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IT MIGHT HAVE BEEN CLASSICAL LOGIC

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Abstract
In this paper, a propositional logic Q is presented. This logic is more attractive than classical propositional logic P for explicating actual proofs. Moreover, while Q and P assign the same consequence set to consistent premise sets, Q assigns a sensible and non-trivial consequence set to inconsistent premise sets.

1. *Aim of this paper*

When Frege designed classical propositional logic, henceforth P, he had basically two sources to rely on. The first source was a large collection of alleged proofs, some recognized as correct, others containing steps that were deemed mistaken. This provided the raw material for which an explication (in the sense of [9]) had to be provided. The second source was a cluster of philosophical views, most of them deriving from traditions that had started with Aristotle. These views formed the theoretical constraints for the new logic.

This paper invites the reader to imagine that the philosophical views had been absent or different, and to join in a project that leads to a different propositional logic, which will be called Q. Both Q and the journey leading there are interesting. For one thing, Q is at least as good an explication of "correct proof" as P. To be more precise, all *sensible* alleged proofs that are classified as correct by P are classified as correct by Q and vice versa.[1] So it will not come as a surprise that P and Q assign exactly the same consequence set to consistent premise sets.

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[1] What is meant here by "sensible" will be explicated in the sequel of the text.

Nevertheless, Q is very different from P. It is so different that some logicians, before seeing Q at work, would consider the combination of its properties impossible. Many proofs sanctioned correct by P cannot be obtained in terms of Q and precisely these 'proofs' display features that proofs should not display. Moreover, Q is defined in terms of a procedure, which provides a decision method at the propositional level and a positive test for derivability at the predicative level.

A remarkable feature of Q is that it invalidates *Ex Falso Quodlibet*. So although, as said above, everything derivable from a *consistent* premise set by P is derivable from the same premise set by Q, Q is a *paraconsistent* logic, and actually a strictly paraconsistent one.[2] So it does not lead from inconsistency to triviality. To the contrary, it assigns a sensible consequence set to every inconsistent premise set.

That Q assigns the same consequence set as P to consistent premise sets makes it a very unusual paraconsistent logic. And indeed, Q is remarkable. Unlike most paraconsistent logics, Q validates Disjunctive Syllogism: $A \vee B, \neg A \vdash_{\mathsf{Q}} B$. However, unlike the known paraconsistent logics that validate Disjunctive Syllogism, Q also validates Addition: $A \vdash_{\mathsf{Q}} A \vee B$. Any logician will realize that something unexpected is going on here.

Notwithstanding all this, Q is a simple system and its application is easy. A warning is in place here. We have all been made so familiar with the theory behind the proofs of Tarski logics, that we consider most of it as obvious. Mastering the theory behind Q-proofs requires a couple of definitions and conventions. After this, however, all is simple and easy.

I shall not argue that P has to be replaced by Q. Nor shall I try to present an elaborate set of philosophical views in order to underpin Q. All I want to argue is that Q is a fascinating system, which deserves careful attention and further study.

In subsequent sections, I shall spell out Q, provide it with a semantics, and study its central properties. I shall keep the discussion at the propositional level, as the typical difference with classical logic resides there. It is useful to mention, however, that the corresponding predicative logic is called CL^- (pronounce it C-L-minus) in Ghent and that results on it are forthcoming (or meanwhile published), including an 'axiomatization' of CL^- in [21]. Most of that work was done by Peter Verdée and Dagmar Provijn.

Before moving on, let me mention that the present paper had a long history. The idea of a prospective dynamics — see below — was first applied to P in [7]. There it became clear that *Ex Falso Quodlibet* is isolated from the other rules. It took a while before I realized that the logic Q, which is obtained by

[2] A logic L is *paraconsistent* iff $A, \neg A \nvdash_{\mathsf{L}} B$. This means that there is an A such that not all formulas are L-derivable from $\{A, \neg A\}$. A logic L is *strictly paraconsistent* iff there is no A such that all formulas are L-derivable from $\{A, \neg A\}$.

dropping this isolated rule, has impressive properties that largely compensate the lack of some usual properties. The logic Q was first presented in the Third World Congress on Paraconsistency (Toulouse, 2003). Not much later, I wrote a draft paper on Q, containing a semantics and some metatheory. More urgent work, mainly on adaptive logics, prevented me from finishing the paper, until 2010 when the present volume was planned. An advantage of this situation is that other people have worked on the matter, studying the system and extending it, and that the present version is in line with those results. Moreover, many results presented here are new, for example the three-valued semantics and the connected metatheory.

2. *Some Ideas Behind the Logic*

Engaging in constructing an explication for alleged proofs, it would come to mind that proofs are goal-directed sequences: they follow a path that leads from premises to the conclusion. Moreover, proofs are the result of goal-directed search processes. In searching for a proof, one follows a certain procedure, which may be more or less deterministic and may or may not lead to success. Obviously, there is a difference between the successful result of a proof search and a published proof. The latter is cleaned up and polished; unsuccessful search branches are removed and the result is transformed in such a way that it looks neat to the reader, that its line is easy to follow, and that it provides insight into the proven statement. That published proofs are cleaned up, does not mean that the search process that led to them and the resulting 'unclean' proof are uninteresting or unimportant. Proof heuristics is an important aspect of logic — logic teachers who neglect it are justly unpopular.

Trying to derive p from the premise set $\{q \supset \neg(t \vee \neg r),\ t,\ q \vee s, q \vee (t \supset p)\}$, one might reason as follows. Clearly, p can be obtained from $q \vee (t \supset p)$ provided we obtain both $\neg q$ and t; t is a premise, so we are still looking for $\neg q$; this follows from $q \supset \neg(t \vee \neg r)$ provided we obtain $t \vee \neg r$; this formula can be obtained from either t or $\neg r$; and t is a premise. So the desired proof is found.

If one follows a usual proof format, not much can be written down before the proof is found — just the premises. There is a way around this: to push part of the heuristics into the proof. This is realized by means of *prospective expressions*, which have the form $[\Delta]\, A$, and are interpreted as: the formula A can be obtained from the premises by obtaining the members of the finite set Δ from them. The finite set Δ will be called the *condition* of $[\Delta]\, A$, even if $\Delta = \emptyset$ as is explained below; A will be called the *formula* of $[\Delta]\, A$.[3] With this convention, the previous proof search leads to the following result — some explanation follows.

1	$[p]\, p$	Goal	R^{11}
2	$q \vee (t \supset p)$	Prem	
3	$[\neg q]\, t \supset p$	2; $\vee$E	
4	$[\neg q, t]\, p$	3; $\supset$E	R^{6}
5	t	Prem	
6	$[\neg q]\, p$	4, 5; Trans	R^{11}
7	$q \supset \neg(t \vee \neg r)$	Prem	
8	$[t \vee \neg r]\, \neg q$	7; $\supset$E	R^{10}
9	$[t]\, \neg q$	8; C$\vee$E	R^{10}
10	$\neg q$	5, 9; Trans	
11	p	6, 10; Trans	

The prospective expression $[p]\, p$ on line 1 states the truism that p can be obtained from the premises if p can be obtained from them. The function of this expression is to indicate that we are trying to derive the formulas in the condition, viz. p. Next we introduce a premise from which p can be obtained — this is made precise in the next section. On line 3 the premise is analysed: the disjunction is eliminated in such a way that p can be obtained from the formula of the resulting prospective expression. This expression states that $t \supset p$ is obtained if $\neg q$ is obtained. As we are trying to derive p, the *formula* $t \supset p$ is further analysed, which leads to line 4. The formula of this line is p and this is what we were after because p is an (actually the only) element of the condition of line 1. The second member of the condition of 4 is the premise t. It is introduced on line 5 and next is removed from the condition of line 4 by Trans, resulting in 6. At this point a mark is added to line 4. This line became redundant in view of line 6: $\neg q$ alone is sufficient to obtain p. At line 7, a premise is introduced in view of the target $\neg q$. The premise is analysed at line 8. As the condition of 8 cannot be 'obtained' from a premise,[4] the condition is analysed by eliminating the disjunction. There are two ways for doing so; only the first occurs in the present proof: the disjunction in the condition is eliminated by relying on the fact that t is

[3] Prospective expressions are used in order to keep proofs between margins. An alternative is to write only the formula in the second column of the annotated proofs and to write the condition in a fourth column. The idea of prospective proofs originated as an inversion of the conditions that occur in the dynamic proofs of adaptive logics.

[4] What I mean is that it is impossible to introduce a premise and to obtain $[\Delta]\, t \vee \neg r$ from it, for some Δ, in the same way as 4 is obtained from 2 and 9 is obtained from 7. The matter is made precise in the next section.

sufficient to obtain $t \vee \neg r$. The only member of the condition of line 9 occurs at line 5. So we apply Trans and mark lines 8 and 9 as redundant. As the formula of line 10 is the only member of the condition of line 6, Trans is once more applied, whence p is derived unconditionally. At this point, line 1 is marked as redundant, which means that the goal is reached.

As we shall see in Section 4, the resulting sequence of prospective expressions may be turned into a standard proof and there is an algorithm for doing so. However, the sequence may be regarded as a proof itself.

Certain choices were made in constructing 1–11. After line 3, one might have searched for $\neg q$, postponing the application of $\supset$E. Similarly, one might have derived $[\neg r]\, \neg q$ instead of $[t]\, \neg q$ at line 9 (by C$\vee$E). This would have led to a *dead end* as $\neg r$ cannot be obtained from the premises. As such choices concern computational matters rather than conceptual matters, I shall not discuss them in the present paper.

It is useful to consider the proof 1–11 as containing eleven prospective expressions. So where a condition is absent, as at line 2, the formula is taken to be preceded by an *empty condition*, as in $[\emptyset]\, q \vee (t \supset p)$. The prospective expression introduced by the Goal rule is called the *goal expression*. A proof consisting of just an application of the Goal rule will be said to be at *stage* 1, the sequence of lines obtained by extending a proof at stage n with one line, will be said to be stage $n+1$ of the proof.

3. *The Procedure*

An *instruction* is a rule with a deontic restriction attached to it. The restriction is a permission or obligation to apply the rule in view of the stage of the proof. A *procedure* is a set of instructions. An attempted proof for $\Gamma \vdash_{\mathsf{Q}} G$ (see below) will be the result of applying a procedure. The restrictions will depend on the stage of the proof, viz. on the prospective expressions that occur in the proof. Incidentally, while a procedure is described here as a complication of a system of rules, the latter (a formal system of the standard kind) may also be seen as a borderline case of a procedure: a set of rules that come with a universal permission.

That attempted proofs are defined in terms of a procedure is the natural outcome of the fact that proofs are the result of a goal-directed process. Many procedures are equivalent in the sense that they lead to the same consequence set. They may differ in efficiency, elegance, etc. Once a procedure for explicating a set of alleged proofs is described, one may expect that improvements for it are proposed. Certain steps that look sensible at one point may be deemed not sensible as insights in the procedure are gained. In order to make my point in the present paper, I shall present a rather permissive procedure, neglecting matters of efficiency.

Let me first mention the rules without their deontic restrictions. There are three kinds of rules: formula analysing rules, condition analysing rules, and 'structural' rules. The first two kinds will be introduced in a unified form (varying on a theme from [20]). Let $*A$ denote the 'complement' of A, viz. B if A has the form $\neg B$ and $\neg A$ otherwise.[5] To each formula two other formulas are assigned according to the following table:

$\mathfrak{a}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$		$\mathfrak{b}$	$\mathfrak{b}_1$	$\mathfrak{b}_2$
$A \wedge B$	A	B		$\neg(A \wedge B)$	$*A$	$*B$
$A \equiv B$	$A \supset B$	$B \supset A$		$\neg(A \equiv B)$	$\neg(A \supset B)$	$\neg(B \supset A)$
$\neg(A \vee B)$	$*A$	$*B$		$A \vee B$	A	B
$\neg(A \supset B)$	A	$*B$		$A \supset B$	$*A$	B
$\neg\neg A$	A	A				

The formula analysing rules for $\mathfrak{a}$-formulas and $\mathfrak{b}$-formulas are respectively:[6]

$$\frac{[\Delta]\,\mathfrak{a}}{[\Delta]\,\mathfrak{a}_1 \quad [\Delta]\,\mathfrak{a}_2} \qquad \frac{[\Delta]\,\mathfrak{b}}{[\Delta \cup \{*\mathfrak{b}_2\}]\,\mathfrak{b}_1 \quad [\Delta \cup \{*\mathfrak{b}_1\}]\,\mathfrak{b}_2}$$

The condition analysing rules for $\mathfrak{a}$-formulas and $\mathfrak{b}$-formulas are respectively:

$$\frac{[\Delta \cup \{\mathfrak{a}\}]\,A}{[\Delta \cup \{\mathfrak{a}_1, \mathfrak{a}_2\}]\,A} \qquad \frac{[\Delta \cup \{\mathfrak{b}\}]\,A}{[\Delta \cup \{\mathfrak{b}_1\}]\,A \quad [\Delta \cup \{\mathfrak{b}_2\}]\,A}$$

Here are the four further rules, in which Γ refers to the premise set and G to the intended conclusion (the Goal):

Goal To introduce $[G]\,G$.

Prem To introduce A for some $A \in \Gamma$.

