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NATURAL LANGUAGE SETS

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Abstract
Oliver and Smiley have tried to demonstrate how irreplaceable plural predications are, by showing specifically that certain plural predications cannot be handled in terms of sets, or any other kind of singular object. This paper first points out some grammatical facts bearing on the adequacy of the current mathematical account of sets, and then goes on to inspect the adequacy of Oliver and Smiley's defence of their account of plurals. It is something wrong with the mathematical theory of sets which leads these writers, first to think of set expressions as referring to abstract objects in some objectionable way, and then to try to replace these ontologically questionable objects with something with a clearly non-abstract reference, like a plural term. Oliver and Smiley have not been thinking in terms of ordinary speech, but in terms of the standard professional theory, and it is something to do with that theory that is making them come to their erroneous judgements. I here describe a conservative extension of standard Set Theory that avoids the abstraction problem with 'sets', and end by applying it in the evaluation of Oliver and Smiley's arguments.

As I have pointed out before (Slater 2006) the principal philosophical problem with the current mathematical account of sets is the difficulty in identifying singleton sets, as writers like D.K. Lewis, Penelope Maddy, and Michael Potter have seen. For instance, while one can see an apple on a table, where is the singleton set of that apple? Is it in the same place as the apple? But in that case, how is it distinguished from it? The solution to the problem, we shall see, is found by looking first at non-singleton sets, such as pairs. For consider, instead, a pair of apples. The common assumption has been that a pair of apples is a novel kind of individual object distinct from each of the apples. There is certainly a single object in the vicinity, namely the mereological sum of the two apples. But that cannot be the pair of apples, since that whole might be carved up in more than one way, and in some ways with more than two components. The expectation, as a consequence, has been that one must look elsewhere for another object to be the pair of apples. But this supposed other object is a grammatical mirage. The crucial question with respect to a pair of apples is: is that pair of apples a single object? The answer is 'no': it is simply a pair of objects, instead.

There are, in the area, at least three types of expression, with quite different grammars, and without a close inspection it is far to easy to run them all, or even just some of them together. Thus there are collective nouns like 'shoal', 'herd', 'pack', 'tribe', collective nouns like 'pair', 'triplet', 'quartet', 'dozen', and plural collective nouns like 'the fish', 'the beasts', 'the cards', 'the savages'. It is from collective nouns of the first sort that Set Theory draws the idea of collections of objects, but the distinctive thing about those natural language terms is that they are species specific, to one extent or another, so that they each describe certain mereological sums under a certain aspect. That is to say, the principle of division of the whole mereological sum is provided through the further count noun they are normally associated with — 'shoal' with 'fish', 'herd' with 'beasts', 'pack' with 'cards', 'tribe' with 'savages', etc. The fact that such collections are mereological sums is also shown by the fact that shoals, herds, packs, tribes, and the like, are located and can move around in physical space, just like their members.

There is not the same to be said with regard to collective nouns of the second sort, and not only because a complete description of the intended set must be given — the pair is maybe of fish, the triplet of beasts, etc. For these collective nouns can also be used when no physical objects are involved, and so when there are no mereological sums of the associated objects. They therefore only indicate the number of some things, and we have to remember the general grammar of 'y is one of a number of Ss'. This is not of the form '$y \in s$', with a singular term in place of 's'. A specification of it, for instance, would be 'y is one of 2 Ss', which relates 'y' to a plural term, and the original is just '$(\exists n)$(y is one of n Ss)'.

John Burgess (Burgess 2004, 197–9, 211) thinks there are two senses of 'is one of', so that one might say, for instance, not only

y is a member of the set of apples,

but also

y is amongst the apples,

even though the expressions are equivalent. That would allow a singular 's' to occupy '$y \in s$', while a plural 'xx' occupies '$y \;\alpha\; xx$'. Certainly y is a member of a/the/that pair of apples

is the same as

y is amongst 2/the 2/those 2 apples,

but 'is one of' could be used in both cases, and in the same sense. For the equivalences show that 'a/the/that pair of apples' still refer to the same things as '2/the 2/those 2 apples' — the former merely refer to them (*sic*) in a different manner, namely collectively, i.e. taking them as a unit. One must take care about what is added to the bare

y is one of some apples:

clearly

y is one of 2 apples

may speak about the same apples, but is more specific about their number, and

y is one of a pair of apples

likewise. But the latter does not invoke a further object, 'a pair', in addition to the two apples, it merely introduces a certain numerical measure of the apples, by taking the two as a unit. 'A pair of apples', in other words, differs from '2 apples' simply in changing '2 times 1 apple' into '1 times 2 apples'. We can count with such units, by counting in pairs, but we are then not counting something other than apples; we are merely not counting the apples singly, i.e. one by one. We say

