Logique & Analyse **200** (2007), 367–389

AXIOMS FOR ACTION

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In a preceding paper [LT], starting from von Wright's ideas, we have proposed to investigate the idea that action depends in a fundamental way on a set of conditions. This has led us to define action as a mapping from a set of conditions, essentially described as a set of incompatible formulas, to results, described as formulas. The aim of the present paper is to explore some axiomatizations inspired by that approach. The systems will be presented in the algebraic style, because we think that it is well adapted to our setting, but the interested reader will have no difficulty in translating it into a more traditional propositional language style. In any case, that algebraic style will not prevent us from giving intuitive support to our axioms, when we deem it necessary or simply useful.

In the first section, we slightly generalize the second part of [LT], by giving an algebraic definition of "explicit algebras of actions". No proofs will be given there, because they are rather standard and are already hinted at in our preceding paper. The definitions however are important, because they support the intuition of action conceived as a mapping from a set of conditions to results and they give an accurate account of their rich structure ; that very intuition remains our "concrete" guide for the rest of the paper and hopefully for future work.

The second section defines the notion of "support algebra", which reproduces in a unique more familiar structure the behavior of our explicit algebras of action, which need three different structures for their description. Support algebras are not yet the structures which we think best suited to the study of action, but they are presupposed in more adequate richer structures presented in sections 3, 4 and 5. We show that in simple cases, support algebras correspond exactly to explicit algebras of actions.

The third section defines "support algebras with truth-value supports", taking $S_0^{=0}$ as a primitive symbol ; $S_0^{=0}$ represents the set of conditions on which an action would lead to a contradiction, i.e. it represents the set of conditions on which an action is impossible. A notable feature of $S_0^{=0}$ is that it is axiomatized by a very simple adjointness property and that allows one to define all

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sorts of supports and all operations of [LT], in particular a typical negation $\sim$ of action and an implication $\rightarrow$ between actions.

Truth-value supports however represent a rather extrinsic point of view on action and they are still too close mimics of explicit algebras of actions. A more intrinsic point of view would be welcomed. That is why we propose in the fourth section a definition of "support algebras with complex negation", which takes the typical negation $\sim$ of action as a primitive. Here again, it is to be noted that complex negation is axiomatized by a familiar adjointness property and that support algebras with complex negation are equivalent to support algebras with truth-value supports.

We propose in the fifth section yet another axiomatization based on implication and its familiar characterization in terms of adjointness. Support algebras with implication are equivalent to support algebras with complex negation, hence also equivalent to support algebras with truth-value supports.

Apart from the elementary theory of Boolean algebras, the present paper is largely self-contained. That does not mean however that it has no relation with other works. We mention some of them in our conclusion, together with some indications on the similarities and the differences and we briefly mention some potential extensions of our approach.

1. *Explicit algebras of actions*

We think that an action α should be considered as a mapping defined on a set Σ of conditions and as associating to each condition $\sigma \in \Sigma$ a certain result $\alpha(\sigma)$. In a first approach, conditions and results are states of the world which may be described by formulas in usual classical propositional logic, or by elements of a Boolean algebra in the algebraic approach which we propose here. We translate the idea that the action should be coherent by asking that, should conditions σ and σ' be compatible, the results $\alpha(\sigma)$ and $\alpha(\sigma')$ should then be the same. And we define an ordering between actions which is well-adapted to further discussions of obligation : $\alpha \leq \beta$, "doing the action α logically implies doing the action β", or "α entails β", if, roughly speaking, the conditions for β entail conditions for α, and for compatible conditions σ and π for α and β respectively, $\alpha(\sigma)$ entails $\beta(\pi)$. With that ordering, the set of actions inherits a very rich structure which we described in some detail in the second part of [LT] to which we refer the reader. We will however repeat here a bunch of precise definitions, dropping parts of the structure, slightly abstracting and putting things in an algebraic setting.

The definitions are relative to two Boolean algebras : B, the *Boolean algebra of conditions*, representing the structure of conditions, and C, the *Boolean algebra of results*, representing the structure of results. Distinguished elements, operations and relations of B and C are denoted by 0, 1, $\neg$, $\wedge$, $\vee$, $\leq$, occasionally with indices B or C when clarity makes it desirable.

Definition 1.1: *(1) An* action from B to C *is a mapping* $\alpha\ : dom\alpha \longrightarrow C$, *where* $dom\alpha \subseteq B$, $dom\alpha$ *is finite and* α *satisfies a* coherence condition *: for all* $\sigma, \sigma' \in dom\alpha$, *if* $\sigma \wedge \sigma' \neq 0$, *then* $\alpha(\sigma) = \alpha(\sigma')$.
(2) Actions are pre-ordered *by the relation* $\leq$ *defined by :* $\alpha \leq \beta$ *iff* $\bigvee dom\beta \leq \bigvee dom\alpha$ *and for all* $\sigma \in dom\alpha$, $\pi \in dom\beta$, *if* $\sigma \wedge \pi \neq 0$, *then* $\alpha(\sigma) \leq \beta(\pi)$.
(3) The pre-ordering $\leq$ *induces an* equivalence $\approx$ *of actions characterized by :* $\alpha \approx \beta$ *iff* $\bigvee dom\alpha = \bigvee dom\beta$ *and for all* $\sigma \in dom\alpha$, $\pi \in dom\beta$, *if* $\sigma \wedge \pi \neq 0$, *then* $\alpha(\sigma) = \beta(\pi)$.
(4) The empty *action* 1 *is defined by :* $dom(1) = \emptyset$ *and* 1 *is the empty mapping from* $dom(1)$ *to* C.
(5) The everywhere nul *or* zero *action* 0 *is defined by :* $dom(0) = \{1_B\}$ *(*1_B *representing the greatest element of* B*) and* $0(1_B) = 0_C$ *(*0_C *representing the smallest element of* C*).*
(6) Operations $\cdot$, $-$ *and* $+$ *on finite subsets* Σ *and* Π *of* B *are defined by :*
$\Sigma \cdot \Pi = \{\sigma \wedge \pi \mid \sigma \in \Sigma, \pi \in \Pi, \sigma \wedge \pi \neq 0\}$
$-\Sigma = \{\neg \bigvee \Sigma\}$
$\Sigma + \Pi = (\Sigma \cdot \Pi) \cup (-\Sigma \cdot \Pi) \cup (\Sigma \cdot -\Pi)$.
(7) The conjunction $\alpha \wedge \beta$ *of actions* α *and* β *is defined by :* $dom(\alpha \wedge \beta) = dom\alpha + dom\beta$ *and for all* $\omega \in dom(\alpha \wedge \beta)$, $(\alpha \wedge \beta)(\omega)$ *is defined according to the form of the domain by three cases :*
(a) if $\omega \in dom\alpha \cdot dom\beta$, *then* $\omega = \sigma \wedge \pi$ *for some* $\sigma \in dom\alpha$ *and* $\pi \in dom\beta$ *and one lets* $(\alpha \wedge \beta)(\omega) = \alpha(\sigma) \wedge \beta(\pi)$ *;*
(b) if $\omega \in -dom\alpha \cdot dom\beta$, *then* $\omega = \neg \bigvee dom\alpha \wedge \pi$ *for some* $\pi \in dom\beta$ *and one lets* $(\alpha \wedge \beta)(\omega) = \beta(\pi)$ *;*
(c) if $\omega \in dom\alpha \cdot -dom\beta$, *then* $\omega = \sigma \wedge \neg \bigvee dom\beta$ *for some* $\sigma \in dom\alpha$ *and one lets* $(\alpha \wedge \beta)(\omega) = \alpha(\sigma)$.
(8) The disjunction $\alpha \vee \beta$ *of actions* α *and* β *is defined by :* $dom(\alpha \vee \beta) = dom\alpha \cdot dom\beta$ *and for all* $\omega \in dom(\alpha \vee \beta)$, $\omega = \sigma \wedge \pi$ *for some* $\sigma \in dom\alpha$ *and* $\pi \in dom\beta$ *and one lets* $(\alpha \vee \beta)(\omega) = \alpha(\sigma) \vee \beta(\pi)$.
(9) The 0-cosupport $C_0\alpha$ *of action* α *is defined by :* $dom(C_0\alpha) = -dom\alpha = \{\neg \bigvee dom\alpha\}$ *and* $(C_0\alpha)(\neg \bigvee dom\alpha) = 0_C$.
(10) The negation $\neg\alpha$ *of action* α *is defined by :* $dom(\neg\alpha) = dom\alpha$ *and for all* $\sigma \in dom\alpha$, $(\neg\alpha)(\sigma) = \neg(\alpha(\sigma))$, *the latter negation being taken in* C.

We repeat here the main properties of that structure. The equivalence relation is compatible with the operations $\neg$, $\wedge$, $\vee$ and C_0, a fact which allows us to consider the quotient structure ; in practice, we identify equivalent actions, writing $\alpha = \beta$ instead of the more formal $\alpha \approx \beta$ and speak accordingly of the ordering $\leq$.

The everywhere nul action 0 is the smallest action and the empty action 1 is the greatest action for the ordering $\leq$, two facts which justify our notation for them. The conjunction $\alpha \wedge \beta$ is the infimum of α and β ; we emphasize here that it is so to speak a "long conjunction" ; it is defined not only on the common set of circumstances $dom\alpha \cdot dom\beta$ where it prescribes to do both α and β, but when α is defined and β is not, it prescribes to do α and similarly when β is defined and α is not, $\alpha \wedge \beta$ prescribes to do β. By contrast, the disjunction $\alpha \vee \beta$ is defined only on the common set of circumstances $dom\alpha \cdot dom\beta$ and there, it prescribes to do at least one of the actions α and β. The negation $\neg\alpha$ is essentially "the opposite action of α in the same circumstances". The 0-cosupport $C_0\alpha$ of α is in general best considered as a sort of canonical representation of the complement of the (union of the) domain of α as an action.

