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HOW TO CATCH ACHILLES:* AN INTRODUCTION TO THE THEORY OF INFINITALS

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Abstract
In this paper I argue that cardinality and ordinality do not exhaust our pretheoretic notion of size when applied to infinite collections. Drawing on the work of Paul du Bois-Reymond, I develop a measure of relative size that is able to capture pretheoretic intuitions that are missed by both ordinality and cardinality. I argue that this new measure is appropriate to certain sorts of questions about size, and that it should be accepted along with cardinality and ordinality.

In this article I will be arguing for pluralism with respect to views about infinity. Specifically, I want to see what can be said for the claim that some infinite collections are bigger than others, even when they can be put in to a one-to-one correspondence with each other. I won't be arguing that the Cantorian notion is wrong. Rather, I will be arguing for the weaker claim that cardinality does not exhaust the meaning of 'size'.

To a certain extent, this is already recognized. We currently have two notions of size, cardinality and ordinality. It is well known, for example that ω is a smaller ordinal than $\omega + 1$ even though they both are the same size with respect to cardinality. In this paper I wish to introduce a third measure of size, one that is different than either cardinality or ordinality. It will turn out that on this notion of size it will make sense to say that there are more natural numbers than there are even natural numbers. This is a claim that has considerable pretheoretic appeal. It is often a stumbling block when students are told, on the basis of an appeal to equinumerosity, that the two collections are the same size. Thus I take it that this new notion cashes out at least one important pretheoretic intuition about the sizes of infinite collections. To be sure, pretheoretic intuitions are sometimes just wrong. But when it is

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possible to construct a theory that preserves them, that theory is, other things being equal, to be preferred over theories that violate them.

I will begin by setting out a case in which I think there is a very strong pretheoretic intuition that one collection is larger than another despite their being equinumerous. I will then outline a portion of Paul Du Bois-Reymond's Infinitärcalcül. Finally I will show how a modified version of the Infinitärcalcül can be used to support the pretheoretic intuition.

1. *The Race*

We join Achilles and Tortoise just after their race, made famous by Zeno, in which Achilles caught up with Tortoise after giving him a head start.

"Whew," said Tortoise, "that was some race Achilles. I was sure that I had you after you offered to give me a head start. Who would have thought that you would have had the sheer dogged persistence to complete the infinite series of tasks needed to catch up to me?"

"Oh come now Tortoise, it wasn't that much of a feat. You have to remember, I run a lot faster than you do."

"You are too modest Achilles. And besides, we both know that your speed, impressive as it is, had nothing to do with your catching up to me."

"We do?"

"Of course we do. Had you run as fast as Hermes, you would never had caught me if you hadn't stuck it out through that infinite series of tasks. If your dedication to your goal had ever wavered, had you allowed yourself to be daunted by the enormity of the task before you, then I would be the one wearing the laurel wreath"

"Well, yes, I suppose that's true. But wasn't it my speed that allowed me to do all that?"

"There you go again, trying to distract attention from the real secret to your success. You are a sly one. I wish I had seen through your cunning when you offered to give me a head start, for now I see that it was the head start that allowed you to catch up to me."

"What!?"

"Yes, in fact I'll wager that had we switched positions, I would have caught up with you rather than you catching up with me. It would have taken me longer, but still, I would have done it."

"That's preposterous!"

"Is it? Let's put it to a test. We'll run another race. This time I'll give you a head start and we'll say that if I catch you, then I win. Agreed?"

"Well, um, er, that is, ... Look Tortoise, if you give me a head start, and I run faster than you, we could run all the way to Macedonia and you wouldn't catch up to me."

"What's wrong with running to Macedonia?"

"Well, for one, I have a victory celebration to attend this evening and I could never get back in time."

"Hmmm, well then, let's run on this circular track. That way we won't get too far away from the celebration. I'll give you a one lap head start. If there is ever a time at which I've run as many laps as you, then I'll have caught up to you. Right?"

"Well, yes, that would work. But still it would take too long. Even though we wouldn't go very far away, the race still wouldn't be done in time for the dinner."

"You may have a point there. Let's get Zeus to help us. Every time I run a lap, he can double our speeds. That way each lap I run will only take half as long as the previous one.

"I don't know, it could still take a long time."

"Well then, if I haven't caught up to you by the start of the victory dinner, we'll just say that you won."

"Okay."

