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ALCHOURRÓN AND BULYGIN ON DEONTIC LOGIC AND THE LOGIC OF NORM- PROPOSITIONS: AXIOMATIZATION AND REPRESENTABILITY RESULTS

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Abstract

The paper deals with the problem of *axiomatizing* three formal systems of Deontic Logic / Normative Logic in a sense derived from Alchourrón & Bulygin (1973) and Alchourrón (1969). First of all we consider a version DL of their Deontic Logic (= Logic of Norms), secondly a version NL of their Normative Logic (= Logic of Norm-Propositions), to which they add a number of definitions generating certain normative operators "proper". Thirdly, we deal with a third system NOBL constructed with a view to isolating the normative fragment of NL which arises as a result of adding those definitions to NL. The main outcome of this paper is then a *representability* result to the effect that the set of sentences provable in NOBL is exactly the set of the sentences which are provable in NL on the basis of those definitions. The distinction Logic of Norms vs. Logic of Norm-Propositions is explained and discussed rather informally both in the introduction and in the concluding section of the present paper.

1. *Introduction: The distinction between the Logic of Norms and the Logic of Norm-Propositions*

In their joint paper Alchourrón & Bulygin (1973) — section 7, entitled '*Norms and Norm-propositions*' — the authors observe (p. 679) that deontic sentences exhibit in ordinary language a characteristic ambiguity: sometimes they are used *prescriptively* as norm-formulations, and sometimes they are used *descriptively* to make normative statements. Moreover, our authors observe that in the first case deontic sentences express *norms* and that in the second case they express *norm-propositions* (or normative propositions).

As to the basic distinction *norms* vs. *norm-propositions* Alchourrón & Bulygin (1973, p. 666) understand by a norm a prescription to the effect that something *ought to* or *may* or *must not* be done, i.e. a prescription enjoining, permitting or prohibiting certain actions or states of affairs; whereas they take norm-propositions (p. 680) to convey information about the deontic status of certain actions or states of affairs, like that of being *obligatory* or *permitted* or *prohibited* by a certain legislative authority x, who promulgates (issues) norms to various effects, say that such-and-such an action or state of affairs *ought to* or *may* or *must not* be done or realized. We note that a norm-proposition in this sense does not have to 'literally' *prescribe* anything whatsoever, it may merely *describe* or *record* the empirical fact that a certain norm has actually been promulgated (issued) by the legislative authority in question.

On the basis of their fundamental norm vs. norm-proposition distinction Alchourrón & Bulygin (1973, p. 680) are then in a position to argue, successfully in my opinion, that a number of important consequential distinctions have to be heeded and to be seriously explored:

(I) *Prescriptive vs. descriptive deontic* sentences *and* operators

Our authors stipulate that the usual symbols O and P shall stand for prescriptive deontic operators, to be read respectively as 'it ought to be that' and 'it may be that'. Furthermore, they introduce three new symbols $OBLx$, $PRMx$ and $prmx$ for descriptive deontic operators, to be read (descriptively) as 'it is obligatory according to x that', 'it is strongly permitted by x that', and 'it is weakly permitted by x that'. [Here, I allow myself to use the notation employed below in this paper of mine.] A slightly improved terminological point with respect to the present distinction: in the sequel we shall throughout speak of O and P (and the defined operator Ph, 'it must not be that') just as our deontic (= prescriptive) operators, and we shall speak of the three new operators (together with the defined operator $PROHBx$, 'it is prohibited by x that') as our normative (= descriptive) operators. This point goes back to the earlier paper Alchourrón (1969).

(II) *Prescriptive vs. descriptive* interpretation *(*sense*,* use*) of deontic sentences*

As was already suggested above, when interpreted prescriptively, deontic sentences express norms, when interpreted descriptively, they express norm-propositions.

At this juncture, Alchourrón & Bulygin (1973, p. 680) make the important observation that the logical properties of norms and those of norm-propositions are in fact different, whence they recommend — as a cautious policy — that the logical properties of norms and norm-propositions be *investigated separately* (in order to find out whether and under what circumstances they are 'isomorphic', as has been claimed by quite a few deontic logicians and legal philosophers). This observation leads our authors to emphasize strongly the following distinction:

(III) *Deontic Logic (in the sense of the Logic of Norms) vs. Normative Logic (in the sense of the Logic of Norm-Propositions)*

Deontic Logic in this sense is then taken to be that of the prescriptive operators O and P, while Normative Logic (in the present sense) will be that of the descriptive operators $OBLx$, $PRMx$ and $prmx$. A very fruitful insight of our authors, again going back to Alchourrón (1969), is now to the effect that Normative Logic can be construed as an *extension* of Deontic Logic — a manoeuvre which must be regarded as an ingenious innovation indeed.

My purpose in this paper is to consider the formal systems suggested by Alchourrón & Bulygin (1973) and Alchourrón (1969) for Deontic Logic and Normative Logic (in the special sense just mentioned) from a standpoint a bit different from their own, viz. that of so-called *possible worlds semantics*. As appears from the title of this paper, we shall be especially interested in obtaining *axiomatization* and *representability* results for certain systems which are closely related (though not identical) to those suggested by our authors, but are based on the same fundamental ideas. Our standpoint leads us to emphasize the well known distinction between (i) the syntax, (ii) the semantics or model theory, and (iii) the proof theory or axiomatics, for any logical system to be considered.

The plan of this paper is then as follows.

First of all, in Section 2 *infra* we present a version DL of the Alchourrón-Bulygin Deontic Logic (= their Logic of Norms). In point of expressive power, our version will be seen to be slightly richer than the version originally due to our colleagues.

In Section 3 we then present a version NL of the Alchourrón-Bulygin Normative Logic (= Logic of Norm-Propositions). We must observe, however, that NL is not yet the "descriptive" logic of norm-propositions in the strict sense of being a logic for the three new descriptive operators $OBLx$, $PRMx$ and $prmx$, because NL lacks, in its *primitive* (undefined) logical notation, those three new operators. Instead, Alchourrón and Bulygin construe their analogue of NL as the result of adding to the primitive vocabulary of their "prescriptive" analogue of our DL a new operator N (or indexed family $\{Nx\}$ with x denoting some appropriate *agent*) which they take to represent the dyadic relation of promulgation. Nevertheless, NL proves to be a highly interesting logic inasmuch as the three normative operators $OBLx$, $PRMx$ and $prmx$ turn out to be *definable* in terms of the new operator N plus the old deontic operators "proper", viz. O and P; this is shown in Section 4 *infra*.

In Section 5 we eventually arrive at an extension NOBL of DL, which differs from NL in having the normative operators in its *primitive* (undefined) logical notation, unlike the promulgation operator N, which is not forthcoming in the basic vocabulary of NOBL at all. The manoeuvre of thus working with three logical systems, viz. DL, NL and NOBL, instead of just two as in Alchourrón & Bulygin (1973), enables us to formulate rigorously the problem which occupies us in Sections 6 and 7: *exactly which* NOBL-sentences are provable in NL on the basis of the definitions given in Section 4 above? Our solution to this problem is given in Section 7 in the form of a Translation or Representability Theorem [THEOREM IV], which asserts that the set of sentences provable in NOBL is precisely the set of those sentences which are provable in NL on the basis of the definitions under consideration. This representability result is in effect the main outcome of the present paper. As to the proof of THEOREM IV we shall rely on our soundness & completeness results for the three logics DL, NL and NOBL given in Theorems I–III below (Sections 2, 3 and 5).

Finally, in the concluding Section 8 we discuss some issues of an independent and general interest.

Remarks

(1) Our logics will deviate from those developed by Alchourrón and Bulygin in the following respects: (i) As pointed out above, we work with three systems (DL, NL, NOBL) instead of just two (their analogue of DL, their analogue of NL plus definitions in the style of our Section 4). (ii) The alethic modal operators $\Box$ and $\Diamond$ (for so-called *universal* necessity and possibility, respectively) are primitive in all three of our logics; see also Section 8.3 below). (iii) *Iteration* of prescriptive deontic operators is allowed in all three of our logics; for discussion, see again Section 8.3. (iv) In the system NL we add a primitive operator Mx interpreted as the dual of the doxastic operator Nx (“according to x”). (v) Unlike our colleagues we consistently add an agent-index x on all our normative operators under consideration (both in definitions added to the logic NL as well as on the primitive operators in NOBL). (vi) In Alchourrón & Bulygin (1973, pp. 682 ff.) and Alchourrón (1969, pp. 251 ff.) our authors draw attention to a crucial ambiguity in the notion of Negation in Normative Discourse, which derives from the interesting paper von Wright (1959): they point out the need for a careful distinction between *strong* or *internal* negation and *weak* or *external* (“ordinary” propositional, sentential) negation. Our treatment of this highly useful distinction is to be found in Sections 4 and 5.1 below and will be seen to deviate from that of our colleagues in a certain ‘tricky’ technical respect.