There are 2 pairs of apples,

but this is exactly equivalent to

There are 4 (single) apples,

and

There is a quartet of apples.

To say there is a set of apples is equivalent to saying

(∃n)(there is an n-tuple of apples).

Words like 'pair', 'quartet', '16-tuple' are thus like 'ounce' and 'pound' in

'There are/is 8 ounces/a pound of beef'.

They do not refer to further entities, but are instead the basis for measurements of quantity. Here are two pairs of alphabetic letters:

a b c d.

Notice that each of the pairs is indicated without braces, as with '{a, b}', since the latter, i.e. the common 'set-theoretic' symbolism, does not represent sets as merely numbers or quantities of objects but as independent objects, distinct from their members. However, the four letters above, which were taken as two pairs, also can be taken as a quartet, while the two 'sets' {a, b} and {c, d} cannot be the same as the 'set' {a, b, c, d}. In fact no two 'sets' can be the same as any one 'set', but 2 twos are exactly the same as 1 four.

The reification of a unit of measure as a further separate object, maybe arises through forgetting the difference between the two sorts of collective term. For the mereological 'tribe' does have an objective reference to an independent object, but 'pair' needs supplementing, and then in 'a pair of apples', it merely qualifies the following substantive. Maybe focussing on count terms and forgetting mass terms also has something to do with the misconception. For the same matter of change of units even more clearly arises with fractions than with multiples of individuals. In 'There is a half of a loaf' there is obviously no reference to anything other than bread: there is not, in addition, reference to one of a range of mysterious, further objective entities, 'halves', 'quarters', 'parts', etc. There is merely a specification of how much of a loaf there is, maybe as a prelude to counting half-loaves, or totting up different parts of loaves to find an equivalent sum of complete loaves, etc.

We can now have a first try at tackling the question of what a 'set-theoretic' singleton is, i.e. what '{y}', or '{x: x=y}' might represent. In the natural language locution 'a singleton S', of course, 'singleton' just describes the S as the only one of its kind, and does not refer to any other object. But another way of representing being the sole S as 'being one of' something might well be the prime source of the set-theoretic notion of 'singleton'. For even if the number of things which are S is just 1 then we can still say 'y is one of those things which are S', making *those things which are* S what the sole S is one of. But 'is one of' is then again succeeded by a plural term, not a singular one. If we read set abstraction expressions as such plural terms, therefore,

{x: x=y}

would be

those things which are y,

allowing

y ∈ {x: x=y}

to be

y is one of the things which are y.

But then, identification of such 'singletons' with their single members, in the manner of Maddy, would clearly be ungrammatical, since 'y is those things which are y' does not make sense. Indeed the general identification of set abstracts with plural terms could not be thoroughgoing, since not only would

'y={x: x=y}'

be ungrammatical, so would

'y={x; Px}'

for any 'P', and therefore also

'{{x: Px}}',

and

'{{x: Px}, {x: Qx}}',

because these latter would have to be

'{z: z={x: Px}}',

and

'$\{z\colon z=\{x\colon Px\} \vee z=\{x\colon Qx\}\}$'

respectively.

It might be thought that such points about collective nouns and plural terms presuppose that unless Mathematics can be re-construed back into non-mathematical language then it is unintelligible. But the claim is merely that the 'sets' of 'Set Theory' cannot be interpreted either in terms of collections of things, or in terms of numbers of things, or by translation into plural expressions. Certainly there is the set-theoretic symbolism, and the rules for its manipulation, and maybe it all has some interpretation. But it does not have any of the traditional interpretations, on the basis of which it was developed, and the trouble that Lewis, Maddy, Potter and others still have had with the basic notion of a singleton shows it has yet to be given a clear sense.