Achilles and Tortoise line up at the starting point. Achilles takes off first, Tortoise starts after Achilles has run one lap. Achilles runs ten times faster than the Tortoise, so by the time Tortoise completes one lap, Achilles has run a total of 11. By the time Tortoise has run two laps, Achilles has run 21. Zeus obligingly doubles the speed of the two contestants each time Tortoise completes a lap. Thus even though Tortoise takes an hour to complete the first lap, by the time two hours have passed, the race is over. We rejoin the contestants as they are catching their breaths.

"Well Tortoise, that was quite some race. I've never run so fast. But now it appears that I have two victories to celebrate tonight."

"Oh?"

"Yes. Although the race was a bit of a blur at the end you must admit that I won. I started off with a one lap lead, and I increased that lead throughout the race. By the time you had run one lap, I had a lead of 10 laps. When you finished your second lap, I had a lead of 19. All through the race my lead kept getting bigger and bigger. You never decreased my lead in the slightest. In fact, if you consider the infinite series comprised of my lead after each lap you ran (i.e the series 10, 19, 28, ...) you can see that it approaches infinity. So it seems reasonable to conclude, that by the end of the race I was ahead by an infinite number of laps. So not only did you never catch me, you always just got farther behind and ended by being infinitely far behind."

"Well put Achilles! I see that you've learned something about infinite series from the many discussions of our previous race. But I'm afraid you're missing the point."

"I am?"

"Yes, despite your argument, I did win the race."

"How can that be?"

"Well, we agreed that I would win the race if there were ever a point at which I had run as many laps as you."

"Yes, that's right."

"Let's look at the number of laps we each ran. You started by running your first lap, right?"

"Of course."

"And after that you ran a second lap, and then a third, a fourth, and so on?"

"Yes."

"So the total number of laps you ran could be figured by taking the limit of the series 1, 2, 3, ..., right?"

"Yes, if I understand limits correctly, that seems right."

"But notice that I also ran a first lap, followed by a second, then a third, and so on. So the total number of laps that I ran is given by the limit of the series 1, 2, 3, ... It doesn't matter that I ran my third lap after you had already run your third lap, all we're concerned with is the total number of laps that we each ran."

"Ok."

"So for each of us, the total number of laps that we ran is given by the limit of the same series. And that of course means that we ran the same number of laps."

"I'm not sure I see . . ."

"Let me put it this way. If you ran more laps than I did, then there must be some number such that you ran that many laps and I didn't."

"That seems right."

"But there is no such number. For every number of laps that you ran, I ran that many also. Thus I ran as many laps as you. And we agreed at the beginning that all I needed to do to win was to have run as many laps as you at some point. This is that point, I have now run as many laps as you, ergo I won."

"Hmm, but for each number of laps, you ran that number after I did, so didn't I finish first?"

"Well, we didn't say that I had to finish at the same time as you. But even if we had, it wouldn't make any difference for I did finish at the same time as you."

"How can that be, since I ran every lap before you did?"

"Well, let's see how long it took each of us to run our laps. I ran the first lap in an hour, the second in a half hour, the third in a quarter hour, and so on. So the total time I spent running was two hours. You'll recall this kind of decreasing series from our first race."

"Yes I remember."

"You ran laps two through 11 in one hour, laps 12 through 21 in a half hour, laps 22 through 31 in a quarter hour, and so on. So the total time you spent running was two hours plus the time you spent on your first lap which we don't need to count. So we both finished running two hours after your first lap. Since we both ran the same number of laps, and finished in the same time, it's obvious that I caught up with you. So rather than having two victories to celebrate tonight, I'm afraid you'll have to share the honors of your victory celebration with me."

1.1. *Analysis*

By considering the sequence of laps completed by each contestant, we are led to the conclusion that they have run the same number of laps. But if we instead consider the sequence of Achilles' leads, we are led to the conclusion that Achilles ran more laps than the tortoise. At the end of the race, Achilles is ahead by an infinite number of laps. The problem is that both sequences seem to be natural ways of describing the race, but they provide conflicting answers to the question of who won the race.