The set of all actions from B to C equipped with the distinguished elements 0, 1, with the operations $\neg$, $\wedge$, $\vee$, C_0 and with the relation $\leq$ will be referred to as the explicit algebra of all actions from B to C. For further reference and comparison, we give a formal definition of the more general notion of "explicit algebra of actions" :

Definition 1.2: *An* explicit algebra of actions $\mathcal{B}$ *is a triple* $< B, E, C >$ *where* B *and* C *are Boolean algebras and* E *is a* $< 0, 1, \neg, \wedge, \vee, C_0 >$*-subalgebra of the explicit algebra of all actions from* B *to* C.

Note that in explicit algebras of actions, interesting derived operations may be obtained by duality, via $\neg$: $\alpha \leq^* \beta$ iff $\neg\beta \leq \neg\alpha$, $\alpha \wedge^* \beta = \neg(\neg\alpha \vee \neg\beta)$ (the *short conjunction*, defined on $dom\alpha \cdot dom\beta$), $\alpha \vee^* \beta = \neg(\neg\alpha \wedge \neg\beta)$ (the *long disjunction*, defined on $dom\alpha + dom\beta$), $1^* = \neg 0$ (the *everywhere unit* action), $C_1\alpha = \neg C_0\neg\alpha = \neg C_0\alpha$ (the 1-*cosupport* of α). Note in particular that $\neg 1 = 1$, so that the empty action is auto-dual. Other derived notions are given by : $S_0\alpha = C_0C_0\alpha$ (the 0-*support* of α) and its dual $S_1\alpha = \neg S_0\neg\alpha = \neg S_0\alpha$ (the 1-*support* of α), $\gamma \setminus \beta = C_0\beta \wedge (\gamma \wedge^* \neg\beta)$ (the *difference* of γ and β) and its dual $\beta \rightarrow^* \gamma = C_1\beta \vee^* (\neg\beta \vee \gamma)$ (the *co-implication* from β to γ). For an explicit description of these notions, we refer the reader to [LT].

On the other hand, there are very fundamental operations on actions which make sense in explicit algebras of actions, but do not seem to be derivable from the preceding operations. We will study them in sections 3, 4 and 5, but mention their explicit definition here :

Definition 1.3: *(1) The* 0-value 0-support *of* α, $S_0^{=0}\alpha$, *is defined by* $dom(S_0^{=0}\alpha) = \{\sigma \mid \sigma \in dom\alpha, \alpha(\sigma) = 0\}$ *and for* σ *in that domain,* $(S_0^{=0}\alpha)(\sigma) = 0$.
(2) The complex negation *of* α, $\sim\alpha$, *is defined by* $dom(\sim\alpha) = -\Sigma \cup \Sigma_{\neq 0}$,

where $\Sigma = dom(\alpha)$ *and* $\Sigma_{\neq 0} = \{\sigma \mid \sigma \in dom\alpha, \alpha(\sigma) \neq 0\}$ *; for* $\sigma \in -\Sigma$, *i.e.* $\sigma = \neg \bigvee \Sigma$, *one lets* $(\sim\alpha)(\sigma) = 0$ *; for* $\sigma \in \Sigma_{\neq 0}$, *one lets* $(\sim\alpha)(\sigma) = \neg(\alpha(\sigma))$.
(3) The implication *from* β *to* γ, $\beta \to \gamma$, *is defined by* $dom(\beta \to \gamma) = (-\Pi \cdot \Xi) \cup (\Pi/\Xi)$, *where* $\Pi = dom\beta$, $\Xi = dom\gamma$ *and* $\Pi/\Xi = \{\pi \wedge \xi \mid \pi \in \Pi, \xi \in \Xi, \pi \wedge \xi \neq 0$ *and* $\beta(\pi) \not\leq \gamma(\xi)\}$ *; for* $\omega \in -\Pi \cdot \Xi$, *one has* $\omega = \neg \bigvee \Pi \wedge \xi$ *for some* $\xi \in \Xi$ *and one lets* $(\beta \to \gamma)(\omega) = \gamma(\xi)$ *; for* $\omega \in \Pi/\Xi$, *one has* $\omega = \pi \wedge \xi$ *for some* $\pi \in \Pi$, $\xi \in \Xi$ *and one lets* $(\beta \to \gamma)(\omega) = \beta(\pi) \to \gamma(\xi)$.

Here is why those operations are basic. The operation $S_0^{=0}$ picks in the domain of an action α those conditions σ for which $\alpha(\sigma) = 0$, i.e. for which $\alpha(\sigma)$ is contradictory ; in other words, it codifies our capacity to recognize conditions under which an action is impossible to execute ; it seems fair to admit that that operation is a fundamental ingredient of a theory of action. The complex negation $\sim$ is indeed a combined negation : when applied to α, it exhibits the codomain of α, it drops the part where α is impossible to execute and gives the "usual" negation of α where α is defined and possible to execute. We will see later that $S_0^{=0}$ and $\sim$ are interdefinable on the basis of the preceding operations, but we want to have a close look at $\sim\alpha$ for at least two reasons. The first one is that we think that "the negation" of an action is an ambiguous concept which is in bad need of a clarification : $\sim\alpha$ is a possible explanans, $\neg\alpha$ is another one, $C_0\alpha$ is a third more crude one and we could also consider the duals $\sim^*\alpha = \neg\sim\neg\alpha$ and $C_1\alpha = \neg C_0\neg\alpha = \neg C_0\alpha$. The second reason is that $\sim\alpha$ is a genuine intuitionistic negation ; it indeed satisfies the powerful adjointness property familiar to all students of intuitionism : $\beta \leq \sim\alpha$ iff $\alpha \wedge \beta \leq 0$. The third operation, implication, is a generalization of $\sim$ and it satisfies also a familiar adjointness property : $\alpha \leq \beta \to \gamma$ iff $\alpha \wedge \beta \leq \gamma$. Implication is interdefinable with $\sim$, hence also with $S_0^{=0}$, but we like to mention it, because we do not exclude that it be more adequate to start with when attempting to generalize the present theory to less classical contexts. Note finally that those three operations give many interesting derived operations when combined with $\neg$.

2. *Support algebras*

Support algebras are obtained as a unisorted version of explicit algebras of actions. Here is the definition.

Definition 2.1: *A* support algebra $\mathcal{A}$ *is an octuple* $< A, 0, 1, \neg, \wedge, \vee, C_0, \leq>$ *where* A *is a set,* 0 *and* 1 *are elements of* A, $\neg$ *and* C_0 *are unary operations*

on A, $\wedge$ and $\vee$ are binary operations on A and $\leq$ is a binary relation on A satisfying the following properties :
($Dlatt$) the structure $< A, 0, 1, \wedge, \vee, \leq>$ is a distributive lattice with smallest element 0 and greatest element 1
($AdjC_0$) $C_0\alpha \leq \beta$ iff $1 \leq \alpha \vee \beta$
($AxC_0\vee$) $C_0\alpha \wedge C_0\beta \leq C_0(\alpha \vee \beta)$
($AxC_0\neg$) $\alpha \wedge \neg\alpha = C_0C_0\alpha$
($Ax\neg\neg$) $\neg\neg\alpha = \alpha$
($Ax\neg 1$) $\neg 1 = 1$
($Ax\neg\wedge$) $\neg(\neg\alpha \wedge \neg\beta) = (\alpha \vee \beta) \wedge (\alpha \vee C_0\beta) \wedge (C_0\alpha \vee \beta)$
($Ax\neg Rest$) $\neg(\alpha \vee C_0\beta) \leq \neg\alpha \vee C_0\beta$.

Derived relations and operations may be introduced by the same definitions as before :

Definition 2.2: $\alpha \leq^* \beta$ *iff* $\neg\beta \leq \neg\alpha$
$\alpha \wedge^* \beta = \neg(\neg\alpha \vee \neg\beta)$
$\alpha \vee^* \beta = \neg(\neg\alpha \wedge \neg\beta)$
$1^* = \neg 0$
$C_1\alpha = \neg C_0\alpha$
$S_0 = C_0C_0$
$S_1 = \neg S_0$
$\gamma \setminus \beta = C_0\beta \wedge (\gamma \wedge^* \neg\beta)$
$\beta \rightarrow^* \gamma = C_1\beta \vee^* (\neg\beta \vee \gamma)$.

For all those operations, primitive or derived, we adopt the same terminology as in explicit algebras of actions and we present in the sequel some comments on the axioms and their consequences. Most results will be stated without proofs, because they may be routinely deduced from the axioms. Note also that in our comments, we often speak of $dom\alpha$ as if it were $\bigvee dom\alpha$, speak of $dom\alpha \cdot dom\beta$ as being the intersection of $dom\alpha$ and $dom\beta$, etc. In general, that should not lead to confusion.