We can now bring in the specific interests of Oliver and Smiley in this area. Oliver and Smiley have been concerned with, amongst other things, whether collective predications can be construed as predications on sets — with mathematical sets being what they had in mind. They start from the well-known distinction amongst plural predications between those that are collective, and those that are distributive. Thus

'A and B sum to C'

has a collective predicate, while

'A and B fight C',

for instance, has a distributive predicate. As the term 'collective' indicates, normally it would be said that the difference was a matter of the first, but not the second being about the collection $\{A, B\}$. If so, then both can be mapped onto singular forms, namely

'$\{A, B\}$ sums to C'

and

'A fights C, and B fights C'.

Using '$<$' to mean 'is a part of', and '$\in$' 'is a member of', they can be represented

$\{A, B\} \in \{y\colon y \text{ sums to } C\}$,
$\{A, B\} < \{y\colon y \text{ fights } C\}$.

Of course, the plural terms that can enter into such collective and distributive constructions are not limited to lists; they also include predicative phrases like 'the men'. So we can distinguish, similarly, singular equivalents of

'Those men are causing a riot'

and

'The men are bald',

i.e.

$\{x\text{: Mx}\} \in \{y\text{: y is causing a riot}\}$,

and

$\{x\text{: Mx}\} < \{y\text{: y is bald}\}$,

which might read in the appropriate context

'That mob of men is causing a riot',

and

'Each man is bald'.

One further virtue of this mode of analysis is that it allows a representation of multigrade predicates, to which the same writers, Oliver and Smiley, have recently drawn attention. They think, however, that multigrade predicates cannot be handled by standard formal logic (Oliver and Smiley 2004). For with 'those men are building a house', for instance, there would not seem to be a standard representation in logical notation, since all of the obvious candidates, 'Bxh', 'Bxyh', 'Bxyzh', etc. require the number of builders, x, y, z, to be specified. But if we merely say that a collection of men is building a house, then the number of men in that collection is unspecified, and so is not of any particular size, allowing the plural equivalent to be multigrade in the way Oliver and Smiley want.

There are troubles, however, with this singular style of analysis of collective plurals. The main trouble is that it requires collections, and sets of things to take part in the physical world, while the sets of mathematical set theory are generally thought to be abstract objects without any causal effects. The point is made much of in Oliver and Smiley's original paper on plurals. The largest point Oliver and Smiley repeatedly make concerns the supposed abstraction of sets, and thereby the impossibility of treating plural predication over concrete objects in terms of sets. This would certainly seem to be a strong, and significant point to make. They say, for instance, (Oliver and Smiley 2001, 295):

> It is easy to ridicule this naïve version of changing the [plural subject into a singular one]. Here is Boolos using upper case to great effect: 'It is haywire to think that when you have some Cheerios, you are eating a *set* — what you're doing is: eating THE CHEERIOS'. We *can* get away with saying that a set of premises implies a conclusion, because we are free to enrich the sense of 'implies' to accord with our talk — at least if we are prepared to accept what may be unwelcome consequences. Not so for 'eat'. Sets are abstract objects, and no abstract object is ever eaten. Collective predicates also cause trouble. Tom and Dick weigh over 200 kg, but the set of the two has no weight.

The ridicule misses its target, however, since anyone who has been to a greengrocer, and on a visit in a hospital, knows one can easily weigh a bunch of grapes, and eat a handful of them. So what is wrong is the idea that bunches and handfuls are abstract objects in some pernicious sense, through being examples of what mathematicians have called 'sets'. Certainly, as Oliver and Smiley show, plural collective predication cannot be reduced in any systematic way to plural distributive predication. But their favourite example of the former, namely 'Whitehead and Russell wrote *Principia Mathematica*' still means that that book was written by a couple of people. At the time of William and Mary, also, it may be remembered, a married couple ruled Britain. So that is a couple of examples that immediately come to mind where couples of various descriptions have done things in the world. And that is not to mention the many couples that take to the floor and do the Tango in Buenos Aires, probably every day of the week. As we saw before, 'the book was written by a couple of people' says no more than 'the book was written by two people', even though the grammar of the former expression is easily misinterpreted to suggest that it makes reference to a further individual object beyond the two people, namely the pair of them. But while it does allude to this pair, that pair is not an individual object: it is simply a pair of individual objects. How could Oliver and Smiley, as well as Boolos, have been so misled with regard to the proper grammar of ordinary speech?