Although Tortoise's argument appeals to equinumerosity, there is more at work than simple equinumerosity. When we consider laps run around a track, the most natural way to identify such laps is by their ordinal position. Because Tortoise and Achilles have run the same ordinal sequence of laps, it is much harder for Achilles to make the claim that he has run some laps that Tortoise has not. Achilles and Tortoise have run the same number of laps not only with respect to cardinality, their lap running is the same with respect to ordinality. In fact it is the claim that they have run the same ordinal sequence of laps which grounds Tortoise's claim that they have run the same number of laps. Tortoise claims to have caught up with Achilles not just with respect to cardinality, but also with respect to ordinality. Because they have run the same sequence of laps, they have run the same number of laps.

1.1.1. *Achilles' Case*

For those used to the concept of equinumerosity, it may be tempting to think that there is no problem here - that Tortoise is simply right, and Achilles wrong. In this section I consider what can be said against such a position.

The first thing to note is that despite the demonstrable utility of notions of cardinality and equinumerosity, it is not obvious how these formal notions map onto pretheoretic notions about size. There is considerable intuitive appeal to the claim that the collection of even numbers is smaller than the collection of natural numbers. An appeal that is buttressed by noting that the collection of even numbers forms a proper part of the collection of natural numbers.

Cardinality, a measure of size for infinite sets, is defined in terms of equinumerosity. Two sets have the same cardinality just in case they are equinumerous. Two sets are equinumerous if there is a 1-1 function from one onto the other. Such a function gives us a way of pairing the members of one set with those of the other without leaving anything left over in either set. For finite sets equinumerosity coincides with sameness of size, thus it has seemed reasonable to take equinumerosity to provide a measure of sameness of size for infinite sets also.

But despite its history of use, there is nothing privileged about equinumerosity as a measure of size for infinite sets, other notions will work just as well. We could, for example, begin by defining what it is for one set to be smaller than another. Say that one set is "subnumerous" as compared to another if there is a 1-1 function from the one set onto a proper subset of the other. For finite sets one set is subnumerous than another just in case it is strictly smaller than the other. Hence, as with equinumerosity, we can apply the concept of subnumerosity to infinite sets as a way of saying when one set is smaller than another. Of course this definition will have some odd results. For example, on this definition, the set of natural numbers is smaller than itself. While this does seem odd, it is not on the face of it any odder than saying that the set of natural numbers is the same size as one of its proper subsets, say the set of even numbers[1]. Having defined subnumerosity as a comparative measure of size for infinite sets, we can now use it to do the work that has traditionally been done by equinumerosity - two sets will be equinumerous just in case either both or neither are subnumerous than the other.

The point here is that Tortoise and Achilles are arguing about pretheoretic concepts of size, not about formally defined measures. Because formally defined measures do not map cleanly onto the pretheoretic notions, at least in the infinite case, one cannot resolve the dispute merely by appealing to such measures. In particular, Achilles' claim to have run more laps, as justified by his having run ten laps for every one that Tortoise ran, cannot be overcome by an appeal to equinumerosity anymore than Tortoise's claim can be overcome by an appeal to subnumerosity.

Achilles' claim can be argued for on the basis of two intuitively plausible (I am tempted to say obvious) facts. First, during the time it took Tortoise to run each lap, Achilles increased his lead by nine laps. Second, the lead one

[1] It has been suggested to me that subnumerosity provides a definition of "smaller than or equal to," but this is not quite right. In the finite case subnumerosity corresponds to "strictly smaller than." A different definition would be needed to capture the meaning of "smaller than or equal to." This goes to show that definitions of size are not seamlessly transferrable from finite to infinite cases. For in the finite case no two sets can be subnumerous than each other, but in the infinite case they can.

has at any point in a race, and the lead one has when the race is over, can be given by the sum of the increases and decreases of leads one has made during each segment of the race up to that point. When the rate of increase in leads is constant throughout a race, this sum is given by the limit of the series of leads one has had during the race. Applying these facts to the present case yields the conclusion that by the end of the race, Achilles leads by an infinite number of laps.

There is another obvious principle that is sufficient to ground Achilles' claim. If one has a lead in a race at some point and has lost that lead by some later point, then there must be some intermediate point at which one's lead has diminished but not vanished. Put more simply: leads can only be lost bit by bit and not all at once. But of course Achilles' lead never decreases, hence it is never lost.

With regard to the original race, we may be persuaded that Achilles can catch Tortoise because at each stage of his task he has less to do than before. Throughout the original race, Achilles gets closer and closer to Tortoise until suddenly he has caught up. We may not be able to put our finger on the exact moment at which Achilles catches Tortoise, but we know he must. Quite the opposite holds for Tortoise catching Achilles. At each stage of the new race Tortoise's task grows larger. Throughout the race he gets further and further behind, until suddenly he has caught up! How can it be that Tortoise gets farther and farther behind, never making the slightest gain on Achilles, and ends up even with him?