Trans $$\frac{[\Delta \cup \{B\}]\,A \quad [\Delta']\,B}{[\Delta \cup \Delta']\,A}$$

EM $$\frac{[\Delta \cup \{B\}]\,A \quad [\Delta' \cup \{\neg B\}]\,A}{[\Delta \cup \Delta']\,A}$$

[5] Note that $**\neg p$ is $\neg p$ but that $**\neg\neg p$ is p, just like $**p$.

[6] The rule to the left actually summarizes two rules: both $[\Delta]\,\mathfrak{a}_1$ and $[\Delta]\,\mathfrak{a}_2$ may be derived from $[\Delta]\,\mathfrak{a}$; similarly for the rule to the right and for the condition analysing rule for $\mathfrak{b}$-formulas.

In order to spell out the deontic restrictions, we need three preparatory steps. That A is a *positive part* of another formula is defined as the intersection of all relations fulfilling the following clauses:[7]

1. $\mathrm{pp}(A, A)$.
2. if $\mathrm{pp}(A, \mathfrak{a}_1)$ or $\mathrm{pp}(A, \mathfrak{a}_2)$, then $\mathrm{pp}(A, \mathfrak{a})$.
3. if $\mathrm{pp}(A, \mathfrak{b}_1)$ or $\mathrm{pp}(A, \mathfrak{b}_2)$, then $\mathrm{pp}(A, \mathfrak{b})$.

The set of *goal-descendants* in a prospective proof is defined as the smallest set Σ for which hold:

1. $[G]\, G \in \Sigma$,
2. $[\Delta]\, G \in \Sigma$ if it is obtained by a condition analysing rule from a $[\Delta']\, G \in \Sigma$,
3. $[\Delta \cup \Theta]\, G \in \Sigma$ if it is obtained by EM from $[\Delta \cup \{A\}]\, G$ and $[\Theta \cup \{\neg A\}]\, G$, and $[\Delta \cup \{A\}]\, G \in \Sigma$ or $[\Theta \cup \{\neg A\}]\, G \in \Sigma$,
4. $[\Delta \cup \Theta]\, G \in \Sigma$ if it is obtained by Trans from $[\Delta \cup \{A\}]\, G \in \Sigma$ and $[\Theta]\, A$.

The members of the conditions of lines that are unmarked at stage s will be said to be the *targets* at stage s. We shall see that, with the exception of the Goal rule, rules are applied at a stage in function of the targets at that stage.[8] I shall consider three marking definitions:

Redundant A line in which $[\Delta]\, A$ is derived is *marked as redundant* from a stage s on iff, at stage s, $[\Delta']\, A$ has been derived for some $\Delta' \subset \Delta$. The mark is "R^i" (with i the stage at which the mark is introduced).

Circular A line in which $[\Delta]\, A$ is derived is *marked as circular* iff $\{A\} \subset \Delta$ and $[\Delta]\, A$ is not the goal expression. The mark is "C".

Dead end An (otherwise unmarked) line that has $[A_1, \ldots, A_n]\, B$ as its prospective expression is *marked as a dead end* from a stage s on iff, at stage s, there is an A_i ($1 \leq i \leq n$) to which no condition analysing rule can be applied and that is not a positive part of any premise. The mark is "$A_i|$" (for that specific A_i).

The sense of these definitions is easy to grasp. A line with prospective expression $[\Delta]\, A$ is redundant if another line of the proof reveals that the

[7] Unlike what is done in [18] and [7], I do not introduce negative parts because there is no need for them (and they complicate the predicative case). Note that, on the present definition, $\mathrm{pp}(p, \neg p \supset q)$ but *not* $\mathrm{pp}(\neg\neg p, \neg p \supset q)$.

[8] The requirement will be expressed indirectly for Trans — see below in the text — by the requirement that the lines are unmarked.

formula A may be obtained by obtaining the members of a proper subset of Δ. Circular lines are heuristically useless because obtaining the members of Δ involves obtaining A; but if A is obtained, there is no need to look for the other members of Δ. A line marked as a dead end is heuristically useless in that, where $[\Delta]\,A$ is its prospective expression, there is no point in trying to derive some members of Δ as other members of Δ cannot be analysed — so they are literals (sentential letters or negations of sentential letters) in the present propositional case — and cannot be obtained from the premises anyway.

Not all marked lines are useless. By applying EM, for example, one obtains a prospective expression the condition of which does not contain all formulas from the conditions of the two lines on which the application relies. So one or more applications of EM that start from marked lines may lead to an unmarked line.

Given the above, the (very permissive) procedure may be summarized as follows.

(i) An attempted proof for $\Gamma \vdash_{\mathsf{Q}} G$ starts off by an application of the Goal rule, introducing $[G]\,G$.
(ii) In an attempted proof, a prospective expression $[\Delta]\,A$ can be introduced at most once as a goal-descendant and at most once as a non-goal-descendant.
(iii) A premise may be introduced (by Prem) at a stage if, at that stage, a target is a positive part of it.
(iv) A member of the condition of a line that is not marked as redundant may be analysed (by a condition analysing rule).
(v) A formula analysing rule resulting in $[\Delta]\,A$ may be applied to a prospective expression that is not a goal-descendant, provided a target is a positive part of A at the stage of the proof.
(vi) Trans may be applied to prospective expressions of unmarked lines.
(vii) EM may be applied to prospective expressions that are not marked as redundant.

The expressed permissions are only valid in as far as the application fulfils all obligations. For example, the permission expressed by (v) is restricted by the ban imposed by (ii).

If restrictions that refer to marked lines are dropped (all members of prospective conditions are then targets), one obtains an even more permissive (and less efficient) procedure. Still, if the premise set Γ is finite, a prospective proof for $\Gamma \vdash_{\mathsf{Q}} G$ will either be successful or stop at a finite point.

Some comments on the procedure are useful. Clause (i) is required to get the proof started. Clause (ii) prevents that the same formula is repeated infinitely. The restriction is harmless in that the use of any occurrence of a prospective expression is the same, except that an occurrence that is not a goal-descendant can be used more liberally than one that is a goal-descendant. Premises are only introduced when they are useful in view of (iii). As for (iv), note that a member of a condition may be analysed even if it is itself a positive part of a premise. To see why this is necessary, consider the target $p \vee q$ in the presence of the premise set $\{r \vee (p \vee q), p \vee s, \neg s\}$. Although $p \vee q$ is a positive part of $r \vee (p \vee q)$, the premises do not allow one to obtain it from that premise, whereas analysing the target gives us p and this can be obtained from the premises. Clause (v) warrants that the formulas of goal-descendants are not analysed. This means that G is not analysed whenever it is the formula of a prospective expression that is a goal-descendant. It can indeed be proved that analysing the formula of a goal-descendant, $[\Delta]G$, is useless and moreover would make Theorem 3 false.[9] The deontic restrictions warrant that the formulas of other prospective expressions, even if they are identical to G, are analysed in a way that may be useful for the ongoing proof. The restriction in clauses (vi) and (vii) provably prevent the addition of useless prospective expressions.

Definition 1: *An* attempted Q-proof *for* $\Gamma \vdash_{\mathsf{Q}} G$ *is a list of prospective expressions written by application of the above procedure.*

Definition 2: *An attempted* Q*-proof for* $\Gamma \vdash_{\mathsf{Q}} G$ *is* successful *iff* G *occurs in it on the empty condition.*

Definition 3: *A* Q-proof *of* G *from* Γ *is an attempted* Q*-proof for* $\Gamma \vdash_{\mathsf{Q}} G$ *that is successful.*

Definition 4: $\Gamma \vdash_{\mathsf{Q}} G$ *iff there is a* Q*-proof of* G *from* Γ.

Definition 5: *An attempted* Q*-proof for* $\Gamma \vdash_{\mathsf{Q}} G$ stops *iff the procedure does not allow one to add a prospective expression to the attempted proof.*

In connection with Definition 5, the reader should keep in mind that the procedure is goal-directed. While $[p]\, q$ occurs in some attempted Q-proofs for $p \supset q \vdash_{\mathsf{Q}} q$ and occurs in all such proofs that stop, it does not occur in any attempted proof for $p \supset q \vdash_{\mathsf{Q}} r$.

Proofs of Q-theorems depend essentially on EM. A simple example is the proof of $(p \wedge q) \supset p$:

1	$[(p \wedge q) \supset p]\, (p \wedge q) \supset p$	Goal	R^5
2	$[\neg(p \wedge q)]\, (p \wedge q) \supset p$	1; C$\supset$E	R^5

[9] Peter Verdée gave me the following example: if the restriction were removed, $[p, \neg p]\, q$ would occur in the prospective proof for $p, \neg p \vdash_{\mathsf{Q}} p \vee q$.

3	$[p]\,(p \wedge q) \supset p$	1; C$\supset$E	$p\vert$ R^5
4	$[\neg p]\,(p \wedge q) \supset p$	2; C$\neg\wedge$E	$\neg p\vert$ R^5
5	$(p \wedge q) \supset p$	3, 4; EM	

Another (still very simple) illustration of EM is provided by the proof of $q \vee p$ from $\{\neg p \supset p\}$:

1	$[q \vee p]\, q \vee p$	Goal	R^7
2	$[q]\, q \vee p$	1; C$\vee$E	$q\vert$ R^7
3	$[p]\, q \vee p$	1; C$\vee$E	R^7
4	$\neg p \supset p$	Prem	
5	$[\neg p]\, p$	4; $\supset$E	
6	$[\neg p]\, q \vee p$	3, 5; Trans	R^7
7	$q \vee p$	3, 6; EM	

I claimed that Q is at least as good an explication for correct proofs as P. This was an understatement: Q is much better than P in this respect. First, it excludes repeated steps, like the recurring introduction of the same premise. More importantly, it prevents that proofs are extended with steps that serve no purpose, like 3 in the following official Fitch-style P-proof of q from $\{p, p \supset q\}$:

1	p	Premise
2	$p \supset q$	Premise
3	$p \vee r$	1; Addition
4	q	1, 2; Modus Ponens

Officially, any finite number of lines similar to 3 may be inserted between 3 and 4. The point is not that such lines are superfluous for the resulting successful proof. The point is that the lines are not sensible steps for deriving q from the premises. Allow me to present another example. The following official annotated proof of $p, p \supset q \vdash_{\mathsf{P}} q$ (in which I use the theorem $(A \wedge B) \supset A$ for the sake of brevity) does not contain any step that is superfluous within this proof. Nevertheless, the proof is not the outcome of a sensible goal-directed process.

1	p	Premise
2	$p \supset q$	Premise
3	$p \vee q$	1; Addition
4	$p \vee r$	1; Addition
5	$(p \vee q) \wedge (p \vee r)$	3, 4; Adjunction
6	$p \vee (q \wedge r)$	5; Distributivity
7	$(q \wedge r) \supset q$	Theorem
8	q	6, 2, 7; Dilemma

Before leaving the matter, I add two comments on the procedure. First, the following rule is permissible.

C$*$E $\qquad \dfrac{[\Delta \cup \{*A\}]\, A}{[\Delta]\, A}$

If A is the goal, the rule is even derivable: $[\Delta]\, G$ follows by EM from $[\Delta \cup \{*G\}]\, G$ and $[G]\, G$. If A is a target, the proof contains a prospective expression $[\Delta' \cup \{A\}]\, B$. Applying C$*$E to $[\Delta \cup \{*A\}]\, A$ enables one to derive $[\Delta' \cup \Delta]\, B$ by Trans. If C$*$E is not applied, one obtains $[\Delta' \cup \Delta \cup \{*A\}]\, B$ by Trans, and next $[\Delta' \cup \Delta]\, B$ by EM.[10]

The second comment is that further marking definitions improve the efficiency of the procedure. Some prospective expressions $[\Delta]\, A$ are target-circular in the sense that A is *only* useful to obtain (in one or more steps) a formula B on a different condition, whereas $B \in \Delta$. In this case $[\Delta]\, A$ may be handled as if it were circular.

4. *Some Properties*

In [7], the system Pc is presented, which defines prospective proofs for P. The only difference with Q is that Pc also contains the EFQ rule:

EFQ $\qquad$ Where $A \in \Gamma$, add $[\neg A]\, G$ to the proof.

The recursive definition of "goal-descendant" in Pc is as in Q, except that one should add in item 1 of the definition: if $A \in \Gamma$, $[\neg A]\, G \in \Sigma$. So Pc is an extension of Q. Several lemmas and theorems proved below follow immediately from those proved in [7] or may be proved by a slight modification to those proofs. I shall nevertheless introduce new information below when this provides more insights.[11]

Theorem 1: *If* $\Gamma \vdash_{\mathsf{Q}} A$*, then* $\Gamma \vdash_{\mathsf{P}} A$*.*

Proof. Consider a Q-proof of A from Γ. Transform the list L of prospective expressions (second elements of the lines of the proof) to a list of formulas

[10] So although $[\Delta]\, A$ cannot be obtained without C$*$E, everything one can do with $[\Delta]\, A$ can be obtained. Moreover, deriving $[\Delta]\, A$ from $[\Delta \cup \{*A\}]\, A$ is correct according to the 'interpretation' of prospective expressions — see the paragraph following Theorem 9 for this interpretation. Incidentally, showing that C$*$E is permissible in case A is not a target is more tiresome and is skipped here.

[11] A simpler proof of Theorem 1 is that every prospective Q-proof is a prospective P-proof in view of what is said in this paragraph in the text.