What has mislead them, I believe, is that the current mathematical theory of sets is pre-eminently a theory of 'pure sets' rather than sets of things, and Oliver and Smiley are thinking of 'sets' in terms of this mathematical theory.

The obvious thing is that natural language sets, unlike with the common image of 'mathematical sets', in many cases involve members that are physical objects. Indeed, we shall see that they are mereological sums of such singletons. This is already plausible with respect to such physical sets as tribes, and shoals, which are located in, and move around in space just like individuals. Here, as we saw, the singletons spanning the mereological sum are determined through a further, associated count noun — 'savage' and 'fish'. Such physical objects are not strictly banned from mathematical set theory, but the nature of them has not been as fully investigated as it should have been. In particular it has not been realised that such individual objects are themselves sets — singleton sets. We shall see not only that, on a revised understanding, sets of such have a place in the world, but also, and perhaps even more surprisingly, that it is principally through its inattention to these matters, and its related concentration on 'pure sets', that mathematical set theory got itself into its worst crisis, at the turn of the twentieth century.

The theory was founded slightly before that, and much formal work has ensued, of course, but, in the present connection, it was not until the work of Harry Bunt and David Lewis, in the 1980s, that further significant philosophical developments, relevant to a proper understanding of physical sets, were made. By now it is well known that mathematical sets reduce to mereological sums of 'unicles', or 'singletons', i.e. sets with a single member. That is one important advance these two writers brought about, but the nature of such singletons has remained mysterious, from a philosophical point of view. To make their nature clear, however, it becomes necessary to include a discussion of something not commonly included, namely mass terms, since mereological notions are more commonly brought in in connection with them, and so, for a start, discrimination between the two applications of mereology is required. More to the point, mass terms describe kinds of material, and clarification of natural language singletons only emerges through seeing just how the material in some singleton is distinguished from that singleton itself. For that allows us to see that the singleton is an object with material in it, even while it is not that material itself. Indeed it might be said that 'abstraction', if it is present, begins even with singleton individuals, through their separation out from stuff. And it is that, in reverse, which is required to see that sets, more generally, can be said to have a place in the world, since they have as much reality as individuals, which are commonly taken to be part of 'what is there'. The present analysis has therefore benefited most from Bunt's work, which, although significantly different in several respects, also gives a theory of mass, or 'stuff' terms alongside a theory of count, or 'thing' terms — the latter being the only proper basis for a theory of sets.

The specific point to start from is that predicates that are not count do not determine discrete things. For it is that which solves the major problem about mereological singletons: how they can have substance and yet still be atomic. The prime grammatical distinction to observe is that mass terms do not pluralise in an appropriate way, like count terms. In addition to a gold ring, for example, there may be other rings, but not other golds in the same sense. Indeed there may only be other kinds of gold. When summing some items describable with the same mass term, 'F', we therefore cannot grammatically form the term 'the set of Fs', since there are no appropriate 'Fs'. What we can do is introduce an associated count term, and talk about the gold in a ring, for example, or the flesh in some limbs; more generally, the material in some individual, or individuals. Bunt's unicle-member relation then becomes the relation between an object and the stuff 'in it', i.e. the stuff that comprises it.