It may be replied that if we consider the amount of time that Tortoise has to run in order to catch Achilles, rather than the distance, then Tortoise's task does decrease at each stage. Unless we are willing to countenance a race of infinite duration, this is clearly true. But Tortoise's task is to run as many laps as Achilles, not to run the same amount of time as Achilles. The fact that the remaining time continually decreases explains why the race has a finite duration, but it does not explain how he can catch Achilles at all.

In the remainder of this paper I wish to develop a formal notion of size that can cash out Achilles' intuition that he has won the race. I won't argue that this new notion of size is always more appropriate than cardinality or ordinality, but I do think that there are some cases, such as the current one, in which our pretheoretic intuitions are best seen as expressing the new measure rather than more traditional measures. I will be trading on the work of Paul Du Bois-Reymond in developing my account and want to take this opportunity to thank David McCarty for pointing me in the direction of his work. Actually, since I don't read German I'll be trading on G. H. Hardy's presentation of Du Bois-Reymond's work in [1].

2. *A first attempt at a theory*

One of the main differences between Achilles' argument and Tortoise's is that Achilles rests his claim on the progression of the race, on how his performance and Tortoise's compare throughout the race. Tortoise, on the other hand, doesn't worry about how the contestants have compared during the race, he only considers the collection of laps as a completed whole. We can view the the number of laps each contestant has run throughout the race as the output of a function. I'll use 'A' to name a function representing the number of laps Achilles has run throughout the race and 'T' to name a similar function for Tortoise. The easy way to do this is to measure the progress of both contestants against the progress of Tortoise. Thus T is the identity function and $A(x) = 10T(x) + 1$. Other functions would work as well for T, such as a function which measures Tortoise's progress against a clock.

Seen in this light, Tortoise's argument is that since both A and T approach infinity, the two contestants end up having run the same number of laps. Achilles' argument on the other hand, is that he should be seen as winning the race in virtue of the fact that $A-T$ approaches infinity. But this fact is not sufficient by itself. The theory of cardinality allows for cases in which two sets are equinumerous even though their difference, in some sense, is infinite. The equinumerosity of the integers and the even integers is one such case. In order to provide a theoretical underpinning for Achilles' claim, we need a theory on which the infinite number of laps run by Achilles is greater than the infinite number of laps run by Tortoise.

As a first approach to cashing out Achilles' intuition, we might say that for two functions f and g, the limit of f is greater than the limit of g if $f - g$ approaches infinity. It also seems clear that if the limit of $f - g$ is 0, then the limits of the two functions should be considered equal. Thus if the race had been such that each lap Tortoise ran decreased Achilles' lead by half, then we would say that Tortoise caught up with Achilles even though he was always behind. Now of course these are just two of the possibilities for $f-g$. $f - g$ might have some finite, non-zero limit, or it might have no limit at all. In the latter case the values might be bounded or unbounded. It is less clear what we should say in each of these cases. There is some tendency to say that if $f - g$ never decreases below some fixed positive number, then the limit of f is bigger than limit of g. However there are also reasons why one might prefer to distinguish cases in which the difference approaches infinity from those in which it remains finite. Also one may wish to keep distinct those cases in which the difference is bounded from those in which it isn't. Keeping in mind that Achilles' argument proceeds from the fact that the difference between A and T approaches infinity, we might use the following definitions.

$\lim(f) \succ \lim(g)$ if $f - g \to \infty$.[2]
$\lim(f) \prec \lim(g)$ if $f - g \to -\infty$.

I'll gloss these by saying that $\lim(f)$ is of a higher or lower order of infinity than $\lim(g)$ respectively, or simply that $\lim(f)$ is infinitely bigger or infinitely smaller than $\lim(g)$.

For cases in which $f - g$ tends neither to infinity nor negative infinity, we might use the following definitions.

$\lim(f) \succcurlyeq \lim(g)$ if $f - g > \delta$ cofinally for some constant δ.
$\lim(f) \preccurlyeq \lim(g)$ if $f - g < \Delta$ cofinally for some constant Δ
$\lim(f) \asymp \lim(g)$ if $f - g$ remains between two fixed numbers δ, Δ.