(2) An interesting feature of the Alchourrón-Bulygin method of definitionally introducing the normative operators $OBLx$, $PRMx$ and $prmx$ is this:

by the definitions D3, D1 and D2 given in Section 4 *infra*, it turns out that the *agent-index* x in such locutions as

$OBLxA$ ["it is obligatory according to the agent x that A"]

$PRMxA$ ["it is strongly permitted by the agent x that A"]

$prmxA$ ["it is weakly permitted by the agent x that A"]

can be separated from the underlying prescriptive deontic operators O and P insofar as these locutions amount to, respectively:

$NxOA$ (by D3)

$NxPA$ (by D1)

$MxPA$ (by D2)

the readings of which are given in Section 4, *Remark* (1), below. A definite advantage of this method is, on our view, that the Alchourrón-Bulygin style of definitions automatically leads to satisfaction of the following condition of adequacy that has been advocated by Frändberg (1987, p. 86) with regard to the interpretation of the deontic operators O and P: they should have *the same meaning* whether they occur in the scope of the promulgation operator $Nx/Mx/$ or not [whether, for instance, they appear in a "genuine" legal statement or in the corresponding "non-genuine" legal statement].[1]

(3) We regret being unable in this paper to deal with the authors' very interesting and intriguing discussion of such important notions as *isomorphism*

[1] Frändberg (1987, p. 85 f.) explains the distinction *genuine* vs. *non-genuine* legal statements in a way that makes it virtually coincident with the Alchourrón-Bulygin one *prescriptive* [deontic] vs. *descriptive* [normative] sentences, e.g. legal statements. He then goes on to assert (*ibid.*) that a genuine legal statement is "lacking in true value" and "is neither true nor false", whereas a non-genuine legal statement "as such has true value, that is to say, it is either true or false". However, it is difficult to see how this assertion can be reconciled with Frändberg's adequacy condition to the effect that deontic operators like O and P are to have *the same meaning whether* they appear in a genuine legal statement *or* in the corresponding non-genuine legal statement. Given that we accept the Frändberg adequacy condition as a plausible one (as I think we should), a similar criticism may be directed against various adherents of the view that genuine legal statements are neither true nor false while non-genuine ones are true or false, e.g. Hedenius (1941, ch. II), Wedberg (1951, pp. 252 f., 261), von Wright (1963, p. 104 f.) and von Wright (1989, p. 872). At any rate, it is clear that the Alchourrón-Bulygin method of introducing normative operators by means of definitions in terms of their promulgation operator N (or Nx) provides an elegant escape from the present difficulty.
Let me also mention that my present approach to the *genuine* vs. *non-genuine* distinction was hinted at in the contribution Åqvist (1973), where it was discussed in a less formal way than here.

between the logic of norms and the logic of norm-propositions and the related notions of *consistency* and *completeness* of normative systems. This discussion is heartily recommended, but has to be left to the reader for the time being.

2. *A version* DL *of the Alchourrón-Bulygin Deontic Logic (= Logic of Norms)*

2.1. *Syntax of* DL

The *vocabulary* (morphology, alphabet, language) of the system DL is a structure made up of the following disjoint basic syntactic categories:

(i) An at most denumerable set Prop of *proposition letters*.
(ii) The *propositional constants* T (*verum*) and $\perp$ (*falsum*).
(iii) The *Boolean sentential connectives* $\sim$, $\&$, $\vee$, $\supset$, $\equiv$ with their usual readings.
(iv) A pair of *monadic* (i.e. one-place) *deontic operators*: O and P, where O is read as "it ought to be [it must be, it shall be] the case that" and P as "it may (deontically) be that".
(v) A pair of one-place *alethic modal operators*: $\Box$ (for universal necessity) and $\Diamond$ (for universal possibility).

The set Sent_{DL} of well formed *sentences* of DL is then recursively defined in the straightforward manner, i.e. in such a way that all proposition letters as well as T and $\perp$ will be (atomic) DL-sentences; moreover, Sent_{DL} will be closed under every connective in the categories (iii)–(v) *supra*.

2.2. *Semantics for* DL*: frames, models and truth conditions*

DL-*frames*
By a DL-*frame* we mean any ordered pair

$$F = (W, R)$$

where:

(i) $W(\neq \varnothing)$ is a non-empty set of *possible situations* ("worlds", "points").
(ii) $R(\subseteq W \times W)$ is a binary relation on W, heuristically, of *deontic accessibility* or *ideality*, satisfying the following condition:

Ser : For each w in W there is at least one w' in W such that wRw'. [Read the locution wRw' as "w' is *ideal* relative to w" or as "w' is *deontically accessible from* w".]

DL-*models*
Let $F = (W, R)$ be any DL-frame. By a *valuation on* such a frame we mean any binary function V which associates a truth-value 1 ('truth') or 0 ('falsity') with each ordered pair (p, w), where $p \in$ Prop is a proposition letter and $w \in W$ is a member of W. In technical jargon, V: Prop $\times$ $W \rightarrow \{1, 0\}$.

By a DL-*model* we then mean an ordered triple,

$$M = (W, R, V)$$

where the initial pair (W, R) is a DL-frame and V is a valuation on that frame.

Truth conditions, DL-*validity*, DL-*satisfiability and semantic* DL-*entailment*
Next, we proceed to define recursively what it means for any DL-sentence A to be *true at* a world (situation, "point") $w \in W$ *in* a DL-model $M = (W, R, V)$ [in symbols: $M, w \vDash A$].

(1) $M, w \vDash p$ iff (=if and only if) $V(p, w) = 1$, for any p in Prop.
(2) $M, w \vDash$ T.
(3) not: $M, w \vDash \bot$.

If A is a Boolean compound, the recursive definition goes on as usual. We then handle sentences having the characteristic one-place *deontic* and *alethic modal* operators as their main connective as follows:

(4) $M, w \vDash OA$ iff $M, w' \vDash A$ for each w' in W such that wRw'.
(5) $M, w \vDash \Box A$ iff $M, w' \vDash A$ for each w' in W.

The truth conditions for sentences having the 'possibility' operators P and $\Diamond$ as their main connective are obtained in the dual way: just replace "each" by "some" to the right of the "iff" in the last two conditions!

As usual, then, we say that a DL-sentence A is DL-*valid* (formally: $\vDash_{\mathsf{DL}} A$) iff $M, w \vDash A$ for all DL-models M and all w in W. And we say that a set Γ of DL-sentences is DL-*satisfiable* iff there exists a DL-model M together with a world w in W such that for all sentences A in Γ: $M, w \vDash A$. We also say that a set Γ of DL-sentences *semantically* DL-*entails* a DL-sentence A

(in symbols: $\Gamma \vDash_{\mathsf{DL}} A$) iff $\Gamma \cup \{\sim A\}$ is not DL-satisfiable. One then easily obtains that A is DL-valid iff the empty set (of DL-sentences) semantically DL-entails A.

2.3. *On the proof theory of* DL

The axiomatic system DL is determined by one rule of inference (deduction), one rule of proof, and three groups of axiom schemata. They are as follows:

Rule of inference
mp (modus ponens)

$$\frac{A, A \supset B}{B}$$

Rule of proof
Nec (necessitation for $\Box$)

$$\frac{\vdash A}{\vdash \Box A}$$

Axiom schemata

A0. All truth-functional tautologies over $\mathrm{Sent}_{\mathsf{DL}}$

A1. S5-schemata for $\Box$, $\Diamond$ [i.e. $\Diamond A \equiv \sim\Box\sim A$, $\Box(A \supset B) \supset (\Box A \supset \Box B)$, $\Box A \supset A$, $\Box A \supset \Box\Box A$, $\Diamond\Box A \supset A$]

A2.
(a) $\Box A \supset OA$[2]
(b) $O(A \supset B) \supset (OA \supset OB)$
(c) $PA \equiv \sim O \sim A$
(d) $OA \supset PA$

As usual, the above axiom schemata and rules determine syntactic notions of DL-*provability* and DL-*deducibility* as follows. We say that a sentence A is DL-*provable* [in symbols: $\vdash_{\mathsf{DL}} A$, or just $\vdash A$] iff A belongs to the smallest subset of $\mathrm{Sent}_{\mathsf{DL}}$ which (i) contains every instance of A0, A1, A2(a)–(d) as its member, and which (ii) is closed under the rule of inference mp and the rule of proof Nec. And we say that the sentence A is DL-*deducible from* the set Γ ($\subseteq \mathrm{Sent}_{\mathsf{DL}}$) of *assumptions* [in symbols: $\Gamma \vdash_{\mathsf{DL}} A$] iff there are sentences $B_1, \ldots, B_k$ in Γ, for some natural number $k \geq 0$, such that

[2] See Section 8.3 *infra*.

$\vdash_{\mathsf{DL}}$ (B_1 & ... & B_k) $\rightarrow A$ (i.e. the sentence (B_1 & ... & B_k) $\rightarrow A$ is to be DL-provable in the sense of the preceding definition).

Again, letting $\Gamma \subseteq \mathrm{Sent}_{\mathsf{DL}}$, we say that Γ is DL-*inconsistent* iff $\Gamma \vdash_{\mathsf{DL}} \perp$, and DL-*consistent* otherwise. Finally, we say that Γ is *maximal* DL-*consistent* iff Γ is DL-consistent and, for each A in $\mathrm{Sent}_{\mathsf{DL}}$, either $A \in \Gamma$ or $\sim A \in \Gamma$; where this latter condition is known as requiring Γ to be *negation-complete*.

THEOREM I (Soundness & Completeness of the axiomatic system DL).

Weak version: For each DL-sentence A, A is DL-provable iff A is DL-valid.

Strong version: For each set Γ of DL-sentences, Γ is DL-satisfiable iff Γ is DL-consistent.

Proof. The methodology to be used in the proof of the present result is familiar — see e.g. Åqvist (1987), Chs III and IV.

3. *A version* NL *of the Alchourrón-Bulygin NormativeLogic (= Logic of Norm-Propositions)*

As was pointed out in the Introduction (Section 1 above), Alchourrón & Bulygin (1973) construe their "descriptive" Logic of Norm-Propositions as an *extension* of their "prescriptive" Deontic Logic — an ingenious innovation indeed. Viewed from our standpoint taken here, this means that our version NL of the Logic of Norm-Propositions will be treated as an extension of our version DL of the Logic of Norms (or Deontic Logic). Let us start then by extending the syntax of DL in the following way.

3.1. *Syntax of* NL

The *vocabulary* (morphology, alphabet, language) of the system NL will be a structure made up of the following disjoint basic syntactic items: The categories (i)–(v) in the basic vocabulary of DL, supplemented with these items:

(vi) An at most denumerable set AgPar of *agent parameters*, for which we adopt the meta-linguistic notation: x, y, z, ...