The axiom ($AdjC_0$) is a very powerful adjointness property ; it embodies a fundamental property of supports : the support of β is contained in the cosupport of α if and only if α and β have disjoint supports. To see that, recall that in the explicit case, $\alpha \leq \beta$ entails that the domain of β is contained in the domain of α, 1 is the empty action, $\alpha \vee \beta$ is defined on the common domain of α and β, so that $1 \leq \alpha \vee \beta$ means that the support of $\alpha \vee \beta$ is empty. Together with the axioms of distributive lattice ($Dlatt$) and the definition $S_0 = C_0C_0$, it has the following consequences, which are all obvious when interpreted in terms of supports.

Theorem 2.3: $\alpha \vee C_0\alpha = 1$
$C_0\alpha \vee S_0\alpha = 1$
$S_0\alpha \leq \alpha$
$\alpha \leq \beta$ *entails* $C_0\beta \leq C_0\alpha$
$C_0S_0\alpha = S_0C_0\alpha = C_0\alpha$
$\alpha \leq \beta$ *entails* $S_0\alpha \leq S_0\beta$
$C_0(\alpha \wedge \beta) = C_0\alpha \vee C_0\beta$
$S_0(\alpha \vee \beta) = S_0\alpha \vee S_0\beta$
$C_01 = 0$
$C_00 = 1$
$S_01 = 1$
$S_00 = 0$
$S_0(\alpha \wedge \beta) \leq S_0\alpha \wedge S_0\beta$.

If we want to obtain stronger properties, we have to rely on $(AxC_0\vee)$ which essentially means that the cosupport of the short disjunction $\alpha \vee \beta$ is contained in the union of the cosupports of α and β ; recall indeed that in the explicit case, the long conjunction is defined on the "union" of the domains. This gives us other fundamental properties of supports :

Theorem 2.4: $C_0(\alpha \vee \beta) = C_0\alpha \wedge C_0\beta$
$S_0(\alpha \wedge \beta) = S_0\alpha \wedge S_0\beta$
$C_0\alpha \wedge \beta \leq \gamma$ *iff* $\beta \leq \gamma \vee S_0\alpha$ *(adjointness property* $(AdjC_0S_0)$*)*
$C_0\alpha \wedge S_0\alpha = 0$.

The meaning of $(AxC_0\neg)$ is obvious : the support $S_0\alpha(= C_0C_0\alpha)$ of α is equal to the conjunction $\alpha \wedge \neg\alpha$. To see that, recall that in the explicit case, α and $\neg\alpha$ have exactly the same domain, so that the conjunction $\alpha \wedge \neg\alpha$ is defined on that domain and has 0 as a result. The axiom has indeed the highly desirable consequences that the cosupport of an action coincides with the cosupport of its negation, and similarly for supports :

Theorem 2.5: $C_0\neg\alpha = C_0\alpha$
$S_0\neg\alpha = S_0\alpha$.

The set of 0-supports of elements of a support algebra is stable for the operations C_0, $\wedge$, $\vee$ and contains the elements 1 and 0. It is quite clear that the properties which have been stated so far show that these operations and elements, together with the reverse ordering of $\leq$, endow it with a structure of Boolean algebra ; C_0 plays the role of the complement ; the short disjunction, corresponding to the "intersection" of the domains, plays the role of the infimum ; the long conjunction, corresponding to the "union" of the domains, plays the role of the supremum ; the element 1, the empty action, is the minimum ; the element 0, the everywhere nul action, precisely because it is everywhere defined, is the maximum. Let us state that in a definition and a theorem.

Definition 2.6: *Given a support algebra* $\mathcal{A} =< A, 0, 1, \neg, \wedge, \vee, C_0, \leq>$, *the* structure $S_0\mathcal{A} =< S_0A, 0', 1', \neg', \wedge', \vee', \leq'>$ of 0-supports of $\mathcal{A}$ *is given by* $S_0A = \{S_0\alpha \mid \alpha \in A\}$ *;* $0' = 1$ *;* $1' = 0$ *;* $\neg', \wedge', \vee'$ *are the restrictions to* S_0A *of* $C_0, \vee, \wedge$ *respectively ;* $\leq'$ *is the restriction to* S_0A *of the inverse ordering of* $\leq$ *(i.e.* $\alpha \leq' \beta$ *iff* $\beta \leq \alpha$*).*

Theorem 2.7: *The structure* $S_0\mathcal{A}$ *of* 0*-supports of* $\mathcal{A}$ *is a Boolean algebra.*

The interest of that property is that it shows that a support algebra contains “internally” the structure of supports of its elements, a structure which, in the case of explicit algebras, is given “outside” by the Boolean algebra of conditions B.

If we want to avoid the reversal of ordering, we may use our definitions of derived notions $\leq^*$, $\wedge^*$, $\vee^*$, 1^*, C_1, S_1 together with axioms $(Ax\neg\neg)$ and $(Ax\neg 1)$. The main interest of those notions and axioms is to show that support algebras have a built-in duality given by $\neg$. As usual, we have a duality principle :

Duality principle 2.8: *The dual* T^* *of a term* T *constructed on* $1, 0, \neg, \wedge, \vee,$ $C_0, S_0, 1^*, \vee^*, \wedge^*, C_1, S_1$ *is obtained by exchanging in it dual notions (*0 *and* 1^*, $\wedge$ *and* $\vee^*$, $\vee$ *and* $\wedge^*$, C_0 *and* C_1, S_0 *and* S_1 *;* 1 *and* $\neg$ *are auto-dual and remain unchanged). Every provable inequation* $T \leq U$ *gives by duality the provable* $U^* \leq^* T^*$. *Every provable equation* $T = U$ *gives by duality the provable* $T^* = U^*$.

As an application of that principle, we may dualize algebras of 0-supports $S_0\mathcal{A}$ into algebras of 1-supports $S_1\mathcal{A}$. We will prefer these algebras because they avoid the use of inverse notions :

Definition 2.9: *Given a support algebra* $\mathcal{A} =< A, 0, 1, \neg, \wedge, \vee, C_0, \leq>$, *the* structure $S_1\mathcal{A} =< S_1A, 0', 1', \neg', \wedge', \vee', \leq'>$ of 1-supports of $\mathcal{A}$ *is given by* $S_1A = \{S_1\alpha \mid \alpha \in A\}$ *;* $0' = 1$ *;* $1' = 1^*$ *;* $\neg', \wedge', \vee', \leq'$ *are the restrictions to* S_1A *of* $C_1, \wedge^*, \vee^*$ *and* $\leq^*$ *respectively.*

Theorem 2.10: *The structure* $S_1\mathcal{A}$ *of* 1*-supports of* $\mathcal{A}$ *is a Boolean algebra.*

We now turn to axiom $(Ax\neg\wedge)$ which gives a fundamental connection between negation, long conjunction and short disjunction ; or using dual notions, it may be written as a connection between long disjunction and short disjunction : $\alpha \vee^* \beta = (\alpha \vee \beta) \wedge (\alpha \vee C_0\beta) \wedge (C_0\alpha \vee \beta)$. This means that long disjunction is obtained by piecing together the short disjunction $\alpha \vee \beta$ on the common support $S_0\alpha \vee S_0\beta$, the action α when β is undefined and the action β when α is undefined. As a first consequence, note that for actions having the same support, duality trivializes :

Theorem 2.11: *If α and β have the same support ($S_0\alpha = S_0\beta$ or equivalently $S_1\alpha = S_1\beta$), then $\alpha \vee^* \beta = \alpha \vee \beta$, $\alpha \wedge^* \beta = \alpha \wedge \beta$ and $\alpha \leq^* \beta$ iff $\alpha \leq \beta$.*

Proof. One has :

$$\begin{array}{rll} 1 & = S_0\beta \vee C_0\beta & \text{by Theorem 2.3} \\ & = S_0\alpha \vee C_0\beta & \text{because } S_0\alpha = S_0\beta \\ & \leq \alpha \vee C_0\beta & \text{because } S_0\alpha \leq \alpha. \end{array}$$

Hence $\alpha \vee C_0\beta = 1$. Symmetrically, $\beta \vee C_0\alpha = 1$. We now apply axiom ($Ax\neg\wedge$) to obtain : $\alpha \vee^* \beta = (\alpha \vee \beta) \wedge 1 \wedge 1 = \alpha \vee \beta$. By duality, this last result gives also $\alpha \wedge^* \beta = \alpha \wedge \beta$. Finally,

$$\begin{array}{rll} \alpha \leq^* \beta & \text{iff } \alpha \wedge^* \beta = \alpha & \\ & \text{iff } \alpha \wedge \beta = \alpha & \text{because } \alpha \wedge^* \beta = \alpha \wedge \beta \\ & \text{iff } \alpha \leq \beta. & \end{array}$$

□

An important consequence of that property is that we can show that the set of elements having the same given 0-support $a = S_0a$ is naturally endowed with a structure of Boolean algebra, this time with $\neg$ as negation, $\wedge$ (or $\wedge^*$) as infimum and $\vee$ (or $\vee^*$) as supremum ; the smallest element of that Boolean algebra is given by a and the greatest element is given by $\neg a$. Note that we slightly change the notation here, using Latin letters instead of Greek letters for actions which we like to think of as supports, i.e. actions a such that $a = S_0a$. Sometimes it is also convenient to speak in terms of 1-supports ; we give the corresponding adaptation between parentheses.