Here we might think of the relations as "at least finitely bigger than", "no more than finitely bigger than", and "of the same order". It is important to note that $\lim(f) \succcurlyeq \lim(g)$ is not equivalent to $\lim(f) \not\prec \lim(g)$ or to $\lim(f) \succ \lim(g) \vee \lim(f) \asymp \lim(g)$ as there is no guarantee that $f - g$ tends to any limit nor remains between two bounds. A case where this fails in general to hold is where $f - g$ is an oscillating function of steadily increasing amplitude.

Finally, we may want a stricter notion of equality than that provided by $\asymp$. Stricter notions can be had by requiring that $f - g$ have a definite finite limit or even that the limit be zero. In these cases we get the following definitions.

$\lim(f) \overline{\overline{\asymp}}\ \lim(g)$ if $f - g$ tends to a finite limit.
$\lim(f) \sim \lim(g)$ if $f - g \to 0$.

In what follows I will generally say $f \succcurlyeq g$ rather than $\lim(f) \succcurlyeq \lim(g)$ and similarly for the other defined functions. This both makes the discussion neater and also allows for an easier accommodation of Du Bois-Reymond's notation later.

2.1. *Application to the race*

It should be obvious that under these definitions, $A \succ T$. But just what does this mean? After all, Tortoise is willing to grant that Achilles kept getting further and further ahead, and that's really all $A \succ T$ tells us. Why should we take the winner of the race to be given by the $\succ$ relation rather than by cardinality or ordinality?

Here the use of a race is particularly important. A race is a contest. It involves comparing the performance of two contestants across time. What we are interested in is not primarily the performance of an individual in isolation, but rather the comparison of two or more individuals' performance

[2] I am using 'lim' somewhat loosely here and throughout the definitions in the sense of 'the number yielded by'. Strictly speaking f may not have a limit even when $f - g$ does. I use 'lim' because it gives the right general idea and fits nicely with the informal discussion.

across time. Cardinality provides a static measure. In comparing cardinality we don't take account of any information about the ordering of objects in the collections, nor about the history of how those collections were formed. If we compare the collections with respect to ordinality, we take account of their internal structure, but we still take no account of how they were formed. It is important to note that one collection may be larger than another with respect to ordinality, even though the two are the same size with respect to cardinality. In the cases of both cardinality and ordinality we treat the collections as already completed. The current proposal is different in that it takes into account how the collections were formed. The formation of the collection is represented by the function which forms it.

Imagine that each time Achilles and Tortoise pass a certain point on the track Zeus adds a token to the end of a row representing the number of laps each has run thus far. As the race progresses, the rows grow and grow until by the end they are infinitely long. We can imagine three different judges determining the winner by using three different methods. The first judge watches the creation of the rows throughout the race. He notes that during the race Achilles' row is always longer than Tortoise's and so pronounces Achilles the winner. The second judge arrives just as the race has finished. She looks at the two rows of coins and notices that they stretch for the same length. Thus she pronounces the number of laps the same, and hence Tortoise the winner. The third judge arrives quite late. The tokens have already been swept into two bags. Eager to have his vote counted, he notes that he can pair coins from the two bags in such a way that both bags are emptied with no remainder. He thus declares Tortoise the winner. The third judge has assessed the race with respect to cardinality, the second with respect to ordinality, and the first with respect to orders of infinity. Which judge we take to be right depends on which of the three formal notions of size we think best cashes out the pretheoretic notion of size that was in play when the bet was made. I've already given some argument that orders of infinity are the best choice. I don't think it's possible to give an absolute proof here though. Our pretheoretic notions are often somewhat vague. We develop our pretheoretic notion of size by looking primarily at finite cases. And in finite cases, the three formal notions coincide.

3. *A brief introduction to the Infinitärcalcül*

Thus far what we have is a collection of definitions that match some of our pretheoretic intuitions. The real trick will be turning a few formalized intuitions into something like a mathematical theory. Fortunately we don't have to do all the work. It turns out that the definitions given above fit very nicely as an extension of the Infinitärcalcül developed by Paul Du Bois-Reymond.