(vii) A pair of indexed families $\{Nx\}x \in$ AgPar and $\{Mx\}x \in$ AgPar of one-place *doxastic operators*, where Nx is read as "it follows from what x *accepts* ('rules', 'promulgates') that", and where Mx is read as "it does *not* follow from what x *accepts* (rules, promulgates) that it is *not* the case that", or, more smoothly, "it is compatible with what x *accepts* (rules, promulgates) that". A more general reading of Nx could just be "*according to* what x accepts (etc.) it is the case that:"

We then define the set $\text{Sent}_{\mathsf{NL}}$ of well formed sentences of NL by saying that all members of categories (i)–(ii) in Section 2.1 above are (atomic) sentences in $\text{Sent}_{\mathsf{NL}}$ which, moreover, is closed under every connective in the categories (iii)–(v) and (vii). As far as (vii) is concerned, we stipulate that if x belongs to AgPar and A is in $\text{Sent}_{\mathsf{NL}}$, then NxA and MxA are both in $\text{Sent}_{\mathsf{NL}}$ (this being the only non-standard clause in our recursive definition of $\text{Sent}_{\mathsf{NL}}$).

Remarks

(1) As an obvious corollary of the present definition of $\text{Sent}_{\mathsf{NL}}$ we obtain the result that all sentences of DL are in $\text{Sent}_{\mathsf{NL}}$ [$\text{Sent}_{\mathsf{DL}} \subseteq \text{Sent}_{\mathsf{NL}}$].

(2) In the spirit of Alchourrón (1969) and Alchourrón & Bulygin (1973) NL-sentences of the form $NxOA/NxPA/$ can often be read more fully as

"x has *ruled*, i.e. *issued a norm to the effect, that* it ought to /may/ be the case that A".

Note also in the present context that our authors usually think of the agent denoted by x as an "authority", e.g. as a "legislator" or "law-giver", who promulgates (issues) norms to various effects. For this reason the operators Nx and Mx could often, and more accurately, be characterized as *praxeological* or *praxeological-doxastic* operators rather than simply as *doxastic* ones — as we have done in (vii) above. For convenience, however, we shall continue to speak of them just as *doxastic* operators.

3.2. *Semantics for* NL*: frames, models and truth conditions*

NL-*frames*

By a NL-*frame* we mean an ordered quadruple

$$F = (W, R, U, S)$$

where the initial items W and R constitute a DL-frame (Section 2.2 *supra*) and where:

(iii) $U (\neq \varnothing)$ is a non-empty set of *agents* (*potential users* of our NL-language).

(iv) S is a function defined on U such that for each u in U: $S_u (\subseteq W \times W)$ is a binary *doxastic accessibility* (alternativeness) relation on W.

Remarks

(1) We observe that in clause (iv) the function S determines an indexed family
$$\{Su\}u \in U$$
of binary doxastic accessibility relations on W.

(2) Note here that we don't impose any condition like Ser (Section 2.2 *supra*) on the relations $S_u \subseteq W \times W$.

NL-*models*

Let $F = (W, R, U, S)$ be any NL-frame. By a *valuation on* such a frame we now mean an ordered pair (V, v) where

(a) V is a binary function V: Prop × W → $\{1, 0\}$ just as in the semantics for DL (Section 2.2 above); and

(b) v is a unary function which to each agent parameter x in AgPar assigns a member $v(x)$ of U [in symbols, v: AgPar → U].

Then, by a NL-*model* we mean any ordered sextuple

$$M = (W, R, U, S, V, v)$$

where the initial quadruple (W, R, U, S) is a NL-frame and the concluding pair (V, v) is a valuation on that frame.

Truth conditions, NL-*validity*, NL-*satisfiability* and *semantic* NL-*entailment*

We now proceed to define recursively what it means for any NL-sentence A to be *true at* a world (situation, point) w in W *in* a NL-model $M = (W, R, U, S, V, v)$ [formally: $M, w \vDash A$]. The clauses (1)–(3) for atomic NL-sentences [$p \in$ Prop, T, $\perp$] go over unchanged from those given above in the case of DL-sentences; similarly for Boolean compounds and NL-sentences having deontic or alethic modal operators as their main connective (O, P, $\Box$, $\Diamond$). So the only new clauses in our truth definition will concern NL-sentences having the characteristic one-place *doxastic* operators Nx and Mx as their main connective. They are as follows:

(6) $M, w \vDash NxA$ iff $M, w' \vDash A$ for each w' in W such that $wS_{v(x)}w'$; where x is an agent parameter in AgPar.

(6′) $M, w \vDash MxA$ iff $M, w' \vDash A$ for some w' in W such that $wS_{v(x)}w'$; where x is in AgPar.

Given the above truth conditions, we define the notions of NL-*validity*, NL-*satisfiability* and *semantic* NL-*entailment* in perfect analogy with the matching DL-notions (Section 2.2 *supra*): just replace any reference to DL with a reference to NL!

3.3. *On the proof theory of* NL

The axiomatic system NL is determined by one rule of inference (deduction), one rule of proof, and four groups of axiom schemata, as follows:

Rule of inference
mp (modus ponens)

$$\frac{A, A \supset B}{B}$$

Rule of proof
Nec (necessitation for $\Box$)

$$\frac{\vdash A}{\vdash \Box A}$$

Axiom schemata

A0. All truth-functional tautologies over $\mathrm{Sent}_{\mathsf{NL}}$

A1. S5-schemata for $\Box$, $\Diamond$ [i.e. $\Diamond A \equiv \sim\Box\sim A$, $\Box(A \supset B) \supset (\Box A \supset \Box B)$, $\Box A \supset A$, $\Box A \supset \Box\Box A$, $\Diamond\Box A \supset A$]

A2.
(a) $\Box A \supset OA$
(b) $O(A \supset B) \supset (OA \supset OB)$
(c) $PA \equiv \sim O \sim A$
(d) $OA \supset PA$

A3.
(a) $\Box A \supset NxA$
(b) $Nx(A \supset B) \supset (NxA \supset NxB)$
(c) $MxA \equiv \sim Nx \sim A$

The above rules and axiom schemata determine syntactic notions of NL-*provability* and NL-*deducibility* in the straightforward way — see Section 2.3 *supra*. Similarly for the concepts of NL-*inconsistency*, NL-*consistency* and *maximal* NL-*consistency* (as applied to $\Gamma \subseteq \text{Sent}_{\mathsf{NL}}$).

Crucial caveat
Note carefully that there is *no* analogue in A3 of A2(d)! Thus, we *don't* have

$$\vdash_{\mathsf{NL}} NxA \supset MxA, \text{ or}$$

$$\vdash_{\mathsf{NL}} \sim(NxA \& \sim MxA), \text{ or}$$

$$\vdash_{\mathsf{NL}} \sim(NxA \& Nx\sim A).$$

This is due to the fact that *no* condition like Ser (Section 2.2 above) is imposed on the relations $S_u \subseteq W \times W$ in our semantics for NL.

THEOREM II (Soundness & Completeness of the axiomatic system NL).
Weak version: For each NL-sentence A, A is NL-provable iff A is NL-valid.
Strong version: For each set Γ of NL-sentences, Γ is NL-satisfiable iff Γ is NL-consistent.
Proof. The methodology to be used in the proof of the present result is familiar — see e.g. Åqvist (1987), Chs III and IV.

4. *Definability in* NL *of normative operators and their strongly negated variants*

Let us now add to our version NL of the Alchourrón-Bulygin Logic of Norm-Propositions the following twelve definitions:

D1. $PRMxA =_{\text{df}} NxPA$
D2. $prmxA =_{\text{df}} MxPA$
D3. $OBLxA =_{\text{df}} NxOA$
D4. $oblxA =_{\text{df}} MxOA$

D1′. $\neg PRMxA =_{\text{df}} Nx\sim PA$
D2′. $\neg prmxA =_{\text{df}} Mx\sim PA$
D3′. $\neg OBLxA =_{\text{df}} Nx\sim OA$
D4′. $\neg oblxA =_{\text{df}} Mx\sim OA$

D1″. $\neg\neg PRMxA =_{\text{df}} Nx\sim\sim PA$
D2″. $\neg\neg prmxA =_{\text{df}} Mx\sim\sim PA$
D3″. $\neg\neg OBLxA =_{\text{df}} Nx\sim\sim OA$
D4″. $\neg\neg oblxA =_{\text{df}} Mx\sim\sim OA$

Remarks

(1) As to the readings of the *definienda* in D1–D4, we propose — still in the spirit of Alchourrón (1969) and Alchourrón & Bulygin (1973) — that NL-sentences of the form $PRMxA/OBLxA/$ be read as

> “x *promulgates a norm to the effect that* it may deontically /ought to/ be the case that A”;

and that NL-sentences of the form $prmxA/oblxA/$ be read as

> “*it is compatible with everything promulgated by* x *that* it may deontically /ought to/ be the case that A”.

And we follow our authors in labelling $PRMx$ and $OBLx$ our normative operators of *Strong* Permission and *Strong* Obligation, respectively, whereas we label $prmx$ and $oblx$ our normative operators of *Weak* Permission and *Weak* Obligation, as the case may be.

(2) As to the readings of the *definienda* in D1′–D4′, they are like those just proposed in Remark (1), except that we stick in an ordinary (propositional or sentential) negation “it is not the case that” right *after* the italicized expressions shown above, but *before* the remaining deontic locution — as appears from the *definientia* of D1′–D4′. The *definiendum* of D1′ /D3′/ can then be more conveniently read as

> ”x *promulgates a norm to the effect that* deontically it may not /ought not to/ be the case that A”,

and similarly for the remaining cases.