As a result of this, not only can the marble in a statue (m) be distinguished from the statue (s), since s = $\{m\}$, but also, even more importantly, the parts of some collection can be distinguished from what comprises it. Thus a beast (b) might be a part of a herd (b < h), while the beast consists in some material (n_1), which is not a part of the herd (b=$\{n_1\}$, but not $n_1 < h$), even though the herd consists in a set of such bits of stuff, i.e. h = $\{n_1, n_2, n_3, \ldots\}$. Again, the flesh of some seed (e) might be part of the flesh of some grape (g), so that e < g. But that does not mean either that the seed is part of the grape ($\{e\} < \{g\}$), or that the flesh of the seed is part of the grape (e < $\{g\}$), or that the seed is part of the flesh of the grape ($\{e\}$ < g). Other discriminations also become available. Thus, if the material in Peter's arms is d, then d is a mereological sum, b + c, where $\{b\}$ and $\{c\}$ are Peter's arms; and if the material in Peter's whole body is p, then while Peter = $\{p\}$, it is not the case, for instance, that $\{b\}$ < p, or that $\{c\}$ < p, or that d < $\{p\}$. Certainly the material in Peter's arms is a part of the material in Peter's body, i.e. d < p. But neither of Peter's arms, $\{b\}$, or $\{c\}$, is a part of the material in Peter's body, p. Nor is the material in Peter's arms, b + c, i.e. d, a part of Peter, $\{p\}$. Peter's arms, as a result, constitute a pair of things with only the unicles of each of his arms as proper parts; and Peter is an individual, which is to say he is a singleton, in the sense that he only has himself as a part (an improper part).

That shows that

Anne's flesh was human flesh

has the form

Anne's flesh < the totality of human flesh in her time,

but

Anne ruled Britain in the early 18th century

has the form

Anne $\in$ {y: y ruled Britain in the early 18th century},

where Anne = {Anne's flesh}. The case brings up the most crucial issue about physical sets, however: their transience. For Anne's flesh can appear in another form at some other time. That happened, of course, upon Anne's death, when the material in her body ceased to form a living thing. But it is a dateable matter, also, when the person Anne was formed from the material in her mother's ovum and her father's sperm. Hence, the relation between the unicle, x, and its sole member, y, must be recognised to be a temporally qualified relation. The material, y, in an object at a certain time, forms that object, x, but before that time, and afterwards perhaps, it may not have that form, so that then, as one might say, $x \neq \{y\}$. In those terms, whether it is true or not that $x = \{y\}$ is contingent on the way the world is. But it would be better, in such cases, to suffix the unicle symbol, '{y}', in order to indicate the time at which the material in the object is indeed in it:

'$\{y\}_t$'

would then indicate the object formed from the material y at time t. And then quite possibly

$\{y\}_t \neq \{y\}_{t'}$.

Of course, in other cases there can be no temporal variation, allowing a return to the usual, unadorned set-theoretic symbolism. But the extended symbolism gives us a way, for instance, of referring timelessly not only to physical sets with material members, but also to physical sets that are no longer, or not yet material ones. For even if $\{y\}_t \neq \{y\}_{t'}$, still $\{y\}_t$ may be referred to at time t′. But note that the exact same object $\{y\}_t$ cannot be formed from different material at a different time, i.e.

$\{y\}_t = \{z\}_{t'}$

only if

y=z,

so it is possible there may only be, for example, *counterpart* Ships of Theseus, at times t and t′. That means, for example, that when one looks at old photographs one maybe sees oneself at an earlier age, but never just 'oneself'. For there are properties of oneself at the earlier age that are different from corresponding properties of oneself at the present time.

Formally it must first be remembered that there are collective and distributive predications even with singular expressions. For we must distinguish singular collective predications like

That object is a ring,

from singular distributive predications like

The material in the ring is gold.

If the object is $\{r\}_t$ then we can symbolise them respectively

$\{r\}_t \in \{y\colon Ry\}$,

and

$r < g$,

where 'g' is the totality of all gold. Like all distributive predications the latter admits of variants using quantification, such as

All of the material in the ring is gold,

i.e.

$(x)(x < r \supset x < g)$,

and

Some of the material in the ring is gold,

i.e.

$(\exists x)(x < r.x < g)$.

But the main point to note is that, in the collective case we are not considering *a different piece of material*, merely *the same piece of material differently*, i.e. collectively, and so as a single unit rather than something with parts.