L' by letting every prospective expression $[B_1, \ldots, B_n]\, A$ in L correspond to the formula $*B_1 \vee \ldots \vee *B_n \vee A$. It is easily seen that L' can be extended to a P-proof of A from Γ. An application of the Goal rule is justified by a proof of the P-theorem $*G \vee G$, an application of Prem is justified by the Premise rule, and the transformations of all other Q-rules are derivable rules in P. □

Some will find it easier to transform a prospective expression $[B_1, \ldots, B_n]$ A to $(B_1 \wedge \ldots \wedge B_n) \supset A$. I do not spell out the derivable P-rules because their number is finite and their derivability in P is obvious.

It is instructive to exemplify the transformation described in the present proof of Theorem 1. So let us apply it to the prospective proof in Section 1, freely making use of continuous disjunctions. The line numbers in the justifications are as in the prospective proof. I list the required derivable P-rule for each step.

1	$\neg p \vee p$	P-Theorem
2	$q \vee (t \supset p)$	Prem
3	$q \vee (t \supset p)$	2; A/A
4	$q \vee \neg t \vee p$	3; $A \vee (A \supset C)/A \vee (\neg A \vee C)$
5	t	Prem
6	$q \vee p$	4, 5; $A \vee \neg B \vee C, B/A \vee C$
7	$q \supset \neg(t \vee \neg r)$	Prem
8	$\neg(t \vee \neg r) \vee \neg q$	7; $A \supset B/B \vee \neg A$
9	$\neg t \vee \neg q$	8; $\neg(A \vee B) \vee C/\neg A \vee C$
10	$\neg q$	5, 9; $\neg A \vee B, A/B$
11	p	6, 10; $A \vee B, \neg B/A$

Having proved that P is an upper limit for Q, Theorem 2 holds if it can be shown that: if Γ is consistent and $\Gamma \vdash_{\mathsf{P}} A$, then $\Gamma \vdash_{\mathsf{Q}} A$.

Theorem 2: *For all consistent* Γ, $Cn_{\mathsf{Q}}(\Gamma) = Cn_{\mathsf{P}}(\Gamma)$.

The proof of this theorem in terms of the procedure is longwinded and complicated. However, there is an easy proof in semantic terms. So I postpone the proof to Section 5, making sure that circularity is avoided.

Theorem 3: *If* $[\Delta]\, B$ *occurs in an attempted* Q*-proof for* $\Gamma \vdash_{\mathsf{Q}} A$*, then* $\Gamma \cup \Delta \vdash_{\mathsf{Q}} B$.

This theorem is proved in [7] as Theorem 1. That proof concerns Pc, but is easily adjusted to Q (by dropping all references to the EFQ rule).

Several further properties are immediate in view of the prospective procedure. A premise that is a literal can only be employed in an application of

Trans. So this gives us at once Lemma 1, which is a weak counterpart of Theorem 3.

Lemma 1: *If A is a literal,* $\Gamma \cup \{A\} \vdash_{\mathsf{Q}} G$*, and* $\Gamma \nvdash_{\mathsf{Q}} G$*, then* $[A]\,G$ *occurs in every stopped attempted* Q*-proof for* $\Gamma \vdash_{\mathsf{Q}} G$.

Other properties we are interested in are more general. Let A^B_σ be the result of replacing all occurrences of the sentential letter σ in A by the formula B and let $\Gamma^B_\sigma = \{A^B_\sigma \mid A \in \Gamma\}$.

Theorem 4: Q *is reflexive (*$\Gamma \subseteq Cn_{\mathsf{Q}}(\Gamma)$*), monotonic (*$Cn_{\mathsf{Q}}(\Gamma) \subseteq Cn_{\mathsf{Q}}(\Gamma \cup \Gamma')$*), compact (*$\Gamma \vdash_{\mathsf{Q}} G$ *iff there is a finite* $\Gamma' \subseteq \Gamma$ *for which* $\Gamma' \vdash_{\mathsf{Q}} G$*), structural (if* $\Gamma \vdash_{\mathsf{Q}} G$*, then* $\Gamma^B_\sigma \vdash_{\mathsf{Q}} G^B_\sigma$*), and decidable (there is an algorithm for deciding whether* $A_1, \ldots, A_n \vdash_{\mathsf{Q}} B$*).*

Incidently, the prospective proofs *are* an algorithm for deciding whether $A_1, \ldots, A_n \vdash_{\mathsf{Q}} B$. Indeed, an attempted Q-proof for every such expression either is successful or stops (because at most finitely prospective expressions can occur in the proof in view of the finite number of literals that occur in the premises and conclusion) — if the proof stops, $A_1, \ldots, A_n \nvdash_{\mathsf{Q}} B$.

Let $\mathrm{CNF}(A)$, the *conjunctive normal form* of A, be the last member of the sequence of formulas defined as follows. The first formula in the sequence is A. Every other formula in the sequence is obtained by applying to the previous formula in the sequence the first transformation from the following list that leads to a *different* formula.

$$\ldots \mathfrak{a} \ldots \quad \mapsto \quad \ldots (\mathfrak{a}_1 \wedge \mathfrak{a}_2) \ldots \tag{1}$$

$$\ldots \mathfrak{b} \ldots \quad \mapsto \quad \ldots (\mathfrak{b}_1 \vee \mathfrak{b}_2) \ldots \tag{2}$$

$$\ldots \vee (A \wedge B) \vee \ldots \quad \mapsto \quad (\ldots \vee A \vee \ldots) \wedge (\ldots \vee B \vee \ldots) \tag{3}$$

$$\ldots (A_1 \vee \ldots \vee A_n) \ldots \quad \mapsto \quad \ldots \bigvee \{A_1, \ldots, A_n\} \ldots \tag{4}$$

$$\ldots (A_1 \wedge \ldots \wedge A_n) \ldots \quad \mapsto \quad \ldots \bigwedge \{A_1, \ldots, A_n\} \ldots \tag{5}$$

A few comments are needed. I freely use continuous conjunctions and continuous disjunctions as well as conjunctions and disjunctions of the members of finite sets. This is easily seen to be unproblematic. Next, the obvious understanding of the clauses is that if $\ldots$ is empty, then $\ldots \vee A$ denotes A and $\ldots \wedge A$ denotes A; similarly for $\vee \ldots$ and $\wedge \ldots$. The requirement that the transformation leads to a *different* formula ensures that the list is finite for every A. Note that applying, for example (1) to $p \vee (q \wedge r)$ leads to the same formula. Finally, the outer dots in (5) always denote empty strings in the definition of $\mathrm{CNF}(A)$. In the definition of $\mathrm{DNF}(A)$ — see below in the text — the outer dots in (4) denote empty strings.

So the list starts with a formula A, is finite, and its last formula is $\mathrm{CNF}(A)$ and has the structure

$$\bigwedge\{\bigvee(\Delta_1), \ldots, \bigvee(\Delta_n)\}$$

in which the members of every Δ_i are literals.

The *disjunctive normal form* of A, $\mathrm{DNF}(A)$, is defined by the same transformations except that (3) is replaced by

$$\ldots \wedge (A \vee B) \wedge \ldots \quad \mapsto \quad (\ldots \wedge A \wedge \ldots) \vee (\ldots \wedge B \wedge \ldots)\,.$$

With this replacement, the last formula in the list, $\mathrm{DNF}(A)$ has the structure

$$\bigvee\{\bigwedge(\Delta_1), \ldots, \bigwedge(\Delta_n)\} \tag{6}$$

in which every Δ_i is a set of literals.

The procedures that lead from A to $\mathrm{CNF}(A)$ and to $\mathrm{DNF}(A)$ 'push negations inside' in view of transformations (1) and (2). No other transformation affects negation. So, if a conjunct of $\mathrm{CNF}(A)$ contains both the sentential letter B and its negation $\neg B$, that conjunct is a P-theorem but it is not removed. Similarly, if a disjunct of $\mathrm{DNF}(A)$ contains both B and $\neg B$, the disjunct is a P-contradiction, but is not removed.

Consider two variants of disjunctive normal forms. Where $\mathrm{DNF}(A) = \bigvee(\Sigma)$, so Σ is a set of $\bigwedge(\Delta_i)$, $\mathrm{DNF}^{+}(A) = \bigvee(\Sigma')$ with Σ' defined as the smallest set such that (i) $\Sigma \subseteq \Sigma'$ and (ii) if $\bigwedge(\Delta \cup \{A\}), \bigwedge(\Delta' \cup \{*A\}) \in \Sigma'$, then $\bigwedge(\Delta \cup \Delta') \in \Sigma'$. Moreover, where $\mathrm{DNF}^{+}(A) = \bigvee(\Sigma')$, $\mathrm{DNF}^{\pm}(A) = \bigvee(\Sigma'')$ with $\Sigma'' = \{\bigwedge(\Delta) \mid \bigwedge(\Delta) \in \Sigma'$; for all $\Delta' \subset \Delta$, $\bigwedge(\Delta') \notin \Sigma'\}$.

Obviously $\vdash_{\mathsf{P}} A \equiv \mathrm{CNF}(A)$, $\vdash_{\mathsf{P}} A \equiv \mathrm{DNF}(A)$, $\vdash_{\mathsf{P}} A \equiv \mathrm{DNF}^{+}(A)$, and $\vdash_{\mathsf{P}} A \equiv \mathrm{DNF}^{\pm}(A)$. So a P-valuation (of the standard semantics) assigns the value 1 to A iff it assigns the value 1 to (at least) one literal of each conjunct of $\mathrm{CNF}(A)$. Similarly, a P-valuation assigns the value 1 to A iff it assigns the value 1 to (at least) one disjunct of $\mathrm{DNF}(A)$, to (at least) one disjunct of $\mathrm{DNF}^{+}(A)$, and to (at least) one disjunct of $\mathrm{DNF}^{\pm}(A)$.

Normal forms provide many insights in prospective proofs. For example, B is a positive part of A iff B is a disjunct of a conjunct of $\mathrm{CNF}(A)$.

Consider an attempted Q-proof for $\Gamma \vdash_{\mathsf{Q}} G$ in which one applies the Goal rule and next applies *condition analysing* rules to $[G]\,G$ and its descendants until it is impossible to do so. This results in a set of prospective expressions. Let $[\Delta_1]\,G$, …, $[\Delta_n]\,G$ be the prospective expressions in which the condition contains only literals. The Δ_i are easily seen to be identical to the

Δ_i for which $\bigwedge(\Delta_i)$ is a disjunct of $\mathrm{DNF}(G)$. So a P-valuation assigns the value 0 to G iff it assigns the value 0 to at least one member of each Δ_i.

Let us apply EM as much as possible to these $[\Delta_i]\,G$. This may lead to new lines with prospective expressions $[\Delta_{n+1}]\,G$, …, $[\Delta_{n+m}]\,G$, in which every Δ_i is still a set of literals. The new Δ_i are those for which $\bigwedge(\Delta_i)$ is a disjunct of $\mathrm{DNF}^+(G)$ but not of $\mathrm{DNF}(G)$. In other words, the Δ_i that contain only literals and are the condition of a prospective expression $[\Delta_i]\,G$ are identical to the Δ_i for which $\bigwedge(\Delta_i)$ is a disjunct of $\mathrm{DNF}^+(G)$. Again, a P-valuation assigns the value 0 to G iff it assigns the value 0 to at least one member of each Δ_i.

Finally, consider the $[\Delta_i]\,G$ that occur at lines not marked as redundant. These Δ_i are those for which $\bigwedge(\Delta_i)$ is a disjunct of $\mathrm{DNF}^\pm(G)$. As $\vdash_{\mathsf{P}} \mathrm{DNF}^\pm(G) \equiv G$, a P-valuation assigns the value 0 to G iff it assigns the value 0 to at least one member of each of these Δ_i.

The following considerations are also clarifying. Let $\mathrm{CNF}(G)$ be a disjunction of literals: $A_1 \vee \ldots \vee A_n$. After applying condition analysing rules to $[G]\,G$, one obviously obtains the prospective expressions $[A_1]\,G, \ldots,$ $[A_n]\,G$. Next, let $\mathrm{CNF}(G)$ be the conjunction of two disjunctions of literals, $(A_1 \vee \ldots \vee A_n) \wedge (B_1 \vee \ldots \vee B_m)$. Applying condition analysing rules to $[G]\,G$ results in $[A_1, B_1]\,G$, …, $[A_1, B_m]\,G$, …, $[A_n, B_m]\,G$.[12] Note that $\bigvee\{\bigwedge\{A_1, B_1\}, \ldots, \bigwedge\{A_1, B_m\}, \ldots, \bigwedge\{A_n, B_m\}\}$ is $\mathrm{DNF}((A_1 \vee \ldots \vee A_n) \wedge (B_1 \vee \ldots \vee B_n))$. Such considerations lead to the simple but long-winded proof of the following theorem.

Theorem 5: $\Gamma \vdash_{\mathsf{Q}} G$ *iff, for every conjunct* $\bigvee(\Delta)$ *of* $\mathrm{CNF}(G)$, $\Gamma \vdash_{\mathsf{Q}} \bigvee(\Delta)$.

Every conjunct of the conjunctive normal form of a P-theorem contains both a sentential letter and its negation as a disjunct. So the previous theorem gives us the following corollary — the corollary follows also from Theorem 2.

Corollary 1: Q *and* P *share all theorems* ($\emptyset \vdash_{\mathsf{Q}} A$ *iff* $\emptyset \vdash_{\mathsf{P}} A$).