Again, we follow our authors in using the sign $\neg$ for intuitionistic negation for what they call *strong* or *internal* negation. Obviously, the meaning of $\neg$, as intended here by Alchourrón and Bulygin, is different from the intuitionistic one; we shall have to return below to the matter of explicating it more fully.

(3) The *definienda* of D1″–D4″ differ from those in D1′–D4′ in using a form of *double* strong negation, where the latter use just a form of *single* strong negation. In the sequel we refer to these eight *definienda* as the “strongly negated variants” of the normative operators introduced by D1–D4.

5. *The system* NOBL *with* primitive *normative operators and their strongly negated variants*

5.1. *Syntax of* NOBL

The *vocabulary* of the system NOBL is a structure made up of the following disjoint basic syntactic items: The categories (i)–(v) in the basic vocabulary of DL, supplemented with the following items:

(vi) The set AgPar of *agent parameters* x, y, z, ... as in the basic language of NL.

(viii) The one-place sentence-forming normative operators $PRMx$, $prmx$, $OBLx$, $oblx$ together with their strongly negated variants $\neg PRMx$, $\neg prmx$, $\neg OBLx$, $\neg oblx$ and $\neg\neg PRMx$, $\neg\neg prmx$, $\neg\neg OBLx$, $\neg\neg oblx$, as presented in the last section.

With respect to the operators in category (viii), note that while they were all *defined* in NL by D1–D4, D1′–D4′ and D1″–D4″, they are now, in NOBL, taken as *primitive*, i.e. undefined, logical operators. Furthermore, since these one-place operators are all indexed by AgPar, this means that our present category (viii) will really consist of altogether twelve indexed families of one-place normative operators as follows:

(viii:a) $\{PRMx\}$, $\{prmx\}$, $\{OBLx\}$ and $\{oblx\}$,
with $x \in$ AgPar in each case.
(viii:b) $\{\neg PRMx\}$, $\{\neg prmx\}$, $\{\neg OBLx\}$ and $\{\neg oblx\}$,
with $x \in$ AgPar in each case.
(viii:c) $\{\neg\neg PRMx\}$, $\{\neg\neg prmx\}$, $\{\neg\neg OBLx\}$ and $\{\neg\neg oblx\}$,
with $x \in$ AgPar in each case.

The set $\text{Sent}_{\text{NOBL}}$ of well formed sentences of NOBL is then defined in such a way that all members of categories (i)–(ii) in Section 2.1 above are (atomic) sentences in $\text{Sent}_{\text{NOBL}}$ which, moreover, is closed under every connective in the categories (iii)–(v) and (viii). As far as (viii) is concerned, we stipulate that if x is in AgPar and A is in $\text{Sent}_{\text{NOBL}}$, then $\S xA$ is in $\text{Sent}_{\text{NOBL}}$ as well, where $\S x$ is any member of the twelve indexed families which belong to the category (viii) as subdivided above into (viii:a), (viii:b) and (viii:c).

We then obtain the obvious corollary to the effect that all sentences of DL are in $\text{Sent}_{\text{NOBL}}$ [$\text{Sent}_{\text{DL}} \subseteq \text{Sent}_{\text{NOBL}}$], whereas the corresponding result does not hold for Sent_{NL}, since the operators Nx and Mx are not in the basic vocabulary of NOBL.

Remark

As far as the *strongly negated variants* of our normative operators $PRMx$, $prmx$, $OBLx$ and $oblx$ are concerned, i.e. the indexed families in the subcategories (viii:b) and (viii:c), it is highly important to realize that the operator $\neg$ of strong (internal) negation is not, unlike the operator $\sim$ of ordinary (sentential, external) negation, a connective which forms a new *sentence* when applied to a given *sentence* as its argument; $\neg$ is rather, as we think of it, a connective which forms a new *operator* (connective) when applied to a given *operator* (connective) as its argument. This is immediate from the way we characterized the role of $\S x$ in the above definition of $\text{Sent}_{\mathsf{NOBL}}$, the case of category (viii).

The distinction just emphasized with respect to the syntactic status of $\neg$ as compared to that of $\sim$ can be illustrated as follows in the context of the definitions D1′–D4′ and D1″–D4″ given in Section 4 *supra*. Using square brackets to highlight the role of $\neg$ in this context and ordinary parantheses in the case of $\sim$, we can write e.g. D1′ and D1″ more explicitly in this way:

D1′. $[\neg[PRMx]]A =_{\text{df}} Nx(\sim(PA))$

D1″. $[\neg[\neg[PRMx]]]A =_{\text{df}} Nx(\sim(\sim(PA)))$

and similarly for the remaining definitions at issue. But once we have grasped the import of the present distinction, we adopt the convention of dropping all brackets/parantheses, being confident that no confusion will arise.

Interestingly, we may observe that von Wright (1959), Alchourrón (1969) and Alchourrón & Bulygin (1973) all apparently think of $\neg$ as a *sentential* (propositional, "external") operator which is syntactically on a par with the familiar negation operator $\sim$.[3] My reasons for deviating from this view are mainly of a technical nature and need not concern us here.

5.2. *Semantics for* NOBL*: frames, models and truth conditions*

As far as the semantics for NOBL is concerned, this is quite easy and straightforward. NOBL-*frames* are identical to NL-*frames*, and NOBL-*models* are identical to NL-*models* (see Section 3.2 *supra*).

[3] We ought to note here that Alchourrón & Bulygin (1973, Section 8) contains a most interesting discussion of Negation in Normative Discourse, where they refer to von Wright (1963, pp. 135–141) rather than to von Wright (1959). We should note as well that von Wright (1959) draws attention to some highly intriguing distinctions made by Aristotle that are of crucial importance as far as the present difference between internal and external negation is concerned.

But the *truth conditions* for NOBL-sentences will differ from those we adopted for NL-sentences, simply because these two sets of sentences are different.

So we are to tell what it means for any NOBL-sentence A to be *true at* a world (situation, 'point') w in a NOBL(=NL)-model

$$M = (W, R, U, S, V, v),$$

in symbols: $M, w \vDash A$.

The clauses (1)–(3) for atomic NOBL-sentences ($p \in$ Prop, T, $\bot$) go over unchanged from those given above in the cases of atomic DL- and NL-sentences; similarly for Boolean compounds and NOBL-sentences having deontic or alethic modal operators as their main connective (O, P, $\Box$, $\Diamond$). The only new clauses in our present truth definition will then concern NOBL-sentences having the characteristic one-place normative operators $PRMx$, $prmx$, $OBLx$, $oblx$ as well as $\neg PRMx$, $\neg prmx$, $\neg OBLx$, $\neg oblx$ (and with double strong negations $\neg\neg$ in the place of $\neg$ here) as their main connective. The first four new clauses are then as follows, where the variables w, w' and w'' all range over W:

(7.1) $M, w \vDash OBLxA$ iff for each w' with $wS_{v(x)}w'$ it holds that for each w'' with $w'Rw''$: $M, w'' \vDash A$.
(7.2) $M, w \vDash prmxA$ iff there is a w' with $wS_{v(x)}w'$ such that there is a w'' with $w'Rw''$ and $M, w'' \vDash A$.
(7.3) $M, w \vDash PRMxA$ iff for each w' with $wS_{v(x)}w'$ there is a w'' such that $w'Rw''$ and $M, w'' \vDash A$.
(7.4) $M, w \vDash oblxA$ iff there is a w' with $wS_{v(x)}w'$ such that for each w'' with $w'Rw''$: $M, w'' \vDash A$.

Remark

We observe that the truth conditions for $OBLxA$ and $prmxA$ ((7.1), (7.2)) can readily be expressed in terms of the *relative product* $(S_{v(x)}/R)$, where $w(S_{v(x)}/R)w''$ means that

there is a w' such that $wS_{v(x)}w'$ and $w'Rw''$

so that (7.1) and (7.2) can be written more simply as

(7.1*) $M, w \vDash OBLxA$ iff for each w'' with $w(S_{v(x)}/R)w''$: $M, w'' \vDash A$.
(7.2*) $M, w \vDash prmxA$ iff for some w'' with $w(S_{v(x)}/R)w''$: $M, w'' \vDash A$.

The task of showing the equivalence of these "starred" formulations to the original ones is an elementary exercise left to the reader. Note also that the corresponding simplified formulations are not available in the cases of (7.3) and (7.4).

Proceeding next to the four new clauses pertaining to NOBL-sentences having as their main connective any of the above normative operators prefixed by a *single* strong negation-sign $\neg$, we have the following:

(8.1) $M, w \vDash \neg OBLxA$ iff $M, w \vDash PRMx{\sim}A$ iff
for each w' with $wS_{v(x)}w'$ there is a w''
such that $w'Rw''$ and not: $M, w'' \vDash A$.
(8.2) $M, w \vDash \neg prmxA$ iff $M, w \vDash oblx{\sim}A$ iff there is a w' with $wS_{v(x)}w'$
such that for each w'' with $w'Rw''$ it holds that not: $M, w'' \vDash A$.
(8.3) $M, w \vDash \neg PRMxA$ iff $M, w \vDash OBLx{\sim}A$ iff
for each w' with $wS_{v(x)}w'$ it holds that
for each w'' with $w'Rw''$ we have not: $M, w'' \vDash A$.
(8.4) $M, w \vDash \neg oblxA$ iff $M, w \vDash prmx{\sim}A$ iff there is a w' with $wS_{v(x)}w'$
such that there is a w'' with $w'Rw''$ and not: $M, w'' \vDash A$.

Remarks
(1) The first 'iff' in these four truth conditions reflect the following Laws of Opposition stated on p. 252 in Alchourrón (1969): For the case of (8.1), see his TN-16; for (8.2), his TN-15; for (8.3), his TN-14; and for the case of (8.4), see his TN-17.