The units considered in this case are rings, and one might imagine several such lined up on a display cushion, made of some silky material perhaps. Looking at the display one way it might be seen as a continuous, if lumpy sequence of various sorts of stuff. But by marking divisions between the rings one can be brought to attend to those rings as units, and so, for instance, count them. The function of the lines in the following, however, must be closely noted:

|A||B||C||D||E||F|

The lines naturally serve to mark off the rings from one another, *but also* they mark off the ring '|A|' from its material content 'A', etc. Likewise when we mark the rings another way:

[|A| |B|] [|C| |D| |E| |F|].

Here we indicate, overall, not individual rings as the unit, but collections of rings, and specifically pairs and quartets of them. But we also distinguish the collections from their contained, individual physical objects, and do so while allowing *only the latter* to be individual objects.

This point about the substantiality of units is in line with Frege's point about number words and their dependence on a following count noun. For Frege pointed out that 52 cards are 1 pack, while 4 suits are 1 pack also, and that specific Fregean point was what was extended above to show that such formal set words as 'pair', and 'quartet' are comparable to units of amount, like 'ounce' and 'pound'. Such units of discrete or continuous measure are not terms with a direct physical reference, although when attached to an appropriate count or mass term a compound is formed with such a reference. Thus an ounce of tobacco might be found in a pouch, and a pair of gloves might be located in a drawer. But, as we have seen, the grammar of 'a pair of apples' requires that no further individual object is involved than the apples, even though there is a further singular referential phrase, 'the pair of apples', referring, naturally enough, to *a pair* of individual objects. More significantly still, that pair of individuals can now be seen to be as 'physical' as an individual apple, once such individual objects are distinguished from their material content. Certainly a physical collection is a mereological sum, as we saw before — thus a shoal is the sum of its fish — but once one realises that sets can be physical, one sees that that does not lead to the abandonment of Set Theory's applicability to such cases. For one must not miss the possibility that individual fish, themselves, should be sets — singleton sets — allowing Lewis' mereological analysis in *Parts of Classes* to apply directly. Specifically, if $\{f_1\}$, $\{f_2\}$, $\{f_3\}$, etc. are the fish, then the shoal, s, is

$\{f_1, f_2, f_3, \ldots\}$,

which means that

$\{f_1\} < s$,

etc., as we saw before with beasts in a herd.

One further consequence, however, is that it is now apparent there are no more individuals in a set than in its transitive closure. One might try to distinguish, for instance, the power set of a pair by writing '$\{\emptyset, \{A\}, \{B\}, \{A, B\}\}$' in opposition to '$\{A, B\}$', thinking of the former as containing four individuals. But the reader is encouraged to try doing the same thing with a pair of apples. The difficulty is that, in the power-set symbol the repeated 'A's, etc., are tokens of the same type, but with the actual objects supposedly symbolised there can be no such repetition, since the type-token distinction does not apply. Only a Formalist could miss the difference. With a physical collection of objects all one can do is indicate successively all the single individuals in the collection, and then all the pairs, triplets, quartets, etc. of those individuals. But all this is hand waving in front of the objects, and only the original individuals are involved. Cantor's Theorem, of course, is not in danger, since one can still distinguish the number of parts of a collection from the number of its singleton parts. But its proof becomes an exercise in elementary Permutations and Combinations, rather than the counting of different individuals.

The above discriminations may seem to be concerned merely with the application of Set Theory. But several large formal consequences also emerge. Thus, for a start, they show that there is no problem with any analogue of Russell's Paradox on the present understanding of sets (answering, amongst other things, Oliver and Smiley's arguments in this connection, see Oliver and Smiley 2001, 301–5). For we now have a principled way of avoiding this paradox, and all related paradoxes, because the fault in connection with the Abstraction Axiom has very evidently come from taking it that

$(x)(x \in \{y\colon Fy\} \equiv Fx)$

holds for all 'F', whereas we now know that it only holds for count predicates. So while substitution of

'$\lambda z(z \notin z)[\ ]$'

for 'F', for instance, and then

'{y: y $\notin$ y}'

for 'x', produces a contradiction, the conclusion merely has to be that the substituted lambda expression is not a count predicate. In fact, as we shall see later, it is not a predicate at all, and refers to a state of affairs rather than a property, but still, if