Let us turn to premises. Until now we have concentrated on the question whether a given prospective expression can be derived from a premise *within* a specific prospective proof. The answer to that question depends on (i) the targets that occur in the proof (at the stage) and (ii) the specific premises. Let us now consider the more abstract question which prospective expressions can be obtained from a premise, independent of a specific prospective proof.

[12] If an A_i is identical to a B_j, the corresponding condition is obviously a singleton.

(†) A prospective expression $[\Theta]\, B$, in which B and all members of Θ are literals, can be obtained from a premise A iff there is a conjunct $\bigvee(\Delta)$ of $\mathrm{CNF}(A)$ such that $\Delta = \{B\} \cup \{*C \mid C \in \Theta\}$.

This is most easily seen in the diagrammatic setting of [3]. Seeing it is also simple if one considers an arbitrary formula and goes through the formula analysing rules and the condition analysing rules for $\mathfrak{a}$-formulas and for $\mathfrak{b}$-formulas. The following theorem is a consequence of (†).

Theorem 6: *Where* $\{\bigvee(\Delta_1), \bigvee(\Delta_2), \ldots\}$ *comprises the conjuncts of the conjunctive normal form of the members of* Γ, $\Gamma \vdash_{\mathsf{Q}} G$ *iff* $\{\bigvee(\Delta_1), \bigvee(\Delta_2), \ldots\} \vdash_{\mathsf{Q}} G$.

Note that $\bigvee \Delta_i$ is useless with respect to the target C if $C, *C \in \Delta_i$. Indeed, in this case, the resulting prospective expression is always circular. This is immediately obvious from an example: $p \vee \neg p \vee q \vee r$ leads to the circular expressions $[p, \neg q, \neg r]\, p$ and $[\neg p, \neg q, \neg r]\, \neg p$. This does not mean that $p \vee \neg p \vee q \vee r$ is useless with respect to other targets. Indeed, it also leads to $[p, \neg p, \neg r]\, q$ and to $[p, \neg p, \neg q]\, r$ and there is of course nothing wrong with inconsistent conditions.

We know from Theorem 2 that Q coincides with P for consistent premise sets. What about inconsistent premise sets?

Theorem 7: Q *is a paraconsistent logic.*

To see this, it is sufficient to consider a prospective proof for $p, \neg p \vdash_{\mathsf{Q}} q$. The Goal rule leads to $[q]\, q$ and the prospective proof stops right there: q is not a positive part of any premise. Those who have doubts about the way in which Q handles inconsistent premises should write out Q-proofs for $p, \neg p \vdash_{\mathsf{Q}} p \vee q$ and $\neg p, \neg p \supset p, p \supset r \vdash_{\mathsf{Q}} r$ and $p, \neg p, p \supset q, p \vee r, (q \wedge r) \supset s \vdash_{\mathsf{Q}} s$.

Theorem 8: Q *is not a transitive logic.*

This follows directly from the fact that $p, \neg p \vdash_{\mathsf{Q}} p \vee q$ and $p, \neg p \vdash_{\mathsf{Q}} \neg p$ and $p \vee q, \neg p \vdash_{\mathsf{Q}} q$, but $p, \neg p \nvdash_{\mathsf{Q}} q$. The logic Q is not even cumulatively transitive: there are Γ and Γ' such that $\Gamma' \subseteq Cn_{\mathsf{Q}}(\Gamma)$ but $Cn_{\mathsf{Q}}(\Gamma \cup \Gamma') \nsubseteq Cn_{\mathsf{Q}}(\Gamma)$. The obvious example is that $p, \neg p \vdash_{\mathsf{Q}} p \vee q$ and $p, \neg p, p \vee q \vdash_{\mathsf{Q}} q$, but $p, \neg p \nvdash_{\mathsf{Q}} q$. However, Theorem 2 gives us the following corollary.

Corollary 2: Q *is transitive with respect to consistent premise sets (if* Γ *is consistent and* $\Gamma' \subseteq Cn_{\mathsf{Q}}(\Gamma)$*, then* $Cn_{\mathsf{Q}}(\Gamma') \subseteq Cn_{\mathsf{Q}}(\Gamma)$*).*

The logic Q is a very unusual paraconsistent logic. This was already mentioned: unlike most paraconsistent logics, Q validates Disjunctive Syllogism: $A \vee B, \neg A \vdash_{\mathsf{Q}} B$; unlike the known paraconsistent logics that validate Disjunctive Syllogism, Q also validates Addition: $A \vdash_{\mathsf{Q}} A \vee B$. This is precisely possible because Q is not transitive for inconsistent premise sets.

An interesting open problem concerns the relation between Q and P. We know already that: (1) If Γ is consistent, then $\Gamma \vdash_{\mathsf{Q}} G$ iff $\Gamma \vdash_{\mathsf{P}} G$. From this follows, by the monotonicity of Q: (2) If Γ is consistent and $\Gamma \vdash_{\mathsf{P}} G$, then $\Gamma \cup \Gamma' \vdash_{\mathsf{Q}} G$ (even if $\Gamma \cup \Gamma'$ is inconsistent). Moreover, as Q is structural, it follows from (1) that: (3) If Γ is consistent and $\Gamma \vdash_{\mathsf{P}} G$, then $\Gamma^B_\sigma \vdash_{\mathsf{Q}} G^B_\sigma$ (even if Γ^B_σ is inconsistent). That $p, \neg p, p \vee r, \neg p \vee \neg r \vee q \vdash_{\mathsf{Q}} q$ follows by (3) but not by (2). That $\neg p, \neg q, p \vee q \vdash_{\mathsf{Q}} p \wedge q$ follows by (2) but not by (3). The open problem is whether (2) and (3) together are sufficient to characterize all valid Q-inferences in terms of P.

5. *An Adequate Semantics*

I surmise that many semantic systems are adequate for Q. I have a whole set of them, some deterministic and some indeterministic. The trouble is that I failed to prove for any of them that it is adequate with respect to Q. So I have to come up with the next best thing: a semantics that is adequate under a transformation.

If you are not familiar with this, just read on. The matter is simple and reliable. The idea is that every statement $\Gamma \vdash_{\mathsf{Q}} G$ is reduced to finitely many statements $\Gamma_1 \vdash_{\mathsf{Q}} G_1, \ldots, \Gamma_n \vdash_{\mathsf{Q}} G_n$, in which the Γ_i and G_i fulfil a certain formal requirement $\mathcal{C}$. That $\Gamma \vdash_{\mathsf{Q}} G$ obtains just in case $\Gamma_1 \vdash_{\mathsf{Q}} G_1$ and $\ldots$ and $\Gamma_n \vdash_{\mathsf{Q}} G_n$ obtain is warranted by the *syntactic* metatheory. Next, the semantics is proven adequate for every $\Gamma' \vdash_{\mathsf{Q}} G'$, in which Γ' and G' fulfil the formal requirement $\mathcal{C}$.

Let us first turn to the transformation I shall need for Q. We know from Theorem 5 that $\Gamma \vdash_{\mathsf{Q}} G$ iff $\Gamma \vdash_{\mathsf{Q}} \bigvee(\Delta)$ for every conjunct $\bigvee(\Delta)$ of $\mathrm{CNF}(G)$. We moreover know from Theorem 6 that $\Gamma \vdash_{\mathsf{Q}} G$ iff $\Gamma' \vdash_{\mathsf{Q}} G$, in which Γ' comprises every $\bigvee(\Delta)$ that occurs in the conjunctive normal form of a member of Γ. Putting these bits together, we have a syntactic warrant for the following transformation: $\Gamma \vdash_{\mathsf{Q}} G$ iff, for every conjunct $\bigvee(\Delta)$ of $\mathrm{CNF}(G)$, $\Gamma' \vdash_{\mathsf{Q}} \bigvee(\Delta)$, in which Γ' comprises every $\bigvee(\Theta)$ that occurs in the conjunctive normal form of a member of Γ.

Given this transformation, it is sufficient to devise a semantics that is adequate for $\Gamma \vdash_{\mathsf{Q}} G$ whenever G is a disjunction of literals and every member of Γ is a disjunction of literals.

For those who consider this still rather fast, let me phrase it differently. I shall define a semantic consequence relation $\Gamma \vDash_{\mathsf{Q}} G$ for formulas (G and the members of Γ) that are disjunctions of literals. The function $\vDash_{\mathsf{Q}}$ is extended to all formulas by: $\Gamma \vDash_{\mathsf{Q}} G$ iff, for every conjunct $\bigvee(\Delta)$ of $\mathrm{CNF}(G)$, $\Gamma' \vDash_{\mathsf{Q}} \bigvee(\Delta)$, in which Γ' comprises every $\bigvee(\Theta)$ that occurs in the conjunctive normal form of a member of Γ.

Incidentally, all other connectives can be defined from negation and disjunction in P. The same holds for Q and the definitions are the same. This is another remarkable property of Q. Indeed, P has an adequate two-valued semantics, all connectives are truth-functions with respect to that semantics and the set $\{\neg, \vee\}$ is functionally complete — all possible connectives that are truth-functions with respect to the two-valued semantics can be defined from $\{\neg, \vee\}$. Q to the contrary does not have an adequate two-valued semantics in which negation is a truth function. I shall rely on the usual definitions, for example in Table 1, to restrict attention to negation and disjunction.

Although Q is a propositional logic, I shall use *model* to refer to a valuation function v. Where $\mathcal{W}$ denotes the set of formulas of the propositional CL-language. A model is a valuation function $v\colon \mathcal{W} \to \{0, u, 1\}$ with the following properties:

C1 If A is a literal and $v(A) = 0$ then $v(\neg A) = 1$.
C2 If A is a literal and $v(A) = u$ then $v(\neg A) = u$.[13]
C3 $v(\mathfrak{b}) = 1$ iff ($v(\mathfrak{b}_1) = 1$ or $v(\mathfrak{b}_2) = 1$) and ($v(*\mathfrak{b}_1) = 0$ or $v(\mathfrak{b}_2) = 1$) and ($v(\mathfrak{b}_1) = 1$ or $v(*\mathfrak{b}_2) = 0$).
C4 $v(\mathfrak{b}) = 0$ iff $v(\mathfrak{b}_1) = 0$ and $v(\mathfrak{b}_2) = 0$.
C5 $v(\mathfrak{a}) = 1$ iff $v(\mathfrak{a}_1) = 1$, $v(\mathfrak{a}_2) = 1$ and $v(*\mathfrak{a}_1) = v(*\mathfrak{a}_2)$.
C6 $v(\mathfrak{a}) = 0$ iff $v(\mathfrak{a}_1) = 0$ or $v(\mathfrak{a}_2) = 0$.

A model v *verifies* A iff $v(A) = 1$, *falsifies* A iff $v(A) = 0$, and *verifies* Γ iff it verifies every $A \in \Gamma$. $\Gamma \vDash_{\mathsf{Q}} G$ iff no model verifies Γ and falsifies G.[14]

It is instructive to formulate the semantics in terms of tables — see Table 1. The combinations of values that do not occur in the right hand table are excluded by the semantics — compare C1, C2, the right hand table itself. Note that $v(A \vee B)$ is a truth function of $v(A)$, $v(*A)$, $v(B)$ and $v(*B)$, and that $v(\neg(A \vee B))$ is a truth function of $v(*A)$ and $v(*B)$.

[13] An alternative semantics is obtained if this clause is replaced by "If A is a literal, $v(A) \in \{0, 1\}$." This semantics too is adequate for Q. The proofs that follow in the text hardly need to be adjusted. It is left to the reader to adjust Tables 1 and 2.

[14] Where a *model of* Γ is a model that verifies Γ, $\Gamma \vDash_{\mathsf{Q}} G$ iff no model of Γ falsifies G.

Negation is an indeterministic operator — this is indicated by the $1/0$ in the table for negation. Note that $v(\neg\neg A) = v(A)$ holds for all A by virtue of C5 and C6.

where A is a literal

A	$\neg A$
1	$1/0$
u	u
0	1

A	$\neg\neg A$
1	1
u	u
0	0

A	B	$*A$	$*B$	$A \vee B$	$\neg(A \vee B)$
1	1	1	1	1	1
1	1	1	0	1	0
1	1	0	1	1	0
1	1	0	0	1	0
1	u	1	u	u	u
1	u	0	u	1	0
1	0	1	1	u	u
1	0	0	1	1	0
u	1	u	1	u	u
u	1	u	0	1	0
u	u	u	u	u	u
u	0	u	1	u	u
0	1	1	1	u	u
0	1	1	0	1	0
0	u	1	u	u	u
0	0	1	1	0	1

Table 1. Three-Valued Tables for $\neg A$, $\neg\neg A$, $A \vee B$, and $\neg(A \vee B)$

The value u may be read in two different ways. If one reads it as *undefined*, the valuations are partial functions. The value u may also be read as a weak form of truth, viz. one that does not warrant Disjunctive Syllogism for disjunctive formulas — more generally, a form of truth that does not warrant detachment for $\mathfrak{b}$-formulas. If, for example, $v(*A) = 1$, then $v(A \vee B) = 1$ warrants that $v(B) = 1$, whereas $v(A \vee B) = u$ does not even warrant that $v(B) \neq 0$. Note that, if either A or B is true in the strong sense or in the weak sense, then $A \vee B$ is true in one of both senses. A formula of the form $A \wedge B$ is true in the weak sense iff neither A nor B is false, but either at least one of them is true in the weak sense or $v(*A) \neq v(*B)$.