(2) We observe that the truth conditions for $\neg PRMxA$ and $\neg oblxA$ ((8.3), (8.4)) can again be expressed in terms of *relative products* as follows:

(8.3*) $M, w \vDash \neg PRMxA$ iff
for each w'' with $w(S_{v(x)}/R)w''$: not $M, w'' \vDash A$.
(8.4*) $M, w \vDash \neg oblxA$ iff
for some w'' with $w(S_{v(x)}/R)w''$: not $M, w'' \vDash A$.

Finally, we deal with NOBL-sentences having as their main connective any of the first four normative operators prefixed by a *double* strong negation $\neg\neg$. This is easily accomplished by our saying that in these four cases, i.e. $\neg\neg OBLxA$, $\neg\neg prmxA$, $\neg\neg PRMxA$, $\neg\neg oblxA$, the conditions for their truth at w in M are the same as in (7.1)–(7.4), respectively.

NOBL-*validity*, NOBL-*satisfiability* and *semantic* NOBL-*entailment*
Given our above truth conditions for NOBL-sentences, we define the notions in the present heading in perfect analogy with the matching DL-notions (Section 2.2 *supra*) and NL-notions (Section 3.2 *supra*): just replace any reference to DL or NL with a reference to NOBL!

5.3. *On the proof theory of* NOBL

The axiomatic system NOBL is determined by one rule of inference (deduction), one rule of proof, and six groups of axiom schemata, as follows:

Rule of inference
mp (modus ponens)

$$\frac{A, A \supset B}{B}$$

Rule of proof
Nec (necessitation for $\Box$)

$$\frac{\vdash A}{\vdash \Box A}$$

Axiom schemata

A0. All truth-functional tautologies over $\mathrm{Sent}_{\mathsf{NOBL}}$

A1. S5-schemata for $\Box$, $\Diamond$ [i.e. $\Diamond A \equiv \sim\Box\sim A$, $\Box(A \supset B) \supset (\Box A \supset \Box B)$, $\Box A \supset A$, $\Box A \supset \Box\Box A$, $\Diamond\Box A \supset A$]

A2.
(a) $\Box A \supset OA$
(b) $O(A \supset B) \supset (OA \supset OB)$
(c) $PA \equiv \sim O \sim A$
(d) $OA \supset PA$

A3.
(a) $\Box A \supset OBLxA$
(b) $OBLx(A \supset B) \supset (OBLxA \supset OBLxB)$
cf. C1-CN-6, Alchourrón (1969, p. 265)[4]

[4] We observe here that Alchourrón (1969, p. 265) pertinently points out that his thesis C1-CN-6, i.e. his analogue of our A3(b) in NOBL, does not depend on his special hypothesis (C1-CN), which is then not needed for his proof of C1-CN-6.

(c) $PRMxA \equiv \sim oblx \sim A$ TN-6)
(d) $prmxA \equiv \sim OBLx \sim A$ TN-7)
(e) $OBLxA \supset PRMxA$ TN-32)
(f) $oblxA \supset prmxA$ TN-33)

A4.
(a) $\neg PRMxA \equiv OBLx \sim A$ TN-14)
(b) $\neg prmxA \equiv oblx \sim A$ TN-15)
(c) $\neg OBLxA \equiv PRMx \sim A$ TN-16)
(d) $\neg oblxA \equiv prmx \sim A$ TN-17)

A5. $\neg\neg \S xA \equiv \S xA$, for $\S = PRM, prm, OBL, obl$;
cf. Alchourrón (1969, p. 252 bottom)

Remarks

(1) The references to the right of the axiom schemata A3(c)–(f) and A4(a)–(d) pertain to thesis-schemata listed in Alchourrón (1969, pp. 250 ff.).

(2) We note that our present proof theory for NOBL differs from the axiomatics of NL given in Section 3.3 *supra* in that the three axiom schemata governing the operators Nx and Mx in NL are replaced in NOBL by the three groups A3, A4 and A5 as just presented. The six schemata in A3 list properties of the normative operators $\S x$ in cases where *no* occurrences of the connective $\neg$ of strong negation are forthcoming, whereas those in A4 concern cases involving just *one single* such occurrence, and those in A5 the cases involving occurrences of *double* strong negation.

The above rules and axiom schemata determine syntactic notions of NOBL-*provability* and NOBL-*deducibility* in the straightforward way — see Section 2.3 *supra*. Similarly for the concepts of NOBL-*inconsistency*, NOBL-*consistency* and *maximal* NOBL-*consistency* (as applied to $\Gamma \subseteq \text{Sent}_{\text{NOBL}}$).

THEOREM III (Soundness & Completeness of the axiomatic system NOBL).
Weak version: For each NOBL-sentence A, A is NOBL-provable iff A is NOBL-valid.
Strong version: For each set Γ of NOBL-sentences, Γ is NOBL-satisfiable iff Γ is NOBL-consistent.
Proof. The methodology to be used in the proof of the present result is still familiar — see e.g. Åqvist (1987), Chs III and IV.

6. *The problem of isolating the normative* NOBL*-fragment of the system* NL

Consider the set $\text{Sent}_{\mathsf{NL}}$ of NL-sentences and the set $\text{Sent}_{\mathsf{NOBL}}$ of NOBL-sentences as well as the definitions D1–D4, D1′–D4′ and D1″–D4″ stated in Section 4 above. We now ask the following question

> Q: *Exactly which* NOBL-sentences are provable in NL on the basis of those definitions? In other words,what is the normative NOBL-fragment of NL?

In order to answer this question we have to make precise two important notions, viz. (1) that of the *translation* ϕ from $\text{Sent}_{\mathsf{NOBL}}$ into $\text{Sent}_{\mathsf{NL}}$ which is *induced* by the afore-mentioned twelve definitions, and (2) that of the normative NOBL-fragment of NL *under* the translation ϕ.

6.1. *Definition of the* translation ϕ *from Sent*$_{\mathsf{NOBL}}$ *into Sent*$_{\mathsf{NL}}$ induced by *the definitions in Section 4*

For each sentence A in $\text{Sent}_{\mathsf{NOBL}}$, define $\phi(A) \in \text{Sent}_{\mathsf{NL}}$ by the following recursive conditions:

(i) $\phi(p) = p$, for each proposition letter p in Prop
(ii) $\phi(\mathrm{T}) = \mathrm{T}$
(iii) $\phi(\perp) = \perp$
(iv) $\phi({\sim}A) = {\sim}\phi(A)$
(v) $\phi(A\&B) = (\phi(A)\&\phi(B))$

Similarly for $\phi(A \vee B)$, $\phi(A \supset B)$ and $\phi(A \equiv B)$.

(vi) $\phi(\Box A) = \Box\phi(A)$
(vii) $\phi(\Diamond A) = \Diamond\phi(A)$
(viii) $\phi(OA) = O\phi(A)$
(ix) $\phi(PA) = P\phi(A)$

(x) $\phi(PRMxA) = NxP\phi(A)$
(xi) $\phi(prmxA) = MxP\phi(A)$
(xii) $\phi(OBLxA) = NxO\phi(A)$
(xiii) $\phi(oblxA) = MxO\phi(A)$

(xiv) $\phi(\neg PRMxA) = Nx{\sim}P\phi(A)$
(xv) $\phi(\neg prmxA) = Mx{\sim}P\phi(A)$
(xvi) $\phi(\neg OBLxA) = Nx{\sim}O\phi(A)$
(xvii) $\phi(\neg oblxA) = Mx{\sim}O\phi(A)$

(xviii) $\phi(\neg\neg PRMxA) = Nx{\sim}{\sim}P\phi(A)$

(xix) $\phi(\neg\neg prmxA) = Mx{\sim}{\sim}P\phi(A)$
(xx) $\phi(\neg\neg OBLxA) = Nx{\sim}{\sim}O\phi(A)$
(xxi) $\phi(\neg\neg oblxA) = Mx{\sim}{\sim}O\phi(A)$

Clearly, the most interesting clauses in this definition are (x)–(xxi). In effect, we can even verify by induction on the length of the NOBL-sentence A that $\phi(A) = A$, provided that A does not contain any occurrences of the normative operators $PRMx$, $prmx$, $OBLx$, $oblx$ or of their strongly negated variants (single or double). We observe here how the clauses (x)–(xiii), (xiv)–(xvii) and (xviii)–(xxi) respectively parallel, or correspond to, the NL-definitions D1–D4, D1′–D4′ and D1″–D4″.

In the sequel we shall often write ϕA instead of $\phi(A)$.

6.2. *Definition of the normative* NOBL*-fragment of* NL *under* ϕ

Consider the axiomatic system NL, and let ϕ be the translation from $\text{Sent}_{\mathsf{NOBL}}$ into $\text{Sent}_{\mathsf{NL}}$ as just defined. By the *normative* NOBL*-fragment of* NL *under* ϕ (in symbols: NOBLfrg[NL, ϕ]) we mean the set of sentences A in $\text{Sent}_{\mathsf{NOBL}}$ such that ϕA is provable in NL; more compactly expressed:

$$\mathsf{NOBLfrg}[\mathsf{NL}, \phi] = \{A \in \text{Sent}_{\mathsf{NOBL}} \colon \vdash_{\mathsf{NL}} \phi A\}.$$

Since the translation ϕ is fixed, we may drop the reference to it and speak simply of the normative NOBL-fragment of NL, NOBLfrg[NL], in accordance with the convention:

$$\mathsf{NOBLfrg}[\mathsf{NL}] = \mathsf{NOBLfrg}[\mathsf{NL}, \phi].$$

6.3. *The problem restated*

Suppose that, in answer to the question Q raised at the beginning of the present Section 6, we claim that the set of NOBL-provable sentences is in fact identical to the normative NOBL-fragment of NL. What are we then claiming? Identifying the logic NOBL with the set of its theses, i.e. NOBL-provable sentences, we claim according to our definitions:

$$\mathsf{NOBL} = \{A \in \text{Sent}_{\mathsf{NOBL}} \colon \vdash_{\mathsf{NOBL}} A\} = \{A \in \text{Sent}_{\mathsf{NOBL}} \colon \vdash_{\mathsf{NL}} \phi A\} = \mathsf{NOBLfrg}[\mathsf{NL}].$$

Fortunately, a simpler and more attractive formulation of this claim is the following:

For each sentence A in $\text{Sent}_{\mathsf{NOBL}}$: $\vdash_{\mathsf{NOBL}} A$ iff $\vdash_{\mathsf{NL}} \phi A$

i.e., in plain language, A is provable in NOBL iff its translation ϕA is provable in NL (for any member of $\text{Sent}_{\mathsf{NOBL}}$).