Definition 2.12: *Given a support algebra $\mathcal{A} =< A, 0, 1, \neg, \wedge, \vee, C_0, \leq>$ and an element a of S_0A , the* structure $\mathcal{A}_a =< A_a, 0_a, 1_a, \neg_a, \wedge_a, \vee_a, \leq_a>$ of elements of 0-support a *is given by $A_a = \{\alpha \in A \mid S_0\alpha = a\}$, $0_a = a$, $1_a = \neg a$, and $\neg_a, \wedge_a, \vee_a, \leq_a$ are the restrictions to A_a of $\neg, \wedge, \vee, \leq$ respectively. (Given an element a of S_1A, the* structure $\mathcal{A}'_a$of elements of 1-support a *is the structure of elements of* 0*-support* $\neg a$*).*

Theorem 2.13: *For any* 0*-support $a \in S_0A$, the structure $\mathcal{A}_a$ of elements of* 0*-support a is a Boolean algebra. (For any* 1*-support $a \in S_1A$, the structure $\mathcal{A}'_a$ of elements of* 1*-support a is a Boolean algebra.)*

Proof. Let us check some of the critical properties in the case of 0-supports. The element a is indeed the smallest, because for any $\alpha \in A_a$, $a = S_0\alpha \leq \alpha$, hence $a \leq \alpha$, i.e. $a \leq_a \alpha$. The element $\neg a$ is the greatest, because for any $\alpha \in A_a$, we have $\neg\alpha \in A_a$ and $a \leq \neg\alpha$; hence $\alpha \leq^* \neg a$ by the definition of $\leq^*$ and $(Ax\neg\neg)$, $\alpha \leq \neg a$ by theorem 2.11, i.e. $\alpha \leq_a \neg a$. For any $\alpha \in A_a$, $\alpha \wedge \neg\alpha = 0_a$, because $\alpha \wedge \neg\alpha = S_0\alpha = a = 0_a$. For any $\alpha \in A_a$, $\alpha \vee \neg\alpha = 1_a$, because

$$\begin{aligned} \alpha \vee \neg\alpha &= \alpha \vee^* \neg\alpha && \text{by theorem 2.11} \\ &= \neg(\neg\alpha \wedge \alpha) && \text{by the definition of } \vee^* \\ &= \neg 0_a && \text{by what precedes} \\ &= \neg a \\ &= 1_a. \end{aligned}$$

De Morgan's laws are similarly obtained by trivializing duality :

$$\begin{aligned} \neg(\alpha_1 \wedge \alpha_2) &= \neg\alpha_1 \vee^* \neg\alpha_2 \\ &= \neg\alpha_1 \vee \neg\alpha_2 && \text{by theorem 2.11.} \end{aligned}$$

□

The interest of theorem 2.13 is that it shows that for a given support a, the set of actions with support a is fundamentally classical ; the support algebra embeds thus a whole bunch of algebras $\mathcal{A}_a$, one for each a. More comments on this later.

Here are other consequences of $(Ax\neg\wedge)$ expressed in dual pairs, except for (3), which is auto-dual.

Theorem 2.14: *(1) Distributivity properties :*

$(\alpha \vee^* \beta) \vee \gamma = (\alpha \vee \gamma) \vee^* (\beta \vee \gamma)$

$(\alpha \wedge \beta) \wedge^* \gamma = (\alpha \wedge^* \gamma) \wedge (\beta \wedge^* \gamma)$

(2) Monotony properties :

$\alpha \leq \beta$ *entails* $\alpha \wedge^* \gamma \leq \beta \wedge^* \gamma$

$\alpha \leq^* \beta$ *entails* $\alpha \vee \gamma \leq^* \beta \vee \gamma$

$\alpha \leq \beta$ *and* $\alpha' \leq \beta'$ *together entail* $\alpha \wedge^* \alpha' \leq \beta \wedge^* \beta'$

$\alpha \leq^* \beta$ *and* $\alpha' \leq^* \beta'$ *together entail* $\alpha \vee \alpha' \leq^* \beta \vee \beta'$

(3) Co-disjunction property :

if α *and* β *are co-disjoint (i.e.* $\alpha \vee \beta = 1$ *or equivalently* $\alpha \wedge^* \beta = 1$*), then* $\alpha \vee^* \beta = \alpha \wedge \beta$

(4) Restriction properties :

$\alpha \vee \beta = (\alpha \vee^* \beta) \vee S_0\alpha \vee S_0\beta$

$\alpha \wedge^* \beta = (\alpha \wedge \beta) \wedge^* S_1\alpha \wedge^* S_1\beta$

(5) Comparison of $\wedge^*$ *and* $\vee$ *:*

$\alpha \wedge^* \beta \leq \alpha \vee \beta$

$\alpha \wedge^* \beta \leq^* \alpha \vee \beta$

(6) Comparison of $\leq$ *and* $\leq^*$ *:*

$\alpha \leq^* \beta$ *iff* $S_0\beta \leq S_0\alpha$ *and* $\alpha \leq \beta \vee S_0\alpha$

$\alpha \leq \beta$ *iff* $S_1\beta \leq^* S_1\alpha$ *and* $\alpha \wedge^* S_1\beta \leq^* \beta$.

Proof (sketch). For (1), use $(Ax\neg\wedge)$ and dualize. For (2), first part, reduce $\alpha \leq \beta$ to $\alpha \wedge \beta = \alpha$, apply $- \wedge^* \gamma$ to both sides and distribute by (1). For (2), second part, dualize. For (2), third and fourth parts : apply the first and second parts. For (3), prove that $\alpha \vee \beta = 1$ entails $\alpha \vee C_0\beta = \alpha$ and $C_0\alpha \vee \beta = \beta$, so that $(Ax\neg\wedge)$ will give : $\alpha \vee^* \beta = (\alpha \vee \beta) \wedge (\alpha \vee C_0\beta) \wedge (C_0\alpha \vee \beta) = (\alpha \vee \beta) \wedge \alpha \wedge \beta = \alpha \wedge \beta$. For(4), first part, compute $(\alpha \vee^* \beta) \vee S_0\alpha \vee S_0\beta$ by using $(Ax\neg\wedge)$. For (4), second part, dualize. For (5), first part, start from $\alpha \wedge \beta \leq \alpha \vee \beta$; then, by the monotony already proven in (2), $(\alpha \wedge \beta) \wedge^* S_1\alpha \wedge^* S_1\beta \leq (\alpha \vee \beta) \wedge^* S_1\alpha \wedge^* S_1\beta$; by restriction properties proven in (4), the left side equals $(\alpha \wedge^* \beta)$; the right side equals $(\alpha \vee \beta) \wedge^* S_1(\alpha \vee \beta)$, which is itself equal to $(\alpha \vee \beta)$. For (5), second part, dualize.

We prove (6), first part, with more details :

(a) $\alpha \leq^* \beta$ entails successively $\neg\beta \leq \neg\alpha$, $S_0\neg\beta \leq S_0\neg\alpha$ and $S_0\beta \leq S_0\alpha$;

(b) $\alpha \leq^* \beta$ entails $\alpha \leq \beta \vee S_0\alpha$; to prove that, assume $\alpha \leq^* \beta$ and compute $\alpha \vee \beta$:

$$\begin{aligned}\alpha \vee \beta &= (\alpha \vee^* \beta) \vee S_0\alpha \vee S_0\beta && \text{by (4)}\\ &= \beta \vee S_0\alpha \vee S_0\beta && \text{because } \alpha \leq^* \beta \text{ amounts to } \alpha \vee^* \beta = \beta\\ &= \beta \vee S_0\alpha && \text{because } \beta \vee S_0\beta = \beta \text{ ;}\end{aligned}$$

hence, $\alpha \leq \alpha \vee \beta = \beta \vee S_0\alpha$ and $\alpha \leq \beta \vee S_0\alpha$;

(c) $S_0\beta \leq S_0\alpha$ and $\alpha \leq \beta \vee S_0\alpha$ together entail $\alpha \leq^* \beta$; to prove that, assume the hypotheses and use $(Ax\neg\wedge)$ to compute $\alpha \vee^* \beta$:

$$\alpha \vee^* \beta = ((\alpha \vee \beta) \wedge (\alpha \vee C_0\beta) \wedge (C_0\alpha \vee \beta)$$

but

$$\begin{aligned}\alpha \vee \beta &= (\alpha \vee \beta) \vee S_0\alpha && \text{because } \alpha = \alpha \vee S_0\alpha\\ &= \beta \vee S_0\alpha && \text{because } \alpha \leq \beta \vee S_0\alpha \text{ ;}\end{aligned}$$

on the other hand, $S_0\beta \leq S_0\alpha$ entails $\alpha \vee C_0\beta = 1$; hence,

$$\begin{aligned}\alpha \vee^* \beta &= (\beta \vee S_0\alpha) \wedge 1 \wedge (C_0\alpha \vee \beta)\\ &= (\beta \vee S_0\alpha) \wedge (C_0\alpha \vee \beta)\\ &= \beta \vee (S_0\alpha \wedge C_0\alpha)\\ &= \beta \vee 0\\ &= \beta \text{ ;}\end{aligned}$$

it follows that $\alpha \leq^* \beta$.