(x)(x $\notin$ x),

as is standard in Bernays and von Neumann style set theories, then 'x $\notin$ x' is equivalent to 'x=x', and it is provable quite independently that this is not a count predicate (Hale and Wright 2001, 315). For conjoining a mass term with a count term invariably results in a count term: 'red socks' is count because 'socks' is count. But conjoining a mass term with 'x=x' leaves one with the original mass term, since the addition of 'x=x' produces an equivalent. Hence 'x=x' cannot function like 'socks'. Notice that that does not mean there is 'Limitation of Size' in the usual sense, since such an expression as 'x=x' now does not define a proper class rather than a set. Rather it relates to some stuff. It means that 'λx(x=x)[o]' says not that o is a member of something but simply that it is a part of something. It says that o < u, where 'u' denotes the whole of the material universe.

The reverse point bears on the possibility of creating a theory of 'pure sets', and thereby a theory of what would certainly be entirely abstract objects. For the starting point of this would have to be a definition of the null set, in terms of some paradigm expression that, appropriately enough, cannot have any application. It is common to take 'x $\neq$ x', to do this job, as we have seen, although there are many equivalent others. But the present question, again, is whether any such expression is a count predicate, for only then could it determine a set. No proof of this has been given, to date, or has even been attempted. Indeed, it has simply been presumed that all such expressions are count predicates, and so give rise to sets, or, at least, set-like totalities. But there are also mass terms, and these give rise to urelemente, like the marble in a statue and the gold in a ring, which themselves have no members. That means one cannot define the null set, either, via the Axiom of Extensionality in the form

(x)(x $\in$ y $\equiv$ x $\in$ z) $\equiv$ y=z,

since this would make all urelemente the same, and so must be restricted to the case where y and z have members. Also it must be restricted to sets whose members are not themselves urelemente. For the same material might take different forms at different times, as we have seen, allowing it to be the case that

$$\{m\}_t \neq \{m\}_{t'}$$

even though these two objects have (timelessly) the same single member, i.e.

$$(x)(x \in \{m\}_t \equiv x \in \{m\}_{t'}).$$

That means, in particular, that the sets y and z in the above axiom cannot be physical individuals. But that is quite in tune with standard Set Theory, since the given axiom is commonly applied just to abstract objects, such as numbers.

These requirements might still seem strange, but they parallel some previously made points that connect count terms and sets. For, to allow for the pluralisation of count terms, there has to be at least the possibility of two instantiations before a description is count, ruling out standard definitions of the null set straightaway. But also proper names, and definite descriptions like 'The Queen of Britain in 1710', which identify individuals, necessarily are not pluralised, and cannot be used for counting, so they can only determine degenerate sets. For while one can say 'She was Anne' one cannot say 'She was one Anne', even though one can say 'She was one person called "Anne"' — but that is different.

Thus we see that, while discovery of Russell's paradox was the worst crisis in Set Theory's history, it was not until much later that proper progress was made on the question of just what had caused it, and thereby what adjustment was needed to satisfactorily remedy it. Recent writers like Oliver and Smiley, indeed, have gone so far as to abandon this theory in favour of a logic of plurals, on account of the more widespread, but we can now see intimately related fault of its abstraction. But it is, in fact, through seeing how collective plural constructions are exactly equivalent to ones involving set expressions that we start to remove Set Theory's remoteness from the world. And that removal is completed once we see the significance of the separation between things and stuff, i.e. individuals and material, since we then see that there are collective singular expressions, equivalent to ones that involve set expressions, but also other singular expressions, which are distributive, and have no representation in terms of sets. That clarifies what the material of the world consists in, and leads to a Set Theory that is applicable, where it is applicable, to real life.

Was *Principia Mathematica* written by Russell and Whitehead, or by the pair of them, {R, W}? You can say what you like, as long as you adjust the number of the verb appropriately. If it turns out that Russell and Whitehead have written another book it will be a great surprise, but if it turns out that that pair has written another book it will be exactly the same surprise. More importantly, perhaps, the full analysis of the singular case shows that single individuals are sets already — singleton sets — and so are not the ontological primitives many have imagined them to be. In particular they are not ever some material, even though they might consist in some material, i.e. have some stuff as their only member.