7. *Solution to the restated problem*

In this section we state and sketch the proof of a result [Theorem IV *infra*] which in effect amounts to a solution to our restated problem. Before going into the details of this result, we need the following
LEMMA (Some Derived Rules of Proof and Thesis Schemata in the axiomatic system NL). NL is closed under the following derived rules of proof:

NL0. $\dfrac{\vdash A \supset B}{\vdash OA \supset OB}$ $\quad$ $\dfrac{\vdash A \equiv B}{\vdash OA \equiv OB}$

NL1. $\dfrac{A \supset B}{\vdash NxA \supset NxB}$

NL2. $\dfrac{\vdash A \equiv B}{\vdash NxA \equiv NxB}$

NL3. $\dfrac{\vdash A \supset B}{\vdash MxA \supset MxB}$

NL4. $\dfrac{\vdash A \equiv B}{\vdash MxA \equiv MxB}$

Furthermore, we list some theorem-schemata, or thesis-schemata, of NL, i.e. schemata of which every instance (in $\text{Sent}_{\mathsf{NL}}$) is provable in NL:
NL5. $NxA \equiv {\sim}Mx{\sim}A$
NL6. $NxPA \equiv {\sim}Mx{\sim}{\sim}O{\sim}A \equiv {\sim}MxO{\sim}A$
NL7. ${\sim}OA \equiv P{\sim}A$
NL8. $Mx{\sim}OA \equiv MxP{\sim}A$

Proof.
Ad NL0: Use Nec and mp together with axiom schemata A0, A2(a) and A2(b) in NL!
Ad NL1: Use Nec and mp together with axiom schemata A3(a) and A3(b) in NL!
Ad NL2: Use A0, mp and NL1 just derived!
Ad NL3: Use A0, mp together with NL1 and A3(c) in NL!

Ad NL4: Use A0, mp and NL3!

Ad NL5: Use A3(c) in NL together with A0, mp and NL2!
Ad NL6: Use A2(c) together with A0, mp, NL2 and NL5!
Ad NL7: Use A0, mp together with A2(c) and NL0!
Ad NL8: Use NL7 together with NL4!

THEOREM IV. (Translation Theorem for the axiomatic system NOBL).
$\mathsf{NOBL} = \mathsf{NOBL}$frg[$\mathsf{NL}$]; i.e. for each sentence A in Sent$_{\mathsf{NOBL}}$:
$\vdash_{\mathsf{NOBL}} A$ if and only if $\vdash_{\mathsf{NL}} \phi A$.

Proof. The proof is a bit lengthy and will be divided into an "*only if*" *part* and an "*if*" *part*.

"*Only if*" *part*: We are to show that A is provable in NOBL only if its translation ϕA is provable in NL, for any $A \in$ Sent$_{\mathsf{NOBL}}$. We do so by induction on the length of the supposed NOBL-proof of A.

Induction Basis. The length of the supposed NOBL-proof = 1, so A is an instance of one or other of the axiom schemata A0–A5.

Suppose that A is an axiom under A0 so that A is a truth-functional tautology over Sent$_{\mathsf{NOBL}}$. Then ϕA is a tautology over Sent$_{\mathsf{NL}}$ (the detailed proof of this is left to the reader), hence ϕA is an axiom under A0 in the proof theory of NL, hence $\vdash_{\mathsf{NL}} \phi A$.

Suppose next that A is an axiom under A1, say, under the second S5-schema in our list of such schemata. Then, $A = \Box(B \supset C) \supset (\Box B \supset \Box C)$, for some sentences B, C in Sent$_{\mathsf{NOBL}}$, so that ϕB, ϕC are in Sent$_{\mathsf{NL}}$ and $\phi A = \phi(\Box(B \supset C) \supset (\Box B \supset C))$ is in Sent$_{\mathsf{NL}}$ as well. The following is then an NL-proof of ϕA:

1. $\Box(\phi B \supset \phi C) \supset (\Box\phi B \supset \Box\phi C)$
   by the 2$^{\text{nd}}$ item in A1 in the axiomatics for NL

2. $\Box\phi(B \supset C) \supset (\phi\Box B \supset \phi\Box C)$
   from 1 by the definition of translation ϕ, clauses (v) for $\supset$, and (vi)

3. $\phi\Box(B \supset C) \supset \phi(\Box B \supset \Box C)$
   from 2 by the definition of ϕ, the same clauses

4. $\phi(\Box(B \supset C) \supset (\Box B \supset \Box C))$
   from 3 by the definition of ϕ, clause (v) for $\supset$

where 4 = ϕA = Q.E.D. Hence, $\vdash_{\mathsf{NL}} \phi A$, as desired.

A third illustrative case in the Induction Basis: suppose that A is an axiom under A4(d) so that $A = (\neg oblxB \equiv prmx{\sim}B)$, for some B in $\text{Sent}_{\mathsf{NOBL}}$. Then, ϕB is in $\text{Sent}_{\mathsf{NL}}$ and $\phi A = \phi(\neg oblxB \equiv prmx{\sim}B)$. The following is then an NL-proof of ϕA:

1. $Mx{\sim}O\phi B \equiv MxP{\sim}\phi B$ provable in NL by thesis schema NL8

2. $Mx{\sim}O\phi B \equiv MxP\phi{\sim}B$ from 1 by Def.ϕ, clause (iv)

3. $\phi(\neg oblxB) \equiv \phi(prmx{\sim}B)$ from 2 by Def.ϕ, clauses (xvii), (xi)

4. $\phi(\neg oblxB \equiv prmx{\sim}B)$ from 3 by Def.ϕ, clause (v) for $\equiv$

where 4 = ϕA = Q.E.D. Hence the desired result that $\vdash_{\mathsf{NL}} \phi A$.

The strategy of proof to be used in the remaining cases in the Induction Basis is sufficiently well illustrated already by the present simple cases: we appeal to relevant rules and axiom schemata in NL as well as to the derived rules of proof and thesis schemata in NL given in the Lemma above, together with applicable clauses in the definition of the translation ϕ from $\text{Sent}_{\mathsf{NOBL}}$ into $\text{Sent}_{\mathsf{NL}}$ (Section 6.1 *supra*). These remaining cases can all be left to the reader as exercises. As usual, the division of cases in the Induction Basis follows the axiom schemata in NOBL.

Induction Step. There is a NOBL-proof of A of length > 1, and either (i) A is got by applying mp(modus ponens) to some NOBL-theses B and $B \supset A$, or (ii) A is of the form $\Box B$ and is obtained by applying Nec (necessitation for $\Box$) to some NOBL-thesis B.

Case (i): By the induction hypothesis ϕB and $\phi(B \supset A)$ are both provable in NL. But, by the definition of ϕ, $\phi(B \supset A) = \phi B \supset \phi A$, so that ϕA follows by mp in NL. Hence, $\vdash_{\mathsf{NL}} \phi A$.

Case (ii): By the induction hypothesis we have $\vdash_{\mathsf{NL}} \phi B$ in this case. We then obtain $\vdash_{\mathsf{NL}} \phi(\Box B)$ as follows:

1. $\Box\phi B$ $\vdash_{\mathsf{NL}} \phi B$, Nec

2. $\phi(\Box B)$ from 1 by Def.ϕ, clause (vi)

where 2 = ϕA = Q.E.D. Hence $\vdash_{\mathsf{NL}} \phi A$, as desired.

This completes the proof of the "only if" part.

"*If*" *part*: We must show that if $\vdash_{\mathsf{NL}} \phi A$, then $\vdash_{\mathsf{NOBL}} A$, or, contrapositively, that if A is *not* NOBL-provable [not: $\vdash_{\mathsf{NOBL}} A$], then ϕA is *not* NL-provable [not: $\vdash_{\mathsf{NL}} \phi A$], for any sentence A in $\text{Sent}_{\mathsf{NOBL}}$. This part is harder, because proof-theoretical methods seem to be less "natural" here; however, in view of our soundness and completeness results for the axiomatic systems NOBL and NL, the problem is not too difficult to cope with.

We would like to argue in accordance with the following

Strategy of Argument:

1. not: $\vdash_{\mathsf{NOBL}} A$ [A is not provable in NOBL] hypothesis

2. not: $\vDash_{\mathsf{NOBL}} A$ [A is not valid in NOBL] — from 1 by the completeness of NOBL

3. not: $M, w \vDash A$, for some NOBL(= NL)-model $M = (W, R, U, S, V, v)$ and some w in W — from 2 by the definition of NOBL-validity

We now claim that this NL(= NOBL)-model has the property that for all B in $\text{Sent}_{\mathsf{NOBL}}$ and all w' in W: $M, w' \vDash B$ iff $M, w' \vDash \phi B$. Then:

4. not: $M, w \vDash \phi A$ — from 3 by M having the property claimed

5. not: $\vDash_{\mathsf{NL}} \phi A$ [ϕA is not NL-valid] — from 4 by the definition of NL-validity

6. not: $\vdash_{\mathsf{NL}} \phi A$ [ϕA is not NL-provable] from 5 by the soundness of NL

where 6 is our desired conclusion.