The Q-semantic consequence leads from strong truth to (strong or weak) truth. For example, $p \vDash_{\mathsf{Q}} p \vee q$: if $v(p) = 1$, then $v(p \vee q) \in \{u, 1\}$.

The proof of the following lemmas is immediate in view of C1 and the tables for $A \vee B$ and $\neg(A \vee B)$ in Table 1. If you have doubts about Lemma 3, please realize that there is a trivial model (one that verifies all formulas).

Lemma 2: *For all formulas A, if $v(A) = 0$ then $v(\neg A) = 1$.*

Lemma 3: *Every premise set has a model.*

Of course this semantics is still very tiresome. Even checking whether associativity for disjunction holds requires a long time. In [1] I developed a means to get a grasp on such obstinate semantic systems — notwithstanding the title of that paper, the technique may be used to turn many indeterministic n-ary semantics into a deterministic m-ary tuple semantics; for a couple semantics $m \leq n^2$. In the present case, the three-valued semantics is turned into a four-valued couple semantics. Its valuation function v assigns to every formula A a member of $\{10, uu, 11, 01\}$, the first digit of $\mathsf{v}(A)$ being the value of $v(A)$ and the second digit of $\mathsf{v}(A)$ being the value of $v(\neg A)$.[15] In the four-valued semantics, which is displayed in Table 2, every connective is a truth function. The couple semantics makes the properties of the logic extremely transparent. In view of properties of the three-valued semantics, there is no reason to distinguish between literals and other formulas for negation and there is no need for a separate table handling $\neg\neg A$. I add tables for the explicitly defined connectives. These tables are derivable from the two ones at the top in view of the standard definitions. They are instructive in themselves, but are useless for the subsequent soundness proof and completeness proof — this will soon become clear.

In connection with Disjunctive Syllogism, we have seen the following. If a model verifies $A \vee B$, then it verifies B whenever it verifies $*A$ and it verifies A whenever it verifies $*B$. However, if a model merely does not falsify $A \vee B$, then it may verify $*A$ and nevertheless not verify (and even falsify) B; and it may verify $*B$ and nevertheless not verify (and even falsify) A. The same holds for implication, viz. for Modus Ponens and Modus Tollens. A model that verifies $A \supset B$ verifies B whenever it does not falsify A and falsifies $*A$ whenever it does not verify $*B$. If, however, the model does merely not falsify $A \supset B$, then it may verify A and not verify B; and even falsify B — see the combination "$11 \supset 01$". Similarly for Modus Tollens. This feature is even more striking for equivalence. The only cases in which $A \equiv B$ is verified by a model is where A and B have the same truth value *and* A and B as well as their their negations have values in $\{0, 1\}$.

[15] There are only four values because the couple values ($1u$, $u1$, $u0$, $0u$, and 00) cannot occur.

A	$\neg A$
10	01
uu	uu
11	11
01	10

$\vee$	10	uu	11	01
10	10	10	10	10
uu	10	uu	uu	uu
11	10	uu	11	uu
01	10	uu	uu	01

$\wedge$	10	uu	11	01
10	10	uu	uu	01
uu	uu	uu	uu	01
11	uu	uu	11	01
01	01	01	01	01

$\supset$	10	uu	11	01
10	10	uu	uu	01
uu	10	uu	uu	uu
11	10	uu	11	uu
01	10	10	10	10

$\equiv$	10	uu	11	01
10	10	uu	uu	01
uu	uu	uu	uu	uu
11	uu	uu	11	uu
01	01	uu	uu	10

Table 2. Four-Valued Couple-Semantics

In line with the three-valued semantics, a four-valued model (valuation) v *verifies* A iff $\mathsf{v}(A) \in \{10, 11\}$, verifies Γ iff it verifies every member of Γ, and falsifies A iff $\mathsf{v}(A) = 01$. Again $\Gamma \vDash_{\mathsf{Q}} G$ iff no model verifies Γ and falsifies G.

It is obvious that the four valued couple semantics is equivalent to (defines the same consequence relation as) the three valued one. Every v defines a v that verifies the same formulas as v and falsifies the same formulas as v; and every v is so defined from a v.

Turning the four-valued couple semantics into a simple four-valued semantics (with values 1, 2, 3, and 4) delivers an ingeniously looking semantics, but the couple semantics reveals where it comes from.

Theorem 9: *If* $\Gamma \vdash_{\mathsf{Q}} G$*, then* $\Gamma \vDash_{\mathsf{Q}} G$*. (Soundness)*

In view of the transformation described in the third paragraph of this section, we restrict our attention to cases where G is a disjunction of literals and every member of Γ is a disjunction of literals.

If $[\Delta]\, G$ is a goal-descendant, it is interpreted as: every model that verifies Γ and does not falsify a member of Δ, does not falsify G. So this comes to: $v(A) \neq 1$ for some $A \in \Gamma$ or $v(A) = 0$ for some $A \in \Delta$ or $v(G) \neq 0$.[16] Other prospective expressions $[\Delta]\, A$ are interpreted as: every model that verifies Γ and does not falsify a member of Δ *verifies* A. Note that these other prospective expressions are obtained from one or more premises.

Suppose then that $\Gamma \vdash_{\mathsf{Q}} G$. So a proof from Γ contains the prospective expression $[\emptyset]\, G$ — I write the condition explicitly to make the point clear. We proceed by an obvious induction on the length of the proof. I outline

[16] Still in other words, no valuation verifies Γ, assigns a value in $\{1, u\}$ to all members of Δ and falsifies G — take Lemma 3 into account to interpret this.

the cases. The reader is prayed to check the semantic statements in terms of the three-valued semantics or, which is often easier, in terms of the couple semantics from Table 2.

An application of the Goal rule is justified by: whatever Γ, if a model does not falsify G then it does not falsify G. An application of Prem is justified by: a model that verifies Γ verifies all members of Γ.

Remember that formula analysing rules are never applied to goal-descendants, but only to prospective expressions that are descendants of premises. It follows that applications of the formula analysing rule for $\mathfrak{a}$-formulas are justified by: if every model that verifies Γ and does not falsify a member of Δ verifies $\mathfrak{a}$, then every model that verifies Γ and does not falsify a member of Δ verifies $\mathfrak{a}_1$ (similarly for $\mathfrak{a}_2$). Applications of a formula analysing rule for $\mathfrak{b}$-formulas are justified by: if every model that verifies Γ and does not falsify a member of Δ verifies $\mathfrak{b}$, then every model that verifies Γ and does not falsify a member of Δ and does not falsify $*\mathfrak{b}_2$ verifies $\mathfrak{b}_1$ (and similarly for $\mathfrak{b}_1$ and $\mathfrak{b}_2$ exchanged).[17]

Applications of a condition analysing rule require two subcases, but the justifications can be phrased in one breath. Applications of $\mathfrak{a}$-formulas are justified by: if every model that verifies Γ and does not falsify a member of $\Delta \cup \{\mathfrak{a}\}$ verifies A (respectively does not falsify G), then every model that verifies Γ and does not falsify a member of $\Delta \cup \{\mathfrak{a}_1, \mathfrak{a}_2\}$ verifies A (respectively does not falsify G). Applications of $\mathfrak{b}$-formulas are justified by: if every model that verifies Γ and does not falsify a member of $\Delta \cup \{\mathfrak{b}\}$ verifies A (respectively does not falsify G), then every model that verifies Γ and does not falsify a member of $\Delta \cup \{\mathfrak{b}_1\}$ verifies A (respectively does not falsify G); and similarly for $\mathfrak{b}_2$.

No model falsifies both A and $*A$. So if v falsifies a member of $\Delta \cup \{A\}$ and falsifies a member of $\Theta \cup \{*A\}$, then it falsifies a member of $\Delta \cup \Theta$. From this follows the justification of EM for both goal-descendants and non-goal-descendants.

Applications of Trans are justified by: if every model that verifies Γ and does not falsify a member of $\Delta \cup \{B\}$ verifies A, and every model that verifies Γ and does not falsify a member of Δ' verifies B, then every model that verifies Γ and does not falsify a member of $\Delta \cup \Delta'$ verifies A. If $[\Delta']\, B$ is a goal-descendant(whence B is G) "verifies A" has to be replaced twice by "does not falsify G" in the justification.

Relying on these cases, the result of the induction is that all prospective expressions in the proof are justified on the interpretation. Among the expressions is the goal-descendant $[\emptyset]\, G$ and its interpretation reads: every model

[17] Please check this on the table for disjunction. For example and in terms of the couple semantics: whenever $\mathsf{v}(A \vee B) \in \{10, 11\}$ and $\mathsf{v}(*A) \neq 01$, viz. $\mathsf{v}(A) \neq 10$, then $\mathsf{v}(B) \in \{10, 11\}$.

that verifies Γ does not falsify G. So $\Gamma \vDash_{\mathsf{Q}} G$. This completes the proof of Theorem 9.

The present semantics distinguishes between true in the strong sense (verification) and true in the weak sense (non-falsification). This seems required to handle the non-transitivity of Q: if $p \vee q$ were verified in all models of $\{p, \neg p\}$, then so would be q, and it shouldn't. So, in a semantics that is adequate for the full language, without the detour of the syntactic transformation, the distinction between verification and non-falsification will be retained or even refined. In view of this, it is useful to stress that $[\Delta]\, A$ is interpreted as: every model that *verifies* Γ and *does not falsify* a member of Δ verifies A (respectively does not falsify G in case $[\Delta]\, A$ is a goal-descendant). To require that the model verifies Δ would lead to a mistaken interpretation of the situation. Even the present semantics enables one to see this. Suppose that $p, \neg p, q \in \Gamma$, that $\neg q$ is not Q-derivable from Γ, and that a proof from Γ contains the prospective expression $[p \wedge q]\, r$, which is later analysed to $[p, q]\, r$. Obviously Γ has models v in which $\mathsf{v}(q) = 10$; if it had only models in which $\mathsf{v}(q) = 11$, then $\neg q$ would be a semantic consequence of Γ. So, on the present semantics, $\mathsf{v}(p \wedge q) = uu$ and this is indeed sufficient to justify the derivability of $[\emptyset]\, r$.

The case of disjunction is similar. Consider a prospective proof in which p is a target and let $(q \vee r) \supset p$ be a premise. Prem will be applied and the formula analysing rule for $\mathfrak{b}$-formulas will give us the prospective expression $[q \vee r]\, p$. From this, the condition analysing rule for $\mathfrak{b}$-formulas will lead to $[q]\, p$ and $[r]\, p$. Suppose that q can be obtained from the premises and that r cannot. So the prospective proof will lead one from $[\emptyset]\, q$ and $[q]\, p$ to $[\emptyset]\, p$. Every model v that verifies the premises will verify $(q \vee r) \supset p$ as well as q, but need not verify r and may even falsify it. So $\mathsf{v}(q \vee r) = uu$ for some models of the premises. These comments are somewhat speculative. They pertain to an adequate semantics which does not require the syntactic transformation.

Theorem 10: *If* $\Gamma \vDash_{\mathsf{Q}} G$ *then* $\Gamma \vdash_{\mathsf{Q}} G$. *(Strong Completeness)*

The proof of this theorem is shorter than that of the previous one. Given the syntactic transformation, we restrict our attention to the case in which G is a disjunction of literals, say $\bigvee(\Lambda)$, and Γ is a set of disjunctions of literals.

Suppose that $\Gamma \nvdash_{\mathsf{Q}} G$. Let Σ comprise all literals that occur in Γ and let $L = \langle B_1, B_2, \ldots \rangle$ be a list of the members of Σ.[18] Define

$$\begin{array}{rcl} \Omega_0 & = & \emptyset \\ \Omega_{i+1} & = & \left\{ \begin{array}{ll} \Omega_i \cup \{B_{i+1}\} & \text{iff } \Gamma \cup \Omega_i \cup \{B_{i+1}\} \nvdash_{\mathsf{Q}} G, \\ \Omega_i & \text{otherwise.} \end{array} \right. \\ \Omega & = & \Omega_0 \cup \Omega_1 \cup \ldots \\ \Omega^\dagger & = & \{A \mid A \in \Omega,\ *A \notin \Omega\} \end{array}$$

and let V be a function that maps every literal to a couple value as follows:[19]

$$\begin{array}{rll} \text{if } A \in \Lambda \cup (\Sigma - \Omega), & V(A) = 01 \text{ and } V(*A) = 10 & (1) \\ \text{if } A \in \Omega - \Omega^\dagger, & V(A) = 11 \text{ and } V(*A) = 11 & (2) \\ \text{if } A \in \Omega^\dagger, & V(A) = 10 \text{ and } V(*A) = 01 & (3) \\ \text{if } V(A) \text{ is not fixed by (1)–(3),} & V(A) = V(*A) = 11. & (4) \end{array}$$

I now show that a couple-valuation v (from Table 2) is compatible with V (they assign the same values to literals), that v falsifies G, and that v verifies Γ, whence we are home.