For (6), second part, dualize. □

The relation between $\leq$ and $\leq^*$ given in point (6) of the preceding theorem allows us to prove strong connections between dual notions :

Theorem 2.15: *(1)* $S_0\alpha \leq^* 0$ *; dually,* $1^* \leq S_1\alpha$
(2) $S_0\alpha \leq^* \alpha$ *; dually,* $\alpha \leq S_1\alpha$
(3) $S_0\alpha \leq^* \alpha \wedge^* 0$ *; dually,* $\alpha \vee 1^* \leq S_1\alpha$
(4) $C_0\alpha \leq^* C_1\alpha$ *and* $S_0\alpha \leq^* S_1\alpha$ *; dually,* $C_0\alpha \leq C_1\alpha$ *and* $S_0\alpha \leq S_1\alpha$
(5) $S_1\alpha \wedge^* 0 \leq^* S_0\alpha$ *; dually,* $S_1\alpha \leq S_0\alpha \vee 1^*$

(6) $S_0\alpha = S_1\alpha \wedge^* 0$ *; dually,* $S_1\alpha = S_0\alpha \vee 1^*$
(7) $S_0\alpha = \alpha \wedge^* 0$ *; dually,* $S_1\alpha = \alpha \vee 1^*$
(8) $S_0\alpha \wedge^* \beta = S_0\alpha \wedge^* S_0\beta = S_1\alpha \wedge^* S_1\beta \wedge^* 0$ *; dually,* $S_1\alpha \vee \beta = S_1\alpha \vee S_1\beta = S_0\alpha \vee S_0\beta \vee 1^*$
(9) $\alpha \vee S_0\beta \leq \alpha \wedge^* S_1\beta$ *; dually,* $\alpha \vee S_0\beta \leq^* \alpha \wedge^* S_1\beta$*.*

Proof. Proof of (1), first part : use the connection between $\leq$ and $\leq^*$; second part by duality. Proof of (2), first part : use the connection between $\leq$ and $\leq^*$; second part by duality. Proof of (3) : by (1) and (2). Proof of $S_0\alpha \leq S_1\alpha$, using (3) : $S_0\alpha \leq \alpha \leq \alpha \vee 1^* \leq S_1\alpha$. The rest of (4) is easy. Proof of $S_1\alpha \leq S_0\alpha \vee 1^*$: $S_1\alpha \wedge C_0\alpha \leq S_1\alpha \wedge C_1\alpha$ by (4), $S_1\alpha \wedge C_1\alpha = 1^*$ by properties already proved ; hence, $S_1\alpha \wedge C_0\alpha \leq 1^*$ and $S_1\alpha \leq S_0\alpha \vee 1^*$ by the adjointness property ($AdjC_0S_0$) of theorem 2.4. Proof of (5), first part : by duality. Proof of (6) : by (1), (4) and (5). Proof of (7), first part : $S_0\alpha \leq^* \alpha \wedge^* 0 \leq^* S_1\alpha \wedge^* 0 = S_0\alpha$, by properties already proven ; hence, $S_0\alpha = \alpha \wedge^* 0$. Proof of (7), second part : by duality. Proof of (8) : easy consequences of (6) and (7). To prove (9), first part, it suffices to prove $\alpha \leq \alpha \wedge^* S_1\beta$ (a) and $S_0\beta \leq \alpha \wedge^* S_1\beta$ (b). Proof of (a) : use the connection between $\leq$ and $\leq^*$. Proof of (b) : $0 \leq \alpha$ and $S_0\beta \leq S_1\beta$; hence $0 \wedge^* S_0\beta \leq \alpha \wedge^* S_1\beta$ by an already proven monotony ; on the other hand, $S_0\beta \leq^* 0$ by (1), hence $0 \wedge^* S_0\beta = S_0\beta$; it follows that $S_0\beta \leq \alpha \wedge^* S_1\beta$. Proof of (9), second part : by duality. ☐

It seems desirable to have the reverse inequalities of (9), so that we can establish $\alpha \vee S_0\beta = \alpha \wedge^* S_1\beta$, a trivial but important fact in explicit algebras of actions ; it expresses that restricting an action α to a set of conditions may be done in two equivalent ways : either by "adding" with the short disjunction the 0-characteristic function of that set, or by "intersecting" with the short conjunction the 1-characteristic function of that set. Our last axiom ($Ax\neg Rest$) is precisely designed to do the job ; it is indeed readily seen to be equivalent to $\alpha \wedge^* S_1\beta \leq \alpha \vee S_0\beta$. That axiom has also a very nice interpretation if we go back to theorem 2.13 on structures $\mathcal{B}_a$, the Boolean algebras of elements of 0-support a ; the bunch of algebras $\mathcal{B}_a$ is in fact organized as a presheaf of Boolean algebras : if two 0-supports $a (= S_0a)$ and $b (= S_0b)$ are such that $a \leq b$, then there is a restriction mapping $i_{a,b}$ going from $\mathcal{B}_a$ to $\mathcal{B}_b$, defined by $i_{a,b}(\alpha) = \alpha \vee b = \alpha \vee S_0b$. An essential step in exhibiting the presheaf structure consists in proving that $i_{a,b}$ respects the structure, in particular negation $\neg$: $i_{a,b}(\neg\alpha) = \neg i_{a,b}(\alpha)$; but that amounts precisely to $\neg\alpha \vee S_0b = \neg(\alpha \vee S_0b)$, which is ensured by our axiom ($Ax\neg Rest$). (To recover more usual notations, observe that $a \leq b$ is equivalent to $\neg b \leq^* \neg a$ and that $i_{a,b}(\alpha) = \alpha \vee b = \alpha \vee S_0b = \alpha \wedge^* S_1b = \alpha \wedge^* S_1(\neg b)$, a formula which looks more familiar to indicate restriction.) The interested reader will also note that there is a natural notion of finite covering given by : $(a_i)_{1\leq i\leq n}$

covers a iff $\bigwedge_{1\leq i\leq n} a_i \leq a$ (or dually, in more familiar notation using 1-supports, $(\neg a_i)_{1\leq i\leq n}$ covers $\neg a$ iff $\neg a \leq^* \bigvee^*_{1\leq i\leq n} \neg a_i$). Here we touch the sheaf aspects of support algebras ; they are obvious to all readers acquainted with the notion and they constitute a representation which we have constantly in mind, but we do not insist on them here and remain at an elementary axiomatic level.

We repeat and slightly enlarge the consequences of $(Ax\neg Rest)$ here :

Theorem 2.16: *(1)* $\alpha \wedge^* S_1\beta = \alpha \vee S_0\beta$ *(auto-dual)*
(2) Distributivity properties :
$(\alpha \wedge^* \beta) \vee \gamma = (\alpha \vee \gamma) \wedge^* (\beta \vee \gamma)$
$(\alpha \vee \beta) \wedge^* \gamma = (\alpha \wedge^* \gamma) \vee (\beta \wedge^* \gamma)$
(3) Adjointness properties of $\gamma \setminus \beta$ *and of* $\beta \to^* \gamma$ *:*
$\gamma \setminus \beta \leq \alpha$ *iff* $\gamma \leq \beta \vee \alpha$
$\alpha \leq^* \beta \to^* \gamma$ *iff* $\alpha \wedge^* \beta \leq^* \gamma$
(4) Relations between $\wedge$ *and* $\wedge^*$ *:*
$\alpha \wedge \beta = (\alpha \wedge^* \beta) \wedge (\alpha \vee C_0\beta) \wedge (C_0\alpha \vee \beta)$
$\alpha \wedge \beta \leq \gamma$ *iff* $\alpha \wedge^* \beta \leq \gamma \vee S_0\alpha \vee S_0\beta$ *and* $\alpha \leq \gamma \vee C_0\beta$ *and* $\beta \leq \gamma \vee C_0\alpha$.

Proof. Proof of (1) : we have already commented upon that property. Proof of (2), first part : compute the short conjunctions $\wedge^*$ by using the formulas :

$$\begin{aligned} \delta \wedge^* \varepsilon &= (\delta \wedge \varepsilon) \wedge^* S_1\delta \wedge^* S_1\varepsilon && \text{by restriction properties of theorem 2.14} \\ &= (\delta \wedge \varepsilon) \vee S_0\delta \vee S_0\varepsilon && \text{by (1)} \end{aligned}$$

and use the distributivity of the short disjunction $\vee$ over the long conjunction $\wedge$. Proof of (2), second part : by duality. For the proof of (3), first part, first recall the definition of $\gamma \setminus \beta$:

$$\gamma \setminus \beta = C_0\beta \wedge (\gamma \wedge^* \neg\beta).$$

Proof of (3), first part, from left to right : show $\gamma \leq \beta \vee (\gamma \setminus \beta)$; this can be done as follows :

$$\begin{aligned} \beta \vee (\gamma \setminus \beta) &= \beta \vee (C_0\beta \wedge (\gamma \wedge^* \neg\beta)) \\ &= (\beta \vee C_0\beta) \wedge (\beta \vee (\gamma \wedge^* \neg\beta)) \\ &= \beta \vee (\gamma \wedge^* \neg\beta) && \text{because } \beta \vee C_0\beta = 1 \\ &= (\beta \vee \gamma) \wedge^* (\beta \vee \neg\beta) && \text{by distributivity proven in (2)} \\ &= (\beta \vee \gamma) \wedge^* S_1\beta \\ &= \beta \vee \gamma \end{aligned}$$

this last equality because $\beta \vee \gamma \leq^* S_1(\beta \vee \gamma) = S_1\beta \wedge^* S_1\gamma \leq^* S_1\beta$; the desired result follows easily.