The crux of this argument is obviously isolated at one single point, viz. the claim that the given NOBL(= NL)-model M has the desired property indicated above. On the basis of that claim the crucial step from 3 to 4 is fully justified and the "if" part is seen to go through as a whole. What remains to be done, then, is to prove the following result:

CRUCIAL CLAIM. Let $M = (W, R, U, S, V, v)$ be any NOBL(= NL)-model. Then, for all A in Sent$_{\mathsf{NOBL}}$ and all w in W: $M, w \vDash A$ iff $M, w \vDash \phi A$.

Proof. By induction on the length of A. By the definition of the translation ϕ, the three cases in the induction basis are seen to be trivial. For the same reason, in the induction step, the cases involving Boolean compounds and NOBL-sentences having deontic or alethic modal operators as their main connective (O, P,$\Box$, $\Diamond$) go through easily. For example, in the *Case* $A = OB$:

1. $M, w \vDash OB$ iff for each w' in W with wRw': $M, w' \vDash B$ — by condition (4) [Section 2.2] in the definition of truth at w in M (viewed as a NOBL-model)

2. $M, w \vDash O\phi B$ iff for each w' in W with wRw': $M, w' \vDash \phi B$ — by condition (4) in the definition of truth at w in M (viewed as an NL-model)

3. $M, w' \vDash B$ iff $M, w' \vDash \phi B$ — by the induction hypothesis, w' being any member of W

Hence:

4. (Right member of 1) iff (right member of 2) — from 3 by elementary steps

5. $M, w \vDash OB$ iff $M, w \vDash O\phi B$ — from 1, 2, 4 by transitivity of "iff"

6. $M, w \vDash OB$ iff $M, w \vDash \phi(OB)$ — from 5 by the definition of ϕ, clause (viii)

where 6 = Q.E.D. in the present case, as desired. The remaining possibilities here, i.e. *Cases* $A = PB$, $\Box B$, $\Diamond B$, are handled in the same vein.

Somewhat more excitingly, consider next some cases involving NOBL-sentences having our characteristic primitive one-place normative operators as their main connective.

Case $A = OBLxB$. We are required to show that $M, w \vDash OBLxB$ iff $M, w \vDash \phi(OBLxB)$. Well, for any B in Sent$_{\mathsf{NOBL}}$ and any w in W, we clearly have:

1. $M, w \vDash OBLxB$ iff for each w'' with
$w(S_{v(x)}/R)w''$: $M, w'' \vDash B$ by (7.1*) in the definition of truth at w in M (viewed as a NOBL-model)

2. $M, w \vDash NxO\phi B$ iff for each w'' with
$w(S_{v(x)}/R)w''$: $M, w'' \vDash \phi B$ by (7.1*) in the definition of truth at w in M (viewed as an NL-model)

3. $M, w'' \vDash B$ iff $M, w'' \vDash \phi B$ by the induction hypothesis, w'' being any member of W

Hence:

4. (Right member of 1)
iff (right member of 2) from 3 by elementary reasoning

5. $M, w \vDash OBLxB$
iff $M, w \vDash NxO\phi B$ from 1, 2, 4 by transitivity of "iff"

6. $M, w \vDash OBLxB$
iff $M, w \vDash \phi(OBLxB)$ from 5 by the definition of ϕ, clause (xii)

where 6 = Q.E.D.

The *Cases* $A = PRMxB$, $A = prmxB$, and $A = oblxB$ are handled in the same spirit and can be left to the reader.

The remaining cases involving normative operators prefixed by a *single* strong negation-sign can also be left to the reader. Similarly for the cases where we have a *double strong* negation $\neg\neg$ as a prefix to any of the four original normative operators in NOBL.

This completes the inductive proof of our Crucial Claim. Hence, the proof of the Translation Theorem IV is complete as well. □

8. *Some glimpses beyond*

Throughout this section and its subsections I will allow myself to use the notation introduced in the present paper, even when directly quoting from contributions by our colleagues.

8.1. *Useful thesis-schemata provable in* NOBL *and due to Alchourrón (1969)*

We now list some thesis-schemata all instances of which can be seen to be provable in the system NOBL. We leave the proof of this fact to the reader. *Hint*: Use the formulation of the proof theory of NOBL given in Section 5.3 together with the main result of this paper — the Translation Theorem IV for NOBL, as stated and proved in Section 7 *supra*! In the right column we indicate the names given in Alchourrón (1969, pp. 250 ff.) to the thesis-schemata under consideration here.

(α) $OBLxA \equiv {\sim}prmx{\sim}A$ TN-8)
(β) $oblxA \equiv {\sim}PRMx{\sim}A$ TN-9)

(γ) ${\sim}PRMxA \equiv oblx{\sim}A$ TN-10)
(δ) ${\sim}prmxA \equiv OBLx{\sim}A$ TN-11)

(ε) ${\sim}OBLxA \equiv prmx{\sim}A$ TN-12)
(ζ) ${\sim}oblxA \equiv PRM{\sim}A$ TN-13)

(η) $\neg PRMxA \equiv OBLx{\sim}A$ TN-14) [= A4(a) in Section 5.3 *supra*]
(θ) $\neg prmxA \equiv oblx{\sim}A$ TN-15) [= A4(b)]

(ι) $\neg OBLxA \equiv PRMx{\sim}A$ TN-16) [= A4(c)]
(κ) $\neg oblxA \equiv prmx{\sim}A$ TN-17) [= A4(d)]

(λ) $PRMxA \equiv \neg OBLx{\sim}A$ TN-18)
(μ) $prmxA \equiv \neg oblx{\sim}A$ TN-19)

(ν) $OBLxA \equiv \neg PRMx{\sim}A$ TN-20)
(ξ) $oblxA \equiv \neg prmx{\sim}A$ TN-21)

(π) ${\sim}\neg PRMxA \equiv prmxA$ TN-22)
(ρ) ${\sim}\neg prmxA \equiv PRMxA$ TN-23)
(σ) ${\sim}\neg OBLxA \equiv oblxA$ TN-24)
(τ) ${\sim}\neg oblxA \equiv OBLxA$ TN-25)

(φ) $OBLx(A\&B) \equiv (OBLxA\&OBLxB)$ TN-26)
(ψ) $prmx(A \vee B) \equiv (prmxA \vee prmxB)$ TN-27)

(χ) $(PRMxA \vee PRMxB) \supset PRMx(A \vee B)$ TN-28)
(υ) $(oblx(A\&B) \supset (oblxA\&oblxB)$ TN-29)

Alchourrón (1969, p. 253) observes that the "full" distribution principles for O over & and for P over $\vee$ in Deontic Logic are also valid for the operators $OBLx$ and $prmx$ in his Normative Logic — as indicated by his TN-26) and TN-27) [= our (φ) and (ψ)], where the main connective is $\equiv$. The situation is different with respect to the principles TN-28) and TN-29): he carefully points out that their converses fail to be valid, so that for his notions of strong permission and weak obligation the implication in (χ) and (υ) holds in only one direction, viz. the one just indicated.

8.2. *The notion and the principle of prohibition: some validities and non-validities in* NOBL *and* NL

Alchourrón (1969, p. 253) proposes the following definition of a strong notion of *normative prohibition*:

Df-PROHB. $PROHBxA =_{\text{df}} \neg PRMxA$

Forgetting for the time being about a possible weak variant of this notion, we may think of Df-PROHB as being added to our system NOBL. Bearing in mind our definition D1′ in Section 4 above we clearly obtain the following result in NL extended with Df-PROHB, where we then think of it as being added to NL supplemented with D1′:

(ω) $PROHBxA \equiv Nx{\sim}PA$

Furthermore, let us now state some laws valid for $PROHBx$, which we formulate in our system NOBL, and where we indicate the matching Alchourrón (1969) names immediately to the left of the thesis-schemata concerned:

TN-30) $PROHBxA \equiv \neg PRMxA$
TN-31) $PROHBxA \equiv OBLx{\sim}A$

TN-32) $OBLxA \supset PRMxA$ cf. A3(e) in our axiomatics for NOBL
TN-33) $oblxA \supset prmxA$ cf. A3(f) in our axiomatics for NOBL
TN-34) $PROHBxA \supset PRMx{\sim}A$ cf. A2(d) in DL with ${\sim}A$ for A

With respect to the last three schemata Alchourrón (1969, p. 254) points out that obligation implies permission of the same kind, and that prohibition implies the strong permission of the negation. He then goes on (*ibid.*) to make the following interesting remarks:

> Many authors have expressed their doubts about the legitimacy of accepting as logically true such principles. They argue in the following way: If obligation logically implies permission then it is logically impossible for something to be obligatory and not permitted, *i.e.*, prohibited. This, however, is not only not impossible but it is frequently found in experience. It is not uncommon that states of affairs are qualified as obligatory and also as prohibited. Of course, this is a regrettable situation, but not an impossible one, and lawyers know how frequent it is.
>
> This argument has been directed against
>
> $$\vdash (Op \supset Pp)$$
>
> which, in deontic logic, is equivalent to
>
> $$\vdash \sim(Op \;\&\; Php)\ [scil. \text{ where, in our DL, } Php \text{ means } {\sim}Pp)$$
>
> But if the argument is analyzed, it may be seen that the concepts referred to in it are the normative and not the deontic ones. With respect to the normative operators the incompatibility of obligation and prohibition does not hold.
>
> The following is not a law of normative logic:
>
> $$\sim(OBLxp \;\&\; PROHBxp),$$
>
> and this formula is not equivalent to TN-32).
>
> Strong obligation implies strong permission, but prohibition is compatible with strong permission. The following is not valid:
>
> $$\sim(PROHBxp \;\&\; PRMxp).$$
>
> Sometimes, the preceding argument has not been considered conclusive. The explanation for this opinion must, I think, be sought in the ambiguity of the notion of permission, because when "permission" is used in the weak normative sense, then it is true that it is incompatible with prohibition.
>
> Alchourrón (1969, pp. 254–255)

Alchourrón then goes on to point out that, in spite of the non-validity in his normative logic of the schema $\sim(PROHBxp \;\&\; PRMxp)$, the situation just described above is reflected in the validity in his logic of the following theorem:

TN-35) $\sim(PROHBxA \,\&\, prmxA)$

which is clearly valid in NOBL + Df-PROHB as well. He also explains why the schema

$$\sim(PROHBxp \,\&\, PRMxp)$$

fails to be valid in his normative logic; we leave to the reader the task of establishing the same result for our system NOBL + Df-PROHB.