(1) A couple-valuation v is compatible with V. To see this the following are sufficient. (i) If $A \in \Lambda$ then $*A \notin \Lambda$. This follows from $\Gamma \vdash_{\mathsf{Q}} \bigvee(\Delta \cup \{A, *A\})$. (ii) $\Omega \cap \Lambda = \emptyset$. This follows from $\Gamma \cup \{A\} \vdash_{\mathsf{Q}} \bigvee(\Delta \cup \{A\})$. (iii) If $A, *A \in \Sigma$, then $A \in \Omega$ or $*A \in \Omega$. To see this, suppose that it were false. So there is an Ω_i for which $\Gamma \cup \Omega_i \cup \{A\} \vdash_{\mathsf{Q}} G$ and an Ω_j for which $\Gamma \cup \Omega_j \cup \{*A\} \vdash_{\mathsf{Q}} G$. Where k is the maximum of $\{i, j\}$, $\Gamma \cup \Omega_k \cup \{A\} \vdash_{\mathsf{Q}} G$ and $\Gamma \cup \Omega_k \cup \{*A\} \vdash_{\mathsf{Q}} G$ by the monotonicity of Q. In view of Lemma 1, it follows that $\Gamma \cup \Omega_k \vdash_{\mathsf{Q}} G$,[20] which is excluded by the construction (the definition of Ω).

(2) v falsifies G. This follows from the fact that $\mathsf{v}(A) = 01$ for all $A \in \Lambda$.

(3) v verifies Γ. Consider a $\bigvee(\Delta) \in \Gamma$ and suppose that v does not verify it. It follows that $\Delta \cap \Omega^\dagger = \emptyset$ (otherwise $\mathsf{v}(\bigvee(\Delta)) = 10$) and that $\Delta \not\subseteq \Omega - \Omega^\dagger$ (otherwise $\mathsf{v}(\bigvee(\Delta)) = 11$). So $\bigvee(\Delta) = \bigvee(\{A_1, \ldots, A_n\} \cup \{C_1, \ldots, C_m\})$ with $n \geq 0$, $m > 0$, $A_1, \ldots, A_n \in \Omega - \Omega^\dagger$, and $C_1, \ldots, C_m \in \Sigma - \Omega$. We know from the construction that $\Gamma \cup \Omega \nvdash_{\mathsf{Q}} G$. Consider a stopped

[18] For the sake of generality, I do not exclude that Σ and hence L are denumerably infinite.

[19] The value assigned in (4) is arbitrary. The present choice also suits the couple semantics derived from the variant semantics mentioned in footnote 13.

[20] Indeed, if $\Gamma \cup \Omega_k \nvdash_{\mathsf{Q}} G$, then both $[A]\,G$ and $[*A]\,G$ occur in the attempted Q-proof for $\Gamma \cup \Omega_k \vdash_{\mathsf{Q}} G$ by Lemma 1. So $[\emptyset]\,G$ occurs in that proof in view of EM.

attempted Q-proof for $\Gamma \cup \Omega \vdash_{\mathsf{Q}} G$. In view of Lemma 1, $[C_1]\, G$, $\ldots$, $[C_m]\, G$ all occur in the prospective proof. As every C_i is a target, the premise $\bigvee(\{A_1, \ldots, A_n\} \cup \{C_1, \ldots, C_m\})$ was introduced in the proof and, for every C_i, the prospective expression $[*D \mid D \in (\{A_1, \ldots, A_n\} \cup \{C_1, \ldots, C_m\}) - \{C_i\}]\, C_i$ occurs in the proof. Moreover, as $A_1, \ldots, A_n \in \Omega - \Omega^{\dagger}$, $*A_1, \ldots, *A_n \in \Omega - \Omega^{\dagger}$. So $*A_1, \ldots, *A_n$ are introduced by Prem whence, by Trans, $[*D \mid D \in (\{C_1, \ldots, C_m\} - \{C_i\})]\, C_i$ occurs in the proof for every C_i. But then, again by Trans, $[*D \mid D \in (\{C_1, \ldots, C_m\} - \{C_i\})]\, G$ occurs in the proof for every C_i. As these as well as $[C_1]\, G$, $\ldots$, $[C_m]\, G$ occur in the proof, so does $[\emptyset]\, G$ (by EM). But this contradicts $\Gamma \cup \Omega \nvdash_{\mathsf{Q}} G$. This ends the proof of Theorem 10.

I postponed the proof of Theorem 2 to the present section. The most transparent proof seems to proceed in terms of the couple semantics from Table 2. A *classical* Q-model is one in which every literal (and hence every formula) has a value in the set $\{10, 01\}$. Iff v is classical, the corresponding v is a valuation of the standard P-semantics. Note that consistent premise sets have classical (as well as non-classical) Q-models.[21]

Incidentally, Theorem 1 also follows from Theorem 10 and the completeness of P with respect to its standard semantics. Indeed, if no Q-model that verifies Γ falsifies G, then no classical Q-model that verifies Γ falsifies G.[22] So if $\Gamma \vDash_{\mathsf{Q}} G$, then $\Gamma \vDash_{\mathsf{P}} G$.

Let us turn to the proof of Theorem 2. In view of the syntactic transformation, we take G and all members of Γ to be disjunctions of literals. Consider a consistent Γ and a G such that $\Gamma \nvdash_{\mathsf{Q}} G$. So $\Gamma \nvDash_{\mathsf{Q}} G$ by Theorem 10. Hence, there is a Q-model v that verifies Γ and falsifies G. As G and the members of Γ are disjunctions of literals, (i) every literal that is a disjunct of G has the v-value 01 and (ii) for every $A \in \Gamma$, either a disjunct of A has the v-value 10 or all disjuncts of A have the v-value 11.

We first transform v to a Q-model v_1 by replacing every occurrence of u in the v-value of a literal by 1 — this means that uu is replaced by 11. I leave it to the reader to check that v_1 verifies Γ and falsifies G. The following hold: (i) every disjunct of G has the v_1-value 01 and (ii) for every $A \in \Gamma$, either a disjunct of A has the v_1-value 10 or all disjuncts of A have the v_1-value 11.

If a disjunct of a premise has the v_1-value 10, the value of the premise will remain 10 when disjuncts that have the v_1-value 11 are given the value 10 or 01. So let $\bigvee(\Delta_1), \bigvee(\Delta_2), \ldots$ be the premises all disjuncts of which have the

[21] If, for example, v assigns the value 11 to some sentential letters and 10 to all others, then it is a non-classical model of $\{p, q\}$. Note also that the trivial model, in which $\mathsf{v}(A) = 11$ for every sentential letter A, is a model of every premise set.

[22] If Γ is inconsistent, no classical Q-model verifies it; if Γ is consistent, it has classical Q-models and they all verify G.

v_1-value 11 and let $\langle B_1, B_2, \ldots\rangle$ be a list of the members of $\Delta_1 \cup \Delta_2 \cup \ldots$. Define $\Omega_0 = \emptyset$, $\Omega_{i+1} = \Omega_i \cup \{B_{i+1}\}$ if $\Gamma \cup \Omega_i \cup \{B_{i+1}\}$ is P-consistent[23] and $\Omega_{i+1} = \Omega_i$ otherwise; finally $\Omega = \Omega_0 \cup \Omega_1 \cup \ldots$. Obviously Ω contains a member of every Δ_i and if $B, *B \in \Delta_1 \cup \Delta_2 \cup \ldots$, then $B \in \Omega$ or $*B \in \Omega$. Let v_2 be exactly like v_1 except that $\mathsf{v}_2(B) = 10$ for all $B \in \Omega$ and $\mathsf{v}_2(B) = 01$ for all $B \in (\Delta_1 \cup \Delta_2 \cup \ldots) - \Omega$. Note that v_2 is a classical model that verifies Γ and falsifies G. So $\Gamma \nvDash_{\mathsf{P}} G$ and, by the soundness of P with respect to its standard semantics, $\Gamma \nvdash_{\mathsf{P}} G$. From this together with Theorem 1 follows Theorem 2.

6. *Analysing Logic*

This section is a digression about a potentially very interesting problem. Consider again the Q-semantics. The semantic consequence relation was defined as follows: $\Gamma \vDash_{\mathsf{Q}} G$ iff no model that verifies Γ falsifies G. This definition differs from the usual one: every model that verifies the premises also verifies the conclusion. If the usual definition is combined with the models of the Q-semantics, we obviously obtain a logic that is a fragment of Q; it assigns to premise sets a consequence set that is a subset of the Q-consequence set and sometimes the subset if proper. Moreover, the fragment would be transitive and *a fortiori* cautiously transitive: if all models that verify Γ verify Δ and all models that verify Δ verify A, then all models that verify Γ verify A.

Considerations of a syntactic nature suggest that the fragment is called *analysing logic*. An obvious property of Q is: if $\Gamma \vdash_{\mathsf{Q}} A$, then $\Gamma \vdash_{\mathsf{Q}} A \vee B$ for all B. Phrased more generally: for all $\mathfrak{b}$, if $\Gamma \vdash_{\mathsf{Q}} \mathfrak{b}_1$, then $\Gamma \vdash_{\mathsf{Q}} \mathfrak{b}$; and if $\Gamma \vdash_{\mathsf{Q}} \mathfrak{b}_2$, then $\Gamma \vdash_{\mathsf{Q}} \mathfrak{b}$. It is this property that we want to remove in analysing logic. Thus from $\{p \wedge q, r \supset \neg p, r \vee s\}$ analysing logic should enable one to derive s but not to derive $s \vee t$.

Two warnings are in place at this point. First, there is absolutely no problem with Q that should be repaired by analysing logic. However, there seems to be a fragment of Q that is closer to usual logics and it seems interesting to delineate it. Next, members of the Ghent logic group use "analysing logic" to denote several logics. One of the reasons for this, which will be revealed in the present section, is that several logics are close to Q and that it was not settled wether, with respect to Q, one of them deserves the name analysing logic more that others. The candidates differ from each other both with respect to derivability and with respect to Tarski properties. I will briefly

[23] Actually P-consistency coincides with Q-consistency.

sketch some of them, mentioning a few properties only and leaving open whether any of them is superior to others.

Let us first look at the problem in terms of the semantics. On the present Q-semantics, analysing logic does not have any valid formulas. For example $\nvDash p \vee \neg p$ because $\mathsf{v}(p \vee \neg p) = uu$ if $\mathsf{v}(p) = uu$. More generally, *no* formula is verified by the model that assigns uu to every sentential letter.

We have seen that the Q-semantics remains adequate iff sentential letter are given values in $\{01, 11, 10\}$. With this change, analysing logic has valid formulas, for example $\vDash p \vee \neg p$. Note, however, that we still have $\nvDash p \vee \neg p \vee q$: this formula is assigned the value uu if $\mathsf{v}(p) = 11$ and $\mathsf{v}(q) = 01$. There is a weirder feature. So defined, analysing logic is not structural (in the usual sense of the term). Indeed, $(p \wedge q) \vee \neg(p \wedge q)$ follows by Uniform Substitution from $p \vee \neg p$. Yet, if $\mathsf{v}(p) = 11$ and $\mathsf{v}(q) = 01$, then $\mathsf{v}((p \wedge q) \vee \neg(p \wedge q)) = uu$. Note that the formula is not even valid under the syntactic transformation because this reduces the formula to $p \vee \neg p \vee \neg q$ and $q \vee \neg p \vee \neg q$ and both are invalid.[24]

An adequate semantics that does not require a syntactic transformation may very well enable one to define a nice analysing logic, which is structural, transitive, and so on. Such a semantic system will apparently require more couple values, including values like $1u$ and $u0$. Still, there may be several such semantic systems and they may lead to different results. So let me stop speculating and move on to the syntaxis.

The attentive reader will have seen that the absence of transitivity (for inconsistent premise sets) is related to the condition analysing rule for $\mathfrak{b}$-formulas. This rule, applied to the goal, enables one to derive $p \vee q$ from, for example, $\{p\}$. In Fitch-style terms the rule corresponds to Addition, which enables one to introduce the arbitrary formula q as a disjunct of the conclusion. Analysing logic would typically invalidate Addition. And indeed syntactic considerations suggest that there is a systematic fragment of Q that encompasses the analysing part of Q.

Obviously, $p \vee q, \neg p \vdash_{\mathsf{Q}} q$ as well as $p \vee q, \neg p \vdash_{\mathsf{Q}} r \vee q$. In the context of Q, there is a clear distinction between $p \vee q$, which is a premise, and $r \vee q$, which is a conclusion. Indeed, if we add $\neg q$ to the premises, we can derive p but we cannot derive r — formally: $p \vee q, \neg p, \neg q \vdash_{\mathsf{Q}} p$ but $p \vee q, \neg p, \neg q \nvdash_{\mathsf{Q}} r$.

The situation can even be put more sharply by means of the following example. Although $p \vee q, p \vee (q \vee s), \neg p \vdash_{\mathsf{Q}} q \vee s$ and $p \vee q, p \vee (q \vee s), \neg p \vdash_{\mathsf{Q}} q \vee r$, adding $\neg q$ to the premises, results in the derivability of s but not in the derivability of r — formally: $p \vee q, p \vee (q \vee s), \neg p, \neg q \vdash_{\mathsf{Q}} s$ but $p \vee q, p \vee (q \vee s), \neg p, \neg q \nvdash_{\mathsf{Q}} r$.

[24] Whether literals may have the value uu or not, without the syntactic transformation, analysing logic would not even validate Adjunction. Indeed, if $\mathsf{v}(p) = 11$ and $\mathsf{v}(q) = 10$, then $\mathsf{v}(p \wedge q) = uu$. So $p, q \nvDash p \wedge q$.

This reveals, as the reader will have understood before, that disjunctions that occur in the conclusion of a Q-inference may be weaker than disjunctions that occur in the premises. Analysing logic could select those Q-consequences in which disjunctions have the same strength as they have in the premises. Actually the matter is slightly more complicated. So let me explain.