Proof of (3), first part, from right to left ; assume $\gamma \leq \beta \vee \alpha$; then

$$\begin{array}{rll}
\gamma \wedge^* \neg\beta & \leq\ (\beta \vee \alpha) \wedge^* \neg\beta & \text{by monotony}\\
& =\ (\beta \wedge^* \neg\beta) \vee (\alpha \wedge^* \neg\beta) & \text{by distributivity}\\
& =\ S_0\beta \vee (\alpha \wedge^* \neg\beta) & \text{because } \beta \wedge^* \neg\beta = \beta \wedge \neg\beta = S_0\beta\\
& =\ \alpha \wedge^* \neg\beta & \text{because } S_0\beta \leq S_0\alpha \vee S_0\beta =\\
& & \qquad S_0(\alpha \wedge^* \neg\beta) \leq \alpha \wedge^* \neg\beta\\
& \leq\ \alpha \wedge^* S_1\beta & \text{using } \neg\beta \leq S_1(\neg\beta) = S_1\beta \text{ and}\\
& & \qquad \text{monotony}\\
& =\ \alpha \vee S_0\beta & \text{by (1) ;}
\end{array}$$

hence, $\gamma \wedge^* \neg\beta \leq \alpha \vee S_0\beta$ and $C_0\beta \wedge (\gamma \wedge^* \neg\beta) \leq \alpha$ by the adjointness property $(AdjC_0S_0)$ of theorem 2.4. Proof of (3), second part ; recall the definition of $\beta \rightarrow^* \gamma$ and observe that it is the dual of $\gamma \setminus \beta$: $\beta \rightarrow^* \gamma = \neg(\neg\gamma \setminus \neg\beta)$.
Proof of (4), first part : dualize $(Ax\neg\wedge)$, apply $(Ax\neg Rest)$ and the codisjunction property of theorem 2.14. Proof of (4), second part : expand $\alpha \wedge \beta$ using the formula which has just been obtained and look at the restrictions to $S_0\alpha \vee S_0\beta$, to $C_0\alpha$ and to $C_0\beta$. □

Before leaving this elementary study of support algebras, we would like to compare explicit algebras of actions and support algebras ; as can already be guessed from the preceding theorems, the connection is close for what concerns actions and their supports or domains ; on the other hand, the notion of support algebra leaves aside the comparison of results of actions when they concern different circumstances : in support algebras, for disjoint elements a and b of $S_1\mathcal{A}$, there is no built-in comparison between $\alpha \wedge^* a$ and $\beta \wedge^* b$, while for concrete algebras of actions, for any (even disjoint) pair of conditions σ and π, $\alpha(\sigma)$ and $\beta(\pi)$ live inside the Boolean algebra of results and may be compared there ; to say it briefly, there is a common algebra of results in explicit algebras of actions, which is absent in support algebras. The following definitions and theorems give a good feeling of that difference in a restricted but typical case. We have not tried to drop the finiteness conditions, neither have we explored the possibility or the necessity of extending the notion of support algebras to capture more of explicit algebras of actions, because we feel that extensions in other directions are more urgent : see next sections. In the sequel, $At(B)$ denotes the set of atoms of the Boolean algebra B.

Definition 2.17: *(1) An explicit algebra of actions $\mathcal{B} =< B, E, C >$ is* conditions-finite *if B is finite. The algebra $\mathcal{B}$ is* conditions-full *iff for all $\sigma \in At(B)$, there exists an $\alpha \in E$ such that $\bigvee dom(\alpha) = \sigma$. The algebra $\mathcal{B}$ is* results-full *iff for all $c \in C$, there exists an $\alpha \in E$ and a $\sigma \in dom(\alpha)$, $\sigma \neq 0$, such that $\alpha(\sigma) = c$.*
(2) A support algebra $\mathcal{A} =< A, 0, 1, \neg, \wedge, \vee, C_0, \leq>$ is support-finite *if*

$S_1\mathcal{A}$ is finite. A support-finite support algebra with common algebra of results *is given by a support-finite support algebra $\mathcal{A} =< A, 0, 1, \neg, \wedge, \vee, C_0, \leq>$ together with a family $< \mathcal{A}'_a \xrightarrow{i_a} D >_{a\in At(S_1\mathcal{A})}$, where $\mathcal{A}'_a$ is the already defined Boolean algebra of elements of 1-support a, D is a Boolean algebra, each i_a is an embedding of Boolean algebras and the family of i_a's is jointly epimorphic ($D = \bigcup_{a\in At(S_1\mathcal{A})} i_a[\mathcal{A}_a]$).*

Theorem 2.18: *There is a natural bijective correspondence between (a) and (b) :*
(a) explicit algebras of actions which are conditions-finite, conditions-full and results-full
(b) support-finite support algebras with common algebra of results.

Sketch of proof. (1) Every explicit algebra of actions $\mathcal{B} =< B, E, C >$ determines a support algebra $\mathcal{A} =< A, 0_\mathcal{A}, 1_\mathcal{A}, \neg_\mathcal{A}, \wedge_\mathcal{A}, \vee_\mathcal{A}, C_{0\mathcal{A}}, \leq_\mathcal{A}>$ by letting $A = E$ and using the definitions of section 1, e.g. for $\alpha \in E$, $\neg_\mathcal{A}\alpha$ is defined by $dom(\neg_\mathcal{A}\alpha) = dom\alpha$ and for all $\sigma \in dom\alpha$, $(\neg_\mathcal{A}\alpha)(\sigma) = \neg_C(\alpha(\sigma))$. If moreover $\mathcal{B}$ is conditions-finite and conditions-full, then $S_1\mathcal{A}$ is isomorphic to B via the mappings $(\ )^\wedge$ from B to $S_1\mathcal{A}$ and $(\ )^\vee$ from $S_1\mathcal{A}$ to B defined as follows :

for $\sigma \in B$, $dom(\sigma^\wedge) = \{\sigma\}$ and $\sigma^\wedge(\sigma) = 1_C$;
for $a \in S_1\mathcal{A}$, $a^\vee = \bigvee doma$.

Since $S_1\mathcal{A}$ is isomorphic to B, atoms of $S_1\mathcal{A}$ correspond to atoms of B and elements of $\mathcal{A}'_a$ are mappings having $a^\vee$ as a domain, giving sense to the definition of $ev_a\ : \mathcal{A}'_a \longrightarrow C\ : \alpha \longmapsto ev_a(\alpha) = \alpha(a^\vee)$. When $\mathcal{B}$ is results-full, the family of ev_a's ($a \in At(S_1\mathcal{A})$) is jointly surjective, which shows that it is the required common algebra of results.

(2) Every finite-support support algebra $\mathcal{A} =< A, 0, 1, \neg, \wedge, \vee, C_0, \leq>$ with common algebra of results $< \mathcal{A}'_a \xrightarrow{i_a} D >_{a\in At(S_1\mathcal{A})}$ determines an explicit algebra of actions $\mathcal{B} =< B, E, C >$ via the definitions :

$B = S_1\mathcal{A}$
$C = D$
$E = \{\vec{\alpha} \mid \alpha \in E\}$,

where $\vec{\alpha}\ : dom\vec{\alpha} \longrightarrow C$ is itself defined by

$dom\vec{\alpha} = \{a \mid a \in At(S_1\mathcal{A}), a \leq^*_\mathcal{A} S_1\alpha\}$
$\vec{\alpha}(a) = i_a(\alpha \wedge^*_\mathcal{A} a)$.

(3) Long but not difficult computations will show that the constructions given in (1) and (2) are inverses one of another up to isomorphism. $\square$

One last word of caution about that theorem. Restrictions to conditions-finite and conditions-full as stated appear to be very natural, but apparently slight changes have important consequences : if you replace "results-full" by the stronger condition "for all $c \in C$, E contains the mapping $\alpha : \{1_B\} \longrightarrow C$ defined by $\alpha(1_B) = c$", then each ev_a defined in part (1) of the proof will be surjective ; in that way, conditions-finite conditions-full algebras $\mathcal{B}$ satisfying that condition correspond bijectively to support-finite support algebras where all $\mathcal{A}'_a$ ($a \in At(S_1\mathcal{A})$) are isomorphic Boolean algebras.

3. *Support algebras with truth-value supports*

In section 1 we mentioned the operation $S_0^{=0}$ which codifies our capacity to recognize conditions under which an action leads to contradictory results. Since that operation does not seem to be definable on the basis of the operations and relations of support algebras, we add it to their structure with a unique characterizing property, which is yet another adjointness condition.

Definition 3.1: *A* support algebra with truth-value supports *is given by a support algebra* $\mathcal{A} =< A, 0, 1, \neg, \wedge, \vee, C_0, \leq>$ *together with a unary operation* $S_0^{=0}$ *satisfying*
($AdjS_0^{=0}S_0$*)* $S_0^{=0}\alpha \leq \beta$ *iff* $\alpha \leq S_0\beta$.

Note that the adjointness property ($AdjS_0^{=0}S_0$) is indeed satisfied in the case of explicit algebras of actions.

Here is a bunch of basic properties of $S_0^{=0}$ exhibiting its effect on operations and relations of the underlying support algebra :

Theorem 3.2: $S_0^{=0}(\alpha \vee \beta) = S_0^{=0}\alpha \vee S_0^{=0}\beta$
$\alpha \leq \beta$ *entails* $S_0^{=0}\alpha \leq S_0^{=0}\beta$
$S_0^{=0}0 = 0$ *and* $S_0^{=0}1 = 1$
$\alpha \leq S_0^{=0}\alpha$
$S_0^{=0}S_0\alpha = S_0\alpha$
$S_0S_0^{=0}\alpha = S_0^{=0}\alpha$
$\alpha \vee \beta = 1$ *entails* $S_0^{=0}(\alpha \wedge \beta) = S_0^{=0}\alpha \wedge S_0^{=0}\beta$
$S_0^{=0}(\alpha \wedge \beta) = S_0^{=0}(\alpha \wedge^* \beta) \wedge S_0^{=0}\alpha \wedge S_0^{=0}\beta$.