In a section dealing *inter alia* with the so-called principle of prohibition [Alchourrón (1969, p. 258 f.)] our author observes that prohibition and weak permission are not only incompatible with each other but are logically exhaustive as well. This, he notes, shows another analogy between normative and deontic logic, viz.

Normative Logic	*Deontic Logic*
TN-49) $\vdash (PROHBxA \vee prmxA)$	TN-11) $\vdash (PhA \vee PA)$ [where $Ph = \sim P$]

But he is careful to point out that prohibition and strong permission are not logically exhaustive, because the schema

$$(PROHBxA \vee PRMxA)$$

fails of validity in his normative logic (and, we may add, in NOBL + Df-PROHB as well). And his reason for this being so deserves to be quoted (p. 259):

> I think that all this is in agreement with our intuitions, in the sense that it is perfectly possible, for a given authority x, that there exists a possible state of affairs p in relation to which x has not issued any norm permitting nor any norm prohibiting it. In such a case it may be said that x has not determined any normative character for p, though perhaps he has normatively characterized not-p.

8.3. *Two controversial aspects of the Alchourrón-Bulygin Deontic Logic: the deontic status of tautologies and the problem of iteration of deontic operators*

In Alchourrón & Bulygin (1973, p. 679) the authors observe that their deontic logic (= their logic of norms) comes very close to von Wright's first system[5] and that his first deontic calculus proves to be an adequate reconstruction of the *prescriptive* notions of permission and obligation. On p. 679 of their paper they make two interesting, though (in my opinion) debatable, comments:

> i. If we wish to reconstruct the legal prescriptive discourse as closely as possible to ordinary usage, we must reject as meaningless (ill-formed) those deontic expressions in which a deontic operator is followed by a contradictory or a tautological expression, because they do not prescribe (command or permit) any particular state of affairs. On the other hand, such a rejection would lead to a very complicated calculus. It is for such purely formal reasons that most logicians, including von Wright, accept as well-formed formulas (and eventually as axioms) expressions of the form 'Ot' or 'Pt' (where 't' stands for a tautology).
>
> ii. We must exclude expressions in which a deontic operator occurs within the scope of another deontic operator. This is so because the content of a norm must be an action, an activity or a state of affairs which is the result of an action or an activity. Therefore, the expression following a deontic operator must be a *description* of one of these things, but it cannot be a prescription. In other words, deontic operators generate norms out of descriptions of a certain kind, but they cannot generate norms out of norms. This rules out the iteration of deontic operators. The authors' italics.

Let me comment on these two quotations in turn.

Ad i.

The axiom schema A2(a) in the proof theory of our DL is obviously not well formed in the Alchourrón-Bulygin (1973) deontic logic, since the latter does not have the alethic modal operator $\Box$ among its primitive logical constants. However, using Nec, A2(a) and mp in our DL, we readily obtain the rule of proof:

$$\text{from } \vdash A \text{ to infer } \vdash OA$$

as a *derived* rule of proof in DL, which is precisely at issue in the first passage quoted here. In the present paper, then, we certainly belong to the class of "most logicians, including von Wright", spoken of in this passage. Our reason for accepting that rule is not just "purely formal", however, it is also bound up with the issue of adequately explicating the *semantical* notion of

[5] See von Wright (1951).

"state of affairs" or "particular state of affairs". It seems to me that Alchourrón and Bulygin take an unnecessarily narrow view here of this admittedly important notion.

Ad ii.

Nor does the Alchourrón-Bulygin rejection of iteration of deontic operators strike me as entirely convincing. Suppose that we add to the axiom schemata under A2 in DL the following "reduction laws":

$$\mathrm{O}A \equiv \mathrm{OO}A \qquad \mathrm{P}A \equiv \mathrm{PP}A$$
$$\mathrm{O}A \equiv \mathrm{PO}A \qquad \mathrm{P}A \equiv \mathrm{OP}A$$

In the spirit of Chellas (1980, sect. 4.4, pp. 147 ff.) let us mean by a *deontic modality* any sequence of the DL-operators $\sim$, O, and P, and $\cdot$ (= the empty sequence of those operators). Now, by Theorem 4.27 in Chellas (1980, p. 154) the axiomatic system which results from such an addition will contain a so-called *normal KD45-system* having at most *six* distinct *irreducible deontic modalities*, viz. $\cdot$ (the empty sequence), O, P, and their negations $\sim$, $\sim O$, and $\sim P$ (the first three of which are affirmative, the remaining three negative). The moral about deontic modalities in systems of the present sort (e.g. DL + the four reduction laws above) is that *iteration is vacuous* : any sequence of Os and Ps can always be *reduced to its innermost term* [again, see Chellas (1980, p. 154)].

How do these observations affect our assessment of the Alchourrón-Bulygin rejection of iteration of deontic operators? In order to be able to answer this question satisfactorily we must, I suggest, consider a third illuminating quotation from their 1973 paper. On p. 689 they argue as follows (where I may still be allowed the notation used in my present paper):

> The interpretation of deontic formulae with iterated operators presents some difficulties which did not escape von Wright's attention. Let us consider, as an example, the iteration of the operator O.
> In view of the distinction between the prescriptive and the descriptive deontic operators, four different cases are to be considered:
>
> (1) $\mathrm{OO}p$
> (2) $\mathrm{OBL}x\mathrm{O}p$
> (3) $\mathrm{OOBL}xp$
> (4) $\mathrm{OBL}x\mathrm{OBL}yp$
>
> The first formula '$\mathrm{OO}p$' is a norm whose content is another norm ($\mathrm{O}p$). If we accept that only actions or states of affairs resulting

> from actions can be the contents of norms (section 6), then the formula 'OOp' must be rejected as meaningless. This result appears to be in accordance with von Wright's opinion (NA 189; EDL 91); norms cannot be the contents of norms.
>
> For similar reasons we decide to reject the formula (2). In the expanded form it amounts to 'NxOOp'. Part of this formula (OOp) is a meaningless expression; this is why we prefer to consider the whole formula as meaningless.
>
> The formula (3) is a norm whose content is a normative proposition. In the expanded form it amounts to 'ONxOp', *i.e.*, it is a norm to the effect that the authority x ought to issue a norm of the form 'Op'. This is exactly what von Wright understands by 'norms of higher order', that is, norms that prescribe (enjoin, permit or prohibit) the performance of *normative actions*, i.e., actions consisting in promulgating (or derogating) other norms. An example of a norm of higher order of the form OOBLxp may be found among the current prescriptions of a penal code: 'Homicide shall be punished with imprisonment from eight to 25 years' can be interpreted as a norm directed to the judge, enjoining him to issue a norm condemning to imprisonment everybody who has committed homicide.

After commenting on such an important kind of norms of higher order as so-called norms of competence, Alchourrón & Bulygin (1973, p. 690) make the following concluding observation with respect to the formula (4):

> Finally, the formula OBLxOBLyp (*scil.* where $x \neq y$) is a norm-proposition asserting the existence of a norm of higher order, i.e., asserting that the authority x has issued a norm to the effect that another authority y should issue a norm of the form Op; so its expanded form is NxONyOp. Here we must distinguish between different authorities, for the limiting case of a self-prescription (autonomous norm) where $x = y$ is of little interest in law. This fact makes it advisable to use subscripts in order to indicate the corresponding authority: 'OBLxOBLyp' would then correspond to 'NxONyOp'.
>
> Alchourrón & Bulygin (1973, pp. 689–690)

Remarks

(1) As an alternative to our authors' rejection of the formula (1) [in their list above] as meaningless we may well, I suggest, adopt the Chellas strategy

of treating $\mathrm{OO}p$ as equivalent to and reducible to its innermost term $\mathrm{O}p$ on the basis of the reduction thesis

$$\mathrm{OA} \equiv \mathrm{OOA}$$

The problematic formula $\mathrm{OO}p$, i.e. (1), then turns out to mean the very same thing as the plain and simple $\mathrm{O}p$ and hence is perfectly meaningful, as is readily verified in the semantics for so-called *normal KD45-systems* (spoken of above).

(2) A similar alternative strategy works nicely also in the case of the formula (2): as observed by our authors, it amounts in its expanded form to $\mathrm{N}x\mathrm{OO}p$, which reduces to the simpler and perfectly meaningful formula $\mathrm{N}x\mathrm{O}p$ in our system NL extended with appropriate axioms (reduction laws) and definitions, notably D3 in Section 4 *supra*. Thus, nor does the formula (2) have to be rejected as meaningless on our preferred alternative strategy.

(3)–(4) I have no objections whatsoever to Alchourrón's and Bulygin's perceptive discussion of the formulae (3) and (4) in their list; it strikes me as genuinely important and illuminating.

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