If a disjunction occurs in a premise, even as a subformula of the premise, and *if* the disjunction is a positive part of the premise, then the disjunction is indeed strong and detachable. Consider the premise $(p \vee q) \vee r$. If it is possible to obtain $\neg r$ from the other premises, then $p \vee q$ can be obtained in its strong sense: as a detachable disjunction. The matter is different for a premise like $(p \vee q) \supset r$. The disjunction $p \vee q$ is obviously not a positive part of this — the $\mathfrak{a}$-formula $\neg(p \vee q)$ is. So the question whether the disjunction is strong or weak does not even arise. And the same should obviously hold for Q-consequences of premise sets: if $(p \vee q) \supset r$ is a Q-consequence of Γ, the question wether the disjunction in $p \vee q$ is strong or weak does not arise. Incidentally, this the syntactic transformation eliminates all possible sources of confusion.

Before going on, another point should be made clear. If r is a target in a proof and $(p \vee q) \supset r$ is a premise, the premise will be introduced, $[p \vee q]\, r$ will be derived, and from this $[p]\, r$ and $[q]\, r$ will be derived. It may seem that this relies on Addition: if one obtains p (or q), then one obtains $p \vee q$ and hence also r. Addition not being an analysing step, do those derivations have a place in analysing logic? They do. The formula $(p \vee q) \supset r$ is equivalent to $(p \supset r) \wedge (q \supset r)$. From this $p \supset r$ and $q \supset r$ follow by analysing means. Put differently, $(p \vee q) \supset r$ expresses that p as well as q as well as the detachable disjunction $p \vee q$ are sufficient to obtain r. This is an important insight that matches the semantics. Suppose that $p \vee q$ can be obtained from the premises and hence is detachable. The logic Q presupposes that p and $\neg p$ cannot both be false. If p is not false, we have r anyway. If $\neg p$ is not false, the detachable disjunction gives us q and this in turn gives us r.

So, on the syntactic approach, analysing logic is the logic for which $\mathfrak{b}$-formulas that are positive parts of the conclusion have the same force as disjunctions that are positive parts of the premises. If this is so, analysing logic is transitive (for consistent premise sets as well as for inconsistent ones).

Analysing logic clearly cannot be defined by removing condition analysing rules for $\mathfrak{b}$-formulas from Q. To see this, consider line 9 in the example proof in Section 2. The premise $q \supset \neg(t \vee \neg r)$ is equivalent to $(q \supset \neg t) \wedge (q \supset r)$. Analysing this gives us $q \supset \neg t$. So t should be sufficient to deliver $\neg q$. Similarly, $(p \wedge r) \vee q \vdash p \vee q$ is clearly correct in view of analysing means. The same holds for $p \vee q \vdash p \vee q$. It even holds for $p \wedge q \vdash p \vee q$; the conclusion is a weakening of the premise, but no arbitrary letter is introduced. In the last three examples, it moreover holds that adding $\neg p$ to the premises enables one to derive q and adding $\neg q$ to the premises enables one to derive p. Formally: $(p \wedge r) \vee q, \neg q \vdash_{\mathsf{Q}} p$ and $(p \wedge r) \vee q, \neg p \vdash_{\mathsf{Q}} q$ and $p \vee q, \neg q \vdash_{\mathsf{Q}} p$, etc.

So I was implicitly applying a criterion for defining the consequence relation of analysing logic. Roughly the criterion states that $\Gamma \vdash A \vee B$ iff $\Gamma \vdash_{\mathsf{Q}} A \vee B$ and $\Gamma \cup \{\neg A\} \vdash_{\mathsf{Q}} B$ and $\Gamma \cup \{\neg B\} \vdash_{\mathsf{Q}} A$. The criterion should obviously be applied with some care. Indeed, we want $p, q \nvdash p \vee (q \vee r)$ even though $p, q \vdash_{\mathsf{Q}} p \vee (q \vee r)$ and $p, q, \neg p \vdash_{\mathsf{Q}} q \vee r$ and $p, q, \neg(q \vee r) \vdash_{\mathsf{Q}} p$. This suggests that the criterion should only be applied to conclusions that are, once more, disjunctions of literals and that the criterion is applied for each literal separately. This still does not seem to help much. Indeed, $p, q \vdash_{\mathsf{Q}} p \vee q \vee r$ and $p, q, \neg p \vdash_{\mathsf{Q}} q \vee r$ and $p, q, \neg q \vdash_{\mathsf{Q}} p \vee r$ and $p, q, \neg r \vdash_{\mathsf{Q}} p \vee q$ all hold. However, and this seems to provide the required insight, $p, q, \neg p, \neg q \nvdash_{\mathsf{Q}} r$. Given the implicit criterion for disjunctions of literals, Theorem 5 suggests a way to generalize the logic to arbitrary formulas.

Definition 6: $\Gamma \vdash G$ *is recursively defined by*

1. *where* Δ *is a set of literals* $\Gamma \vdash \bigvee(\Delta)$ *iff, for every* $\Delta' \subset \Delta$, $\Gamma \cup \{*A \mid A \in \Delta'\} \vdash_{\mathsf{Q}} \bigvee(\Delta - \Delta')$,[25]
2. *where* G *is not a disjunction of literals,* $\Gamma \vdash G$ *iff, for every conjunct* $\bigvee(\Delta)$ *of* $\mathrm{CNF}(G)$, $\Gamma \vdash \bigvee(\Delta)$.

This definition gives us $\vdash p \vee \neg p$ and $\nvdash p \vee \neg p \vee q$. Incidentally, the valid formulas of analysing logic (as fixed by Definition 6) are the formulas B such that every conjunct $\bigvee(\Delta)$ of $\mathrm{CNF}(B)$ has the following property: for every A, $A \in \Delta$ iff $*A \in \Delta$ — for example $\nvdash p \supset (p \vee q)$ and $\nvdash (p \wedge q) \supset p$. This approach is in line with one of the sketched semantic approaches. While that is nice in itself, it gives us at once the weird outcome that analysing logic is not structural. Indeed, $\vdash p \vee \neg p$ but $\nvdash (p \wedge q) \vee \neg(p \wedge q)$ because $\nvdash p \vee \neg p \vee \neg q$ and $\nvdash q \vee \neg p \vee \neg q$.

Of course, there is a very different road. The syntactic approaches discussed so far concern a way to obtain analysing logic in terms of the Q-consequence relation. Very different results may be obtained by proceeding in terms of prospective proofs. As the prospective proofs for Q are natural and systematic, we need all the help we can get from the previous approaches.

I shall present two approaches. The first introduces *multi-conditions*. Just as the formula of goal-descendants is handled in a special way in Q — no formula analysing rule can be applied to it — we may handle the condition of some goal-descendants in a special way for analysing logic. The easiest

[25] With the obvious convention that $\bigvee\{A\}$ is A.

way to explain the matter is as follows. After applying the Goal rule to obtain $[G]\, G$, we apply condition analysing rules to this prospective expression and to its descendants, until no such rule can be applied. The condition analysing rule for $\mathfrak{a}$ formulas is as before, but the one for $\mathfrak{b}$ formulas is replaced by the following:

$$\frac{[\Delta \cup \{\mathfrak{b}\}]\, A}{[\Delta \cup \{\mathfrak{b}_1][\Delta \cup \{\mathfrak{b}_2\}]\, A}$$

So we write both conditions on the same line, thus introducing multi-conditions. Eventually we reach a prospective expression

$$[\Delta_1] \ldots [\Delta_n]\, G\,. \tag{7}$$

To other formulas, viz. to premises and their descendants, we apply the usual rules. An alternative and longwinded way to introduce the change is to define certain *occurrences* of members of conditions as deriving from a sequence of condition analysing rules applied to the condition G of the goal expression.

It still should be specified in which way Trans and EM are applied to the separate conditions in expressions of the form (7). For applications of Trans, we combine one of the multi-conditions with another prospective expression. To avoid confusion, here is the form:

$$\frac{\begin{array}{c}[\Delta_1] \ldots [\Delta_i \cup \{B\}] \ldots [\Delta_n]\, G \\ [\Delta']\, B\end{array}}{[\Delta_1] \ldots [\Delta_i \cup \Delta'] \ldots [\Delta_n]\, G}$$

If one of the resulting conditions is empty, it is (written as $[\emptyset]$ or simply) eliminated altogether. EM, to the contrary, is applied to members of the *same* multi-condition. I again write the general form. As the order to the separate conditions does not matter, I put the two relevant conditions in adjacent places.

$$\frac{[\Delta_1] \ldots [\Delta_i \cup \{B\}][\Delta_j \cup \{\neg B\}] \ldots [\Delta_n]\, G}{[\Delta_1] \ldots [\Delta_i \cup \Delta_j] \ldots [\Delta_n]\, G}$$

So EM gives one a means to reduce the number of conditions, even to eliminate two at once in case $\Delta_i = \Delta_j = \emptyset$.

The idea is still (see Definition 2) that the proof is successful iff G occurs on the empty condition (or on a set of empty conditions if that notation is followed). This means that all members of a multi-condition are reduced to $\emptyset$.

It is instructive to add a few simple proofs by way of illustration. Let us start with a proof for $p, q \vee r \vdash r \vee (p \wedge q)$. I leave it to the reader to adjust the marking definitions.

1	$[r \vee (p \wedge q)]\, r \vee (p \wedge q)$	Goal	R^9
2	$[r][p \wedge q]\, r \vee (p \wedge q)$	1; C$\vee$E	R^9
3	$[r][p, q]\, r \vee (p \wedge q)$	2; C$\wedge$E	R^5
4	p	Prem	
5	$[r][q]\, r \vee (p \wedge q)$	3, 4; Trans	R^9
6	$q \vee r$	Prem	
7	$[\neg q]\, r$	6; $\vee$E	
8	$[\neg q][q]\, r \vee (p \wedge q)$	5, 7; Trans	R^9
9	$r \vee (p \wedge q)$	8; EM	

There is obviously no successful proof for $p, q \vdash r \vee (p \wedge q)$. This is illustrated by the following attempted proof, which has stopped.

1	$[r \vee (p \wedge q)]\, r \vee (p \wedge q)$	Goal	
2	$[r][p \wedge q]\, r \vee (p \wedge q)$	1; C$\vee$E	R^7
3	$[r][p, q]\, r \vee (p \wedge q)$	2; C$\wedge$E	R^5
4	p	Prem	
5	$[r][q]\, r \vee (p \wedge q)$	3, 4; Trans	R^7
6	q	Prem	
7	$[r]\, r \vee (p \wedge q)$	5, 6; Trans	

I add a final example proof, viz. for $(s \vee p) \supset q, q \supset r \vdash p \supset r$, to illustrate that condition analysing rules function as for Q with respect to premises and their descendants.

1	$[p \supset r]\, p \supset r$	Goal	R^{10}
2	$[\neg p][r]\, p \supset r$	1; C$\supset$E	R^{10}
3	$q \supset r$	Prem	
4	$[q]\, r$	3; C$\supset$E	
5	$[\neg p][q]\, p \supset r$	2, 4; Trans	R^{10}
6	$(s \vee p) \supset q$	Prem	
7	$[s \vee p]\, q$	6; C$\supset$E	
8	$[p]\, q$	7; C$\vee$E	
9	$[\neg p][p]\, p \supset r$	5, 8; Trans	R^{10}
10	$p \supset r$	9; EM	

These prospective proofs define a logic that is non-transitive. Indeed, $\vdash (p \wedge q) \vee \neg(p \wedge q)$ and $(p \wedge q) \vee \neg(p \wedge q) \vdash p \vee \neg p \vee \neg q$,[26] but $\nvdash p \vee \neg p \vee \neg q$. So let me spell out an alternative.

The idea is to use disjunctions of sets of formulas. As in the previous approach, one clearly separates the analysis of the condition of the goal expression from the analysis of premises. So where multi-conditions were introduced in the previous approach, one now applies condition analysing rules resulting in conditions that are sets of disjunctions of sets of literals. These condition analysing rules are rather straightforward, but one needs two rules for each type of formula. Consider first $\mathfrak{b}$-formulas:

$$\frac{[\Delta \cup \{\mathfrak{b}\}]\, G}{[\Delta \cup \{\bigvee(\{\mathfrak{b}_1, \mathfrak{b}_2\})\}]\, G} \qquad \frac{[\Delta \cup \{\bigvee(\Theta \cup \{\mathfrak{b}\})\}]\, G}{[\Delta \cup \{\bigvee(\Theta \cup \{\mathfrak{b}_1, \mathfrak{b}_2\})\}]\, G}$$

For $\mathfrak{a}$-formulas, the rules are (the left rule is simply the standard rule):

$$\frac{[\Delta \cup \{\mathfrak{a}\}]\, G}{[\Delta \cup \{\mathfrak{a}_1, \mathfrak{a}_2\}]\, G} \qquad \frac{[\Delta \cup \{\bigvee(\Theta \cup \{\mathfrak{a}\})\}]\, G}{[\Delta \cup \{\bigvee(\Theta \cup \{\mathfrak{a}_1\}), \bigvee(\Theta \cup \{\mathfrak{a}_2\})\}]\, G}$$

These rules are applied to the goal expression until the (sole) condition is a set of disjunctions of (possibly singleton) sets of literals.[27] The targets at this stage are these disjunctions[28] together with the literals that occur in these disjunctions.