With regard to Oliver and Smiley's justification for a logic of plurals, we have thus seen the invalidity of their main argument for the irreducibility of collective plurals. But there is also a clear formal difficulty with Oliver and Smiley's plural account, connected with this, as well. For, in line with their idea that plural subjects are not replaceable with singular subjects in collective predications, they would want to say, for instance, that nothing sums (sic) to 10, but instead that what sum to 10 are, for instance, 2 and 8, 3 and 7, 4, and 6, etc., where '2 and 8' and the rest are plural terms. But how can they collect into an appropriate set the relevant values of such terms? Once we remember that some pairs of numbers do sum to 10, we can use standard set theory to collect up {2, 8}, {3, 7}, {4, 6}, etc. into a set of sets each of which sums to 10. One subset of this is, for instance, {{2, 8}, {3, 7}}. But Oliver and Smiley must try to talk about 2 and 8, and 3 and 7 in place of this subset, even though this collection of four numbers is the same as 2 and 3 and 8 and 7. And what are they going to say: that 4 and 6 is (sic) a member of the set of things that sum to 10? Surely not, since '4 and 6' is a plural subject. Likewise with

The men, the women, and the children are each 10 in number.

In place of the subset

{{x: Mx}, {x: Wx}}

of

{y: y is 10 in number},

they would have to have the men and the women together, i.e.

{the men, the women}

as a subset of

{y: y are 10 in number}.

But that subset numbers 20, in this case. And they would have to have the men being a member of something, mixing a plural subject with a singular verb, again. Without the 'singularisation' provided by set expressions there is no way that Oliver and Smiley can handle these higher-order idioms, and indeed there is no sign that they have considered them. But clearly, if a set consists in these, and those, and those others, then each of these, those, and those others has to be a member of the set, since any member of it has to be singular.

That is not the only further problem with Oliver and Smiley's account. For trying to re-phrase individual cases so that the subjects are irreplaceably plural, to allow for causal effects, also has its problems. Oliver and Smiley, in fact seem to have produced a rather jumbled, and inconclusive collection of arguments against singular, set-theoretic re-phrasings of collective plurals in particular cases. Certainly this is not possible with

Tom, Dick and Harry are similar to one another / each other,

but then these are distributive cases, surely, since

Each of Tom, Dick and Harry is similar to the others

is

$$(x)(x \in \{T, D, H\} \supset (y)([y \neq x.y \in \{T, D, H\}] \supset Lxy)).$$

Likewise with

Tom, Dick and Harry are shipmates,

once a preceding quantified phrase, 'There is a ship', i.e. '$(\exists z)(Sz\ldots$', is inserted as an extra conjunct before

$$(x)(x \in \{T, D, H\} \supset (y)([y \neq x.y \in \{T, D, H\}] \supset Mxyz))).$$

And what is the difficulty with

Tom, Dick and Harry carry the piano upstairs

being equivalent to something like

The gang {Tom, Dick, Harry}, carries the piano upstairs?

In another case Oliver and Smiley argue in this fashion: Tom and Dick are two, but the pair of them is one, so Tom and Dick cannot be the pair. But is it even grammatical to say

Tom and Dick are two,

as Oliver and Smiley repeatedly do? As before, Frege pointed out that number words are second-order quantifiers, requiring a first-order predicate to be supplied before they are determinate: a yard is 3 feet but 36 inches, a year is 52 weeks but 365 days. Certainly, therefore, in Oliver and Smiley's case, Tom and Dick may be *two men*, but with the unit of the counting then being specified, we see that those two men can easily be a pair of men, without 2 being 1. None of these cases, therefore, shows that plural collective predications are not equivalent to singular ones about the associated sets. Indeed, at one point, Oliver and Smiley admit that a set analysis is appropriate, but then go on (Oliver and Smiley 2001, 297):

> Fair enough, but other cases cause insuperable difficulties, such as plural proper names like "the Hebrides" and "Aberedw Rocks". Please do not say that these names are really descriptions built from the predicates "is a Hebrid" and "is an Aberedw Rock".

Here they seem to forget that maybe these rather arbitrary collections of things are better listed, than described. So all of this more particular argument by Oliver and Smiley is, at the very least, highly debatable.

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