Proof. The first properties are mostly routine consequences of adjointness and simple properties of S_0. For example, prove $\alpha \leq S_0^{=0}\alpha$ by showing that for all β, $S_0^{=0}\alpha \leq \beta$ entails $\alpha \leq \beta$; that is easy : $S_0^{=0}\alpha \leq \beta$, hence $\alpha \leq S_0\beta$ by ($AdjS_0^{=0}S_0$), hence $\alpha \leq \beta$ since $S_0\beta \leq \beta$. The last but one property is a bit less immediate when, assuming $\alpha \vee \beta = 1$, we try to prove $S_0^{=0}\alpha \wedge S_0^{=0}\beta \leq S_0^{=0}(\alpha \wedge \beta)$; to do that, we prove that for every γ, $S_0^{=0}(\alpha \wedge \beta) \leq \gamma$ (1) entails $S_0^{=0}\alpha \wedge S_0^{=0}\beta \leq \gamma$. Starting from (1), we get $\alpha \wedge \beta \leq S_0\gamma$ by $(Adj S_0^{=0} S_0)$, hence $\alpha \leq S_0\gamma \vee S_0\alpha$ and $\beta \leq S_0\gamma \vee S_0\beta$, using $\alpha \vee \beta = 1$ and properties of S_0 ; we then compute in succession :

$$\begin{array}{lll} \alpha \leq S_0(\gamma \vee \alpha) \text{ and } \beta \leq S_0(\gamma \vee \beta) & & \\ S_0^{=0}\alpha \leq \gamma \vee \alpha \text{ and } S_0^{=0}\beta \leq \gamma \vee \beta & & \text{by } (Adj S_0^{=0} S_0) \\ S_0^{=0}\alpha \wedge S_0^{=0}\beta \leq & (\gamma \vee \alpha) \wedge (\gamma \vee \beta) & \\ = & \gamma \vee (\alpha \wedge \beta) & \\ \leq & \gamma \vee S_0\gamma & \text{since } \alpha \wedge \beta \leq S_0\gamma \\ = & \gamma & \text{since } S_0\gamma \leq \gamma \end{array}$$

which gives the desired result. To prove the last property, apply the preceding property to the expression relating $\wedge$ and $\wedge^*$: $\alpha\wedge\beta = (\alpha\wedge^*\beta)\wedge(\alpha\vee C_0\beta)\wedge(C_0\alpha \vee \beta)$ (see Theorem 2.16 (4)). $\square$

In support algebras with truth-value supports, we can reconstruct a truth-value support $S_0^{\neq 0}\alpha$ recognizing that part of the domain of α where α is non-zero and from there, we can obtain an implication $\rightarrow$ and a complex negation $\sim$, obeying characteristic adjointness properties already mentioned at the end of section 1.

Definition 3.3: *(1)* $S_0^{\neq 0}\alpha = C_0 S_0^{=0}\alpha \vee S_0\alpha$
(2) $\beta \rightarrow \gamma = (C_0\beta \vee \gamma) \wedge (\neg\beta \vee \gamma \vee S_0^{\neq 0}(\beta \wedge^* \neg\gamma))$
(3) $\sim\alpha = C_0\alpha \wedge (\neg\alpha \vee S_0^{\neq 0}\alpha)$.

Theorem 3.4: *(1)* $S_0^{=0}\delta \vee S_0^{\neq 0}\delta = 1$; $S_0^{=0}\delta \wedge S_0^{\neq 0}\delta = S_0\delta$
(2) $\varepsilon \leq \neg\delta \vee S_0^{\neq 0}\delta$ *iff* $\varepsilon \wedge \delta \leq S_0\delta$
(3) $\alpha \leq \beta \rightarrow \gamma$ *iff* $\alpha \wedge \beta \leq \gamma$
(4) $\sim\alpha = \alpha \rightarrow 0$
(5) $\beta \leq \sim\alpha$ *iff* $\alpha \wedge \beta \leq 0$.

Proof. Proof of (1), first part : immediate. Proof of (1), second part : easy if you apply the properties $S_0\delta \leq \delta \leq S_0^{=0}\delta$. Proof of (2), from left to right : show that $(\neg\delta \vee S_0^{\neq 0}\delta) \wedge \delta \leq S_0\delta$, by applying properties of $S_0^{=0}$. Proof of (2), from right to left : express $\neg\delta \vee S_0^{\neq 0}\delta$ as the long conjunction of two terms $\tau_1 = \neg\delta \vee S_0^{\neq 0}\delta \vee S_0\varepsilon$ and $\tau_2 = \neg\delta \vee S_0^{\neq 0}\delta \vee C_0\varepsilon$; use $\varepsilon \wedge \delta \leq S_0\delta$ to show that $\varepsilon \leq \neg\delta \vee S_0\varepsilon$, hence that $\varepsilon \leq \tau_1$; use again $\varepsilon \wedge \delta \leq S_0\delta$ to show that $S_0^{=0}\delta \leq \neg\delta \vee C_0\varepsilon$, hence that $\tau_2 = 1$ and $\varepsilon \leq \tau_2$; from $\varepsilon \leq \tau_1$ and $\varepsilon \leq \tau_2$, you get $\varepsilon \leq \tau_1 \wedge \tau_2 = \neg\delta \vee S_0^{\neq 0}\delta$. Proof of (3) : a long but not difficult computation using property (2), the relations between $\wedge$ and $\wedge^*$ and the adjointness property $(Adj C_0 S_0)$. Proof of (4) and (5) : immediate. $\square$

The interest of properties (3), (4) and (5) of Theorem 3.4 is that they show that the complex negation and the implication which have just been defined have an intuitionistic behavior. We can therefore use for them all the well-known properties of intuitionistic negation and implication with their synthetic proofs, without having to return to the explicit definitions given in Definition 3.3 (2) and (3). The reader who tries to prove $\sim\sim\sim\alpha = \sim\alpha$, $\sim\sim(\alpha\wedge\beta) = \sim\sim\alpha\wedge\sim\sim\beta$ or $\sim\sim(\alpha \to \beta) = \sim\sim\alpha \to \sim\sim\beta = \alpha \to \sim\sim\beta$ will appreciate the remark!

It is also very natural to use the negation $\neg$ in order to reconstruct various truth-value supports such as $S_0^{=1}$, $S_0^{\neq 0,1}$, $S_0^{\neq 1}$, with the obvious meanings suggested by the notation :

Definition 3.5: *(1)* $S_0^{=1}\alpha = S_0^{=0}\neg\alpha$
(2) $S_0^{\neq 0,1}\alpha = S_0^{\neq 0}\alpha \vee C_0 S_0^{=1}\alpha$
(3) $S_0^{\neq 1}\alpha = C_0 S_0^{=1}\alpha \vee S_0\alpha.$

A word of caution however if you want to express the expected connections between these different supports. E.g., it is tempting to prove that $S_0^{=0}\alpha$ and $S_0^{=1}\alpha$ are codisjoint, i.e. that $S_0^{=0}\alpha \vee S_0^{=1}\alpha = 1$. The proof could run like this : $S_0^{=0}\alpha \vee S_0^{=1}\alpha = S_0^{=0}\alpha \vee S_0^{=0}\neg\alpha = S_0^{=0}(\alpha \vee \neg\alpha) = S_0^{=0}(S_1\alpha) = S_0^{=0}(S_0\alpha \vee 1^*) = S_0^{=0}S_0\alpha \vee S_0^{=0}1^* = S_0\alpha \vee S_0^{=0}1^*$, but we cannot go on because nothing guarantees that $S_0^{=0}1^*$ has the expected value 1! In fact, in support algebras with truth-value supports, we have to take into account a phenomenon of partial degeneracy : $S_0^{=0}1^*$ represents the set of conditions on which the algebra of results is the degenerate Boolean algebra with one element. In explicit algebras of actions, the existence of a common algebra C of results ensures that, should $S_0^{=0}1^*$ be different from 1, the algebra of elements of support $S_0^{=0}1^*$ would be degenerate and C itself would be degenerate. In view of the case of explicit algebras of actions, one could be tempted to adopt the axiom $S_0^{=0}1^* = 1$; it seems however more advisable to leave things as they are, because there seems to be no difficulty in factorizing support algebras with truth-value supports into two components : the degenerate part, on which most operations trivialize, and a fully non-degenerate part, on which $S_0^{=0}1^* = 1$. We leave those considerations for further work.

4. *Support algebras with complex negation*

Instead of adding a truth-value support operation $S_0^{=0}$, we consider here support algebras supplemented with complex negation $\sim$ as a primitive.

Definition 4.1: *A* support algebra with complex negation *is given by a support algebra* $\mathcal{A} =< A, 0, 1, \neg, \wedge, \vee, C_0, \leq>$ *together with a unary operation* $\sim$ *on* A *satisfying*
(Adj∼) $\beta \leq \sim\alpha$ *iff* $\alpha \wedge \beta \leq 0$.

We have already shown that support algebras with truth-value supports have a complex negation satisfying (*Adj*∼), but the converse is also true : given a complex negation satisfying (*Adj*∼), one can define an operation $S_0^{=0}$ satisfying the basic adjointness property ($AdjS_0^{=0}S_0$) :

Definition 4.2: (In support algebras with complex negation) *(*$DefS_0^{=0}$*)* $S_0^{=0}\alpha = C_0\sim\alpha$.

