptions, is in my opinion practically certain to yield a correct explication within a few years. The charges needed may even be very minor. But this is a hunch of mine which it may be unwise to play too hard in the absence of a conclusive argument (2).

The other three criteria of identity are due to Church, in the article in the Sheffer volume previously mentioned. They are not directly applicable to our System V, being framed in terms of the sense-and-denotation theory.

On Church's Alternative 0 two expressions have different sense unless they can be proved to have the same sense. Thus (with certain specifiable exceptions which need not be cited here) *no* two different expressions have

(2) The writings of Stig Kanger (*Provability in logic*, Stockholm 1957: The morning star paradox, *Theoria* Vol. 23 (1957) pp. 1-11) deserve in this connection more publicity than they have received. While his solutions of the modal and intensional paradoxes are open to some of the criticisms brought against Systems I-III in the present paper, and against Carnap in my article in the Schilpp Carnap volume, (as well as to certain others of their own) my present feeling is that the final untangling of these puzzles (which surely cannot be far away) and in particular the final establishment of a *principium individuationis* for intensions, will probably use methods even closer to Kanger's than to Carnap's.

the same sense. This does not determine the identity or diversity of those senses which are not senses of expressions, but it does suggest a series of axioms concerning the identity and diversity of senses from which various interesting consequences can be deduced.

Unfortunately *too* much can be deduced — the system Church designed to embody Alternative 0 is formally inconsistent! For if f and g are two distinct sets of propositions, the proposition that every proposition belongs to f is distinct, in virtue of the «principle of maximum distinction» characteristic of Alternative 0, from the proposition that every proposition belongs to g. There is thus established a one-one mapping of sets of propositions into propositions, in violation of Cantor's theorem. There is no difficulty in formalizing this argument as a rigorous derivation from Church's axioms. Thus Church's Alternative 0, in conjunction with the sense-and-denotation doctrine, is definitely refuted.

But the same argument shows its incompatibility with the approach of Systems IV-V. For an entirely classical argument yields.

$$(\exists f_{[o]})(\exists g_{[o]})(f \neq g \;\&\; [(p_o) f(p) = (p) g(p)])$$

as a theorem of those systems.

Church's Alternative 1 differs from his Alternative 0 roughly speaking by identifying the senses of two formulae obtainable from one another by λ-conversion. We have not investigated the pros and cons of this Alternative or its applicability to our own systems. (It would require at least the replacement of the axioms (15) by the axioms (9), but we have seen no objection to this other than the general objection to asserting as an axiom something which we have no reason to believe). On the whole we are sympathetic with Alternative 1, insofar as the arguments of the preceding and following paragraphs convince us that the truth must be somewhere between Alternative 0 and Alternative 2. But we see no positive grounds for drawing the line in this particular place, nor do we see how to formalize Alternative 1 apart from its connection with the sense-and-denotation formalism.

Alternative 2, finally, identifies two propositions if they are strictly equivalent. This would involve adding to System V the axiom

$$\Box\, [p_o \longleftrightarrow q_o] \rightarrow p = q$$

This Alternative seems to us definitely to be rejected on the well-known grounds that a person may believe a proposition without believing another that is strictly equivalent to it. Fitch has privately answered this objection by the suggestion that a sentence in the form «x believes that ***A***» is correctly analyzed as «x believes that ***A*** expresses a true proposition», and that even if ***A*** and ***B*** are strictly equivalent, «***A*** expresses a true proposition» is not for that reason strictly equivalent to «***B*** expresses a true proposition». But I am afraid this rejoinder leaves me unconvinced. For it seems quite apparent that it is possible to believe that ***A*** without believing that ***A*** expresses a true proposition, in particular in case one does not understand the language in which $\boldsymbol{A}$ is written.

We thus despite Fitch's rejoinder feel that Church's Alternative 2 makes the criterion for identity of propositions (and, by a similar argument, of identity in intensions generally)too weak, while his Alternative 0 makes it too strong. We are thus inclined to place it in between, but see no special reason for placing it precisely at Alternative 1. System V therefore requires some further axioms which will elucidate the *principium individuations* of intensions. Until this is done, Quine's most damaging criticism of a « logic of intensions » remains unanswered and the very existence of the subject as a bona fide discipline is in jeopardy. What we claim to have demonstrated in this paper is merely that, *if such a discipline exists at all*, it must number amongst its theorems all those listed by us as axioms of System V.

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