In view of the targets, premise rules are introduced by Prem. This clarifies at once the last part of the previous paragraph. More often than not, the disjunctions of sets of literals will not be positive parts of any premises, whereas the literals themselves are. If the literals are, we may need the premises in order to derive the goal. On premises and their descendants, all formula analysing rules and condition analysing rules of Q may be applied as well as Trans and EM.

At this point several choices are possible. Keep in mind that a proof, as described so far, consists of two sequences of prospective expressions: on the one hand the application of the Goal rule and the descendants of the goal expression obtained by the new rules, on the other hand the premises and their descendants. We obviously need a way to connect both.

[26] Analyse the goal expression to obtain $[p][\neg p][\neg q]\, p \vee \neg p \vee \neg q$, introduce and analyse the premise to obtain $[p, \neg p]\, \neg q$, obtain $[p][\neg p][p, \neg p]\, p \vee \neg p \vee \neg q$ by Trans and from this obtain $p \vee \neg p \vee \neg q$ by twice EM.

[27] The disjunction of a singleton set of literals is obviously just a literal.

[28] Officially one may consider all formulas that occur in conditions so far as targets, but the effect is the same and the convention followed in the text is heuristically more transparent.

It is possible to devise rules for deriving disjunctions of sets of literals from the premises. However, this leads to many and complicated new instructions and to a rather messy heuristics. The apparently simplest way to handle the matter is to introduce an instruction that enables one to obtain a disjunction of sets of literals from a prospective expression and next to restrict the rule Trans with respect to goal-descendants. The first rule reads:

D $$\frac{[B_1, \ldots, B_n]\, A}{\bigvee(\{A, *B_1, \ldots, *B_n\})}$$

whereas the restricted rule Trans for goal-descendants is called TransG and looks as follows:

TransG $$\frac{\begin{array}{c}[\Delta \cup \{A\}]\, G \\ A\end{array}}{[\Delta]\, G}$$

in which A is a disjunction of a (possibly singleton) set of literals. The rule EM cannot be applied to goal-descendants in the present prospective proofs.

After presenting two example proofs, I shall comment on this logic. Consider first a prospective proof for $(p \wedge q) \supset r, p \vee s, t \supset s, \neg q \supset t \vdash s \vee r$.

1	$[s \vee r]\, s \vee r$	Goal	R[16]
2	$[\bigvee(\{r, s\})]\, s \vee r$	1; C∨E	R[16]
3	$(p \wedge q) \supset r$	Prem	
4	$[p \wedge q]\, r$	3; ⊃E	
5	$[p, q]\, r$	4; C∧E	
6	$p \vee s$	Prem	
7	$[\neg s]\, p$	6; ∨E	
8	$[\neg s, q]\, r$	5, 7; Trans	
9	$\neg q \supset t$	Prem	
10	$[\neg t]\, q$	9; C⊃E	
11	$t \supset s$	Prem	
12	$[\neg s]\, \neg t$	11; C⊃E	
13	$[\neg s]\, q$	10, 12; Trans	
14	$[\neg s]\, r$	8, 13; Trans	
15	$\bigvee(\{r, s\})$	14; D	
16	$s \vee r$	2, 15; TransG	

The second example proof illustrates the absence of theorems in the present system. Consider the attempted proof for $\vdash (p \wedge q) \vee \neg(p \wedge q)$.

1	$[(p \wedge q) \vee \neg(p \wedge q)]\, (p \wedge q) \vee \neg(p \wedge q)$	Goal
2	$[\bigvee(\{p \wedge q, \neg(p \wedge q)\})]\, (p \wedge q) \vee \neg(p \wedge q)$	1; C∨E
3	$[\bigvee(\{p \wedge q, \neg p, \neg q\})]\, (p \wedge q) \vee \neg(p \wedge q)$	2; C∨E
4	$[\bigvee(\{p, \neg p, \neg q\}), \bigvee(\{q, \neg p, \neg q\})]\, (p \wedge q) \vee \neg(p \wedge q)$	3; C∧E

The proof, which also illustrates the analysis of the goal condition, stops right here. Obviously, every attempted proof without premises stops once the goal condition is fully analysed.

The drawback of the present system is that the prospective dynamics itself does not encompass the proof heuristics. This is illustrated by the first example proof. In order to obtain a successful proof, one needs to derive $\bigvee(\{r, s\})$. So one has to keep in mind that one needs to obtain either $[\neg s]\, r$ or $[\neg r]\, s$ from the premises. Nevertheless, in deriving from the premises all prospective expressions in view of the targets, one will find a successful proof if there is one.

The absence of theorems in the present system should not be seen as a drawback. Logic concerns *inference*. Theorems are merely side effects of inference. Moreover, we do not want to have $(p \wedge q) \vee \neg(p \wedge q)$ as a theorem of analysing logic — $p \vee \neg p \vee \neg q$ follows from it and can obviously not be obtained by analysing means. But then we do not want $p \vee \neg p$ as a theorem either, because otherwise the present version of analysing logic is not structural. So the present prospective version of analysing logic has no theorems, but in return seems to be reflexive, transitive, monotonic, structural, compact, and decidable, and moreover is a strict fragment of Q.

The reason for discussing analysing logic at length is that it will be helpful to articulate a semantics that is adequate with respect to the full language of Q and that does not require a syntactic transformation. Especially the last prospective system seems valuable in this respect. It seems to agree with Definition 6, provided one moreover requires that the conclusion is derived *from* the premises.

7. *In Praise of a Logic*

As announced, I do not pretend that classical logic, in the present context P, should be replaced by Q. Nevertheless Q is worth attention. It is a better explication for proofs than P. At least two aspects of Q justify this claim.

The logic P was intended for consistent premise sets. For those, Q does at least as good because it delivers exactly the same consequence sets. However, Q does better than P in that it prevents one from making heuristically useless moves. This is the first aspect.

Note that P may be repaired in this respect, viz. by devising prospective proofs for it. This was done in [7]. So the advantage is one of prospective proofs, rather than of the specific system Q.[29] Nevertheless, Q has the

[29] The prospective dynamics proved useful in other respects as well, for example it provides criteria for final derivability in adaptive logics, which have a consequence relation with a high computational complexity. See [2, 22] for the criteria.

advantage to be a natural system with respect to prospective proofs. In order to devise prospective proofs for P, one needs to *add* EFQ to the rules of Q. While Q consists of sensible and easily justifiable instructions, EFQ is *ad hoc* and unnatural. In the context of Hilbert proofs, which presuppose transitivity, EFQ is an unavoidable outcome of Addition and Disjunctive Syllogism. In the context of prospective proofs, EFQ does not result from any insight in *inference*, but is merely a means to *force* inconsistent premise sets into triviality. This makes it hard to justify EFQ in terms of prospective proofs and precisely these proofs are superior to Hilbert proofs in view of their goal-directed character.

The second aspect is related to inconsistent premise sets. To these P assigns the trivial consequence set. Logicians have looked for a justification of this property. The only purported justification is that inconsistent premise sets are false anyway. This requires some discussion.

Suppose for a moment that all inconsistent premise sets are false, as the classicist claim goes. No argument in favour of P follows from this claim. Indeed, whenever Γ is inconsistent, Q will reveal this and enable one to classify the premise set as false. The whole point is what happens next.

Upon discovering that a premise set is inconsistent, no one actually derives the trivial set from it. The reason is not only that it is impossible to do so in a human lifetime (or during the existence of mankind), but rather that there is no point in doing so. As soon as you derive an inconsistency from a premise set, you know that the set is inconsistent. If you believe that all inconsistent premise sets are false, you have to consider that premise set as false. But why should one embark in deriving the trivial set, or rather claim that the trivial set is derivable? The statement that inconsistent premise sets are false might justify that it is harmless that P assigns the trivial consequence set to inconsistent premise sets. But the statement does not entail that logics *should* assign the trivial consequence set to inconsistent premise sets. That shooting a corpse does not amount to murder, does not entail the obligation to shoot corpses.

There are reasons to doubt the classicist position on inconsistency. That all inconsistent premise sets are false has been questioned by an increasing number of people — see [17, 6] and many other books and papers. However, the classicist needs a further step to argue that it is harmless that P assigns the trivial consequence set to inconsistent premise sets. Indeed, she needs to show that there is no point in reasoning from inconsistent premise sets. This cannot be shown because it is false.[30] Even if all inconsistencies are false,

[30] There is, for example, overwhelming evidence that inconsistencies occurred in the history of the sciences and that scientists reasoned from them. Examples from mathematics are well-known: Cantor's set theory, Frege's set theory, Newton's infinitesimal calculus, ...; for some examples from the empirical sciences see [8, 10, 11, 13, 14, 15, 16, 19].

our views on the world are often inconsistent. We have to reason from them, were it only to arrive at consistent views. So even if all inconsistencies are false, we still need paraconsistent logics.

I have pointed out that Q is an unusual paraconsistent logic. The reader will have surmised that this is related to Q's being non-transitive (for inconsistent premise sets). And indeed, most paraconsistent logics are Tarski logics. In at least one respect Q does better than all of those.

If a premise set is intended to be consistent but turns out inconsistent and we want to replace it by a consistent one, we need to 'interpret' the premise set "as consistently as possible". This is typically the aim of inconsistency-adaptive logics.[31] In a sense, Q fulfils the same task. It provides a basis for reasoning towards a consistent replacement of inconsistent theories. This does not mean that Q makes inconsistency-adaptive logics superfluous. Handling inconsistency (with the aim to eliminate it) is a methodological matter; different approaches, and hence logics, are sensible and more or less suited in specific situations. More often than not, the approaches provide different bases for reasoning towards a consistent replacement. Which of them is the best cannot be settled beforehand. Sometimes several consistent alternatives differ drastically from each other, but are equally consistent and equally close to the inconsistent original. The presumably consistent replacements of Frege's set theory are an obvious example.

Q provides only one of these approaches and not necessarily the most attractive one. For example, although it does not turn inconsistency into triviality, it spreads inconsistencies. Here are some examples:

$$\begin{gathered}
\mathfrak{a}, *\mathfrak{a} \vdash_{\mathsf{Q}} \mathfrak{a}_1 \wedge *\mathfrak{a}_1 \text{ and } \mathfrak{a}, *\mathfrak{a} \vdash_{\mathsf{Q}} \mathfrak{a}_2 \wedge *\mathfrak{a}_2 \\
\mathfrak{b}, *\mathfrak{b} \vdash_{\mathsf{Q}} \mathfrak{b}_1 \wedge *\mathfrak{b}_1 \text{ and } \mathfrak{b}, *\mathfrak{b} \vdash_{\mathsf{Q}} \mathfrak{b}_2 \wedge *\mathfrak{b}_2 \\
A, \neg A, B \vdash_{\mathsf{Q}} (A \wedge B) \wedge \neg(A \wedge B) \\
A, \neg A \vdash_{\mathsf{Q}} ((A \wedge B) \wedge \neg(A \wedge B)) \vee ((A \wedge \neg B) \wedge \neg(A \wedge \neg B)) \\
(A_1 \wedge \neg A_1) \vee \ldots \vee (A_n \wedge \neg A_n) \vdash_{\mathsf{Q}} (A \wedge \neg A) \wedge \ldots \wedge (A_n \wedge \neg A_n)\,.
\end{gathered}$$

This is not a disaster, but it is not attractive either. The more inconsistencies are spread, the more difficult it is to find and eliminate their source.

Incidentally, Q solves a problem that was posed before in the literature. In [12], Joke Meheus argues that it is unreasonable to expect a scientist to apply, for example, disjunctive syllogism in some cases and not in others. If that is correct, Q seems a good logic to explicate the way in which scientists handle

[31] See [4] and many other papers. A provisional and incomplete version of the upcoming survey book [5] is available on the web.

inconsistency. Moreover, Q avoids certain technicalities that are required by the logic AN, which is proposed in [12].

So the consequence sets Q assigns to inconsistent premise sets are sensible in several respects. Moreover, those sets are defined by simple and natural means: the very instructions that define the consequence sets of consistent premise sets — precisely this is the point of the previous paragraph. In view of this, the absence of transitivity (and even of cumulative transitivity) in Q is not a reason for worry. P and Q agree on consistent premise sets and Q does better for inconsistent premise sets. We have seen that assigning the trivial consequence set to inconsistent premise sets is pointless and not justifiable — at best excusable. The absence of transitivity is a price to pay, but the return is (i) that inconsistent premise sets are assigned a sensible and useful consequence set and (ii) that the set of instructions is systematic and natural.

Some have objected that Q concerns computational matters rather than conceptual ones. This is mistaken. The computational aspects of Q are related to the efficiency of the procedure and have hardly been given any attention in the present paper. The prospective proofs, however, were devised to get a grasp on the goal-directed aspects of logic and this is a *conceptual* matter. This is the more obvious as Q is the direct and natural outcome of the prospective dynamics. I have no proof that Q cannot be defined differently, but doing so will apparently require meta-theoretical technicalities; I see no way to characterize Q by means of a set of *rules* — see Section 3 — that concern the handling of formulas of the object language.

So the circle seems to be closed. The prospective dynamics was originally intended as a neutral means to push part of the proof heuristics into the proof (for any logic). The prospective dynamics was first applied to P, viz. in [7]. There it turned out that EFQ is unnatural and *ad hoc*. It took some time before it was realized that the result of dropping EFQ, viz. Q, is actually superior to P as an explication for actual proofs. This suggests that we question the philosophical views on inference that originated with Aristotle and helped define 'classical logic' at the end of the nineteenth century.

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