Theorem 4.3: *Every support algebra with complex negation is naturally endowed with a structure of support algebra with truth-value supports via (*$DefS_0^{=0}$*), i.e. satisfies :* $S_0^{=0}\alpha \leq \beta$ *iff* $\alpha \leq S_0\beta$.

Proof. The following sequence is made of equivalent assertions : $S_0^{=0}\alpha \leq \beta$, $C_0\sim\alpha \leq \beta$, $1 \leq \sim\alpha \vee \beta$, $C_0\beta \leq \sim\alpha$, $\alpha \wedge C_0\beta \leq 0$, $\alpha \leq S_0\beta$. □

To make the concept of complex negation more palatable, it is worth looking at its connections with the different supports and to keep things transparent, we assume in the following theorem that our support algebra is fully non-degenerate, a fact which is expressed here by $\sim 1^* = 0$. Here is a bunch of interesting facts whose proof is left to the reader :

Theorem 4.4: (In support algebras with complex negation satisfying $\sim 1^* = 0$)
(1) Truth-value supports in terms of $\sim$, $\neg$, C_0, S_0 *:*

$$\begin{aligned}
S_0^{=0}\alpha &= C_0\sim\alpha \\
S_0^{=1}\alpha &= C_0\sim\neg\alpha \\
S_0^{\neq 0,1}\alpha &= S_0\sim\alpha \vee S_0\sim\neg\alpha \vee S_0\alpha \\
S_0^{\neq 0}\alpha &= S_0\sim\alpha \vee S_0\alpha \\
S_0^{\neq 1}\alpha &= S_0\sim\neg\alpha \vee S_0\alpha
\end{aligned}$$

(2) Supports of $\neg\alpha$ *in terms of supports of* α *:*

$$\begin{aligned}
C_0\neg\alpha &= C_0\alpha \\
S_0^{=0}\neg\alpha &= S_0^{=1}\alpha \\
S_0^{=1}\neg\alpha &= S_0^{=0}\alpha \\
S_0^{\neq 0,1}\neg\alpha &= S_0^{\neq 0,1}\alpha \\
S_0\neg\alpha &= S_0\alpha
\end{aligned}$$

(3) Supports of $\sim\alpha$ *in terms of supports of* α *:*

$$\begin{array}{lcl} C_0{\sim}\alpha & = & S_0^{=0}\alpha \\ S_0^{=0}{\sim}\alpha & = & C_0\alpha \wedge S_0^{=1}\alpha \\ S_0^{=1}{\sim}\alpha & = & 1 \\ S_0^{\neq 0,1}{\sim}\alpha & = & S_0^{\neq 0,1}\alpha \\ S_0{\sim}\alpha & = & C_0\alpha \wedge S_0^{\neq 0}\alpha \end{array}$$

(4) Supports of ${\sim}{\sim}\alpha$ *in terms of supports of* α *and expression of* ${\sim}{\sim}\alpha$ *in terms of* α *:*

$$\begin{array}{lcl} C_0{\sim}{\sim}\alpha & = & C_0\alpha \wedge S_0^{=1}\alpha \\ S_0^{=0}{\sim}{\sim}\alpha & = & S_0^{=0}\alpha \\ S_0^{=1}{\sim}{\sim}\alpha & = & 1 \\ S_0^{\neq 0,1}{\sim}{\sim}\alpha & = & S_0^{\neq 0,1}\alpha \\ S_0{\sim}{\sim}\alpha & = & S_0^{\neq 1}\alpha \\ {\sim}{\sim}\alpha & = & \alpha \vee S_0^{\neq 1}\alpha \end{array}$$

(5) Supports of ${\sim}\neg\alpha$ *in terms of supports of* α *and expression of* ${\sim}\neg\alpha$ *in terms of* α *and* ${\sim}{\sim}\alpha$ *:*

$$\begin{array}{lcl} C_0{\sim}\neg\alpha & = & S_0^{=1}\alpha \\ S_0^{=0}{\sim}\neg\alpha & = & C_0\alpha \wedge S_0^{=0}\alpha \\ S_0^{=1}{\sim}\neg\alpha & = & 1 \\ S_0^{\neq 0,1}{\sim}\neg\alpha & = & S_0^{\neq 0,1}\alpha \\ S_0{\sim}\neg\alpha & = & C_0\alpha \wedge S_0^{\neq 1}\alpha \\ {\sim}\neg\alpha & = & C_0\alpha \wedge (\alpha \vee S_0^{\neq 1}\alpha) \\ & = & C_0\alpha \wedge {\sim}{\sim}\alpha \end{array}$$

(6) Connection between α, ${\sim}\neg\alpha$, ${\sim}{\sim}\alpha$ *:*

$$ {\sim}{\sim}\alpha \; = \; \alpha \vee {\sim}\neg\alpha. $$

5. *Support algebras with implication*

Instead of adding $S_0^{=0}$ or $\sim$ to support algebras, we can supplement them with an implication $\rightarrow$, again subject to a characteristic adjointness property :

Definition 5.1: *A* support algebra with implication *is given by a support algebra* $\mathcal{A} =< A, 0, 1, \neg, \wedge, \vee, C_0, \leq>$ *together with a binary operation* $\rightarrow$ *on* A *satisfying :*
(Adj $\rightarrow$*)* $\alpha \leq \beta \rightarrow \gamma$ *iff* $\alpha \wedge \beta \leq \gamma$.

The connection between $\rightarrow$ and $\sim$ is clearly the traditional one in intuitionistic contexts, so that we can recover $\sim$ from $\rightarrow$ by the usual equation and show that support algebras with implication are also support algebras with complex negation :

Definition 5.2: (In support algebras with implication) *(Def∼)* $\sim\alpha = \alpha \rightarrow 0$.

Theorem 5.3: *Every support algebra with implication is naturally endowed with a structure of support algebra with complex negation via (Def∼), i.e. satisfies :* $\beta \leq \sim\alpha$ *iff* $\alpha \wedge \beta \leq 0$.

All connections of the preceding section may of course be reproduced here. Further explorations and generalizations of support algebras should eventually decide which type of axiomatization is better : with truth-value supports, with complex negation or with implication. We leave it for later work.

6. *Conclusion*

The present article originates in von Wright's work where we can find the idea of action as essentially dependent on conditions (see especially his papers [VW1], [VW2], [VW3], [VW4]) and it elaborates on our [LT]. We think that a strong point of that approach is that one can make distinctions which are blurred in ordinary usage : the conjunction of actions may be long or short according to the set of circumstances one considers ; similarly for disjunction. Negation of action in particular covers many different usages : those we denoted by $\neg$ and $\sim$ seem particularly important to us. To explore those ideas we have presented "explicit" algebras of action and axiomatizations thereof, which we think are particularly simple : they are given in a unisorted language and the specific postulates have the very nice feature of being adjointness properties. With a bit more work and as already suggested in [LT], the distinction between the different conjunctions, disjunctions, negations of actions may be carried over to a modal superstructure containing an obligation operator O and where one will be able to distinguish $O(\alpha \wedge \beta)$ and $O(\alpha \wedge^* \beta)$, $O(\alpha \vee \beta)$ and $O(\alpha \vee^* \beta)$, $O\neg\alpha$ and $O{\sim}\alpha$, etc.

The development we have presented owes much to category theoretic ideas developed in other contexts : when dealing with actions, we have constantly in mind the notion of sheaf of functions on a topological space and we have insisted on the powerful adjointness properties which are so fundamental in category-theoretic contexts ; for emblematic references, see [LW], [ML], [MLM].

Actions as morphisms are not far from dynamic logic where they appear as programs and are also considered as primitive notions. There are however two important differences : on the one hand we have insisted on the partial character of actions, while programs are everywhere defined ; on the other hand, our language is much less expressive because we have no composition, no iteration and no predicate of programs. It would certainly be interesting to pursue our analysis of action, using the analogy with programs. For a basic reference, see [HKT].

Our conception of action is also clearly related to conditional logic. In its simplest form, conditional logic contains a binary operator $>$ acting on formulas A and B to form $A > B$ and meaning "B under condition A" ; this clearly corresponds to our consideration of an action γ with $dom\gamma = \{A\}$ and $\gamma(A) = B$. Some of the commonest axioms for conditional logic have a nice interpretation in our context, where they correspond to built-in features of action. Thus, the axiom $((A > B) \wedge (A > C)) \rightarrow (A > (B \wedge C))$ corresponds to the formation of the short or long conjunction of actions having the same domain. Similarly the axiom $((A > C) \wedge (B > C)) \rightarrow ((A \vee B) > C)$ finds echoes in some of our considerations on complex actions, in this case the equivalence $\gamma \approx \delta$ where $dom\gamma = \{A, B\}, \gamma(A) = \gamma(B) = C$, $dom\delta = \{A \vee B\}$ and $\delta(A \vee B) = C$. The connection with conditional logic is certainly worth more considerations than suggested here and it could benefit both partners : a more synthetic point of view for conditional logic, a more expressive language for the theory of action presented here. For a basic reference, see [NC]. See also [CFH] for a set of basic references, especially the papers [KS] and [SK].

A last problem we would like to mention is that of generalizing the present theory of action to non-classical contexts. This is motivated by the fact that in realistic contexts, conditions and results are not always clear-cut. Accordingly, one should consider less bivalent notions of support than our S_0 and negations less classical than the negation $\neg$ which has been considered here.

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