3.1. *Basic Relations*

Du Bois-Reymond’s Infinitärcalcül is designed to compare rates of increases of increases of functions. The goal is to be able to say when one function grows faster than another. For example, even though x and x^2 both approach infinity, intuitively we would like to say that x^2 approaches infinity more rapidly than does x. He proposes the following relations to cash out this intuition:

$f \succ_{DBR} g$ if $f/g \to \infty$.
$f \prec_{DBR} g$ if $f/g \to 0$.
$f \asymp_{DBR} g$ if f/g remains between two fixed positive numbers δ, Δ.
$f \overset{\smile}{\frown}_{DBR} g$ if f/g tends to a definite positive limit.
$f \sim_{DBR} g$ if $f/g \to 1$.

The notion of “order of increase” so defined is relative, just as is the standard notion of equinumerosity. That is to say that the relations only tell us when one function grows faster than another, they do not give us a general way of saying how fast a function grows. In the same way, equinumerosity only tells us when two sets are equinumerous, it does not by itself tell us how big a set is. We can of course choose certain functions as standards against which to measure the remaining functions in either case, thus making the scale absolute. This is what we do when we define cardinality in terms of being equinumerous with certain sets.

3.2. *Defined Relations*

We can further define

$f \succcurlyeq_{DBR} g$ if $f > \delta g$ cofinally for some positive constant δ.
$f \preccurlyeq_{DBR} g$ if $f < \Delta g$ cofinally for some positive constant Δ

As with our earlier definitions $f \succcurlyeq_{DBR} g$ is not equivalent to the negation of $f \prec_{DBR} g$, as $f \not\prec_{DBR} g$ does not imply $f \succcurlyeq_{DBR} g$. Nor is it equivalent to the disjunction ’$f \prec_{DBR} g$ or $f \asymp_{DBR} g$, for it is possible that none of $f \succ_{DBR} g$, $f \prec_{DBR} g$, or $f \asymp_{DBR} g$ hold.

3.3. *Limitations*

It would be nice to now set up a canonical set of functions against which all others could be measured. This would be similar to Cantor’s canonical cardinals or von Neuman’s canonical ordinals. Unfortunately this is not possible. There are a couple of important limitations to the application of scales of orders of infinity. The first is that only regularly increasing functions can be measured. The second is that even with this limitation no scale can measure all functions.

3.3.1. *Irregular Functions*

In his presentation of the Infinitärcalcül, Hardy points out that it is possible for f/g to not tend to 0 or ∞ nor to remain between fixed positive integers. In such a case we have neither $f \succ_{DBR} g$, $f \prec_{DBR} g$, nor $f \asymp_{DBR} g$. This will typically happen when one of f or g is not monotonic. For example, consider two monotonic functions f, g such that $f \succ_{DBR} g$. Now let h be a function which zigzags between f and g. f/h will thus vacillate between 1 and f/g. Since $f/g \to \infty$, there is no upper limit to the value of f/h, but neither does f/h tend to infinity or zero.

As a result we cannot use the Infinitärcalcül to well-order the class of functions. Instead these relations provide a pre-order on the class of functions or a partial order on the class of classes of functions equivalent under $\asymp$.

Hardy notes that if we restrict our attention to functions which are continuous, monotone and tend to either ∞, 0, or some other definite limit, then we have one of $f \succ_{DBR} g$, $f \prec_{DBR} g$, or $f \asymp_{DBR} g$, for any such functions f and g. In such a case the class of functions is ordered and the class of equivalence classes under $\asymp_{DBR}$ is well ordered.

3.3.2. *Off Scale Functions*

As noted above, Du Bois-Reymond's relations provide only a relative measure of rates of increase. We can make an absolute scale by choosing some canonical collection of functions against which to measure other functions, much the same as cardinality is defined as equinumerosity with one of a standard collection of sets. However, the following theorem shows that there is no countable collection of functions which can provide an absolute scale for all functions - there will always be some function which increases more rapidly than any function in the canonical collection.

Theorem 1: (Paul Du Bois-Reymond) *Given any scale of increasing functions g_n, i.e. a series of functions such that $g_1 \prec_{DBR} g_2 \prec_{DBR} g_3 \prec_{DBR} \ldots$, we can always find a function f which increases more rapidly than any function of the scale, i.e. which satisfies the relations $g_n \prec_{DBR} f$ for all values of n.*

A proof of the theorem is contained in [1]. For reasons of space, I won't recreate it here. The upshot of the theorem is that while any two regular functions may be compared, there is no collection of canonical functions that can measure all others.

In a way this should not be too surprising. After all, similar things can be said of cardinality and ordinality. There are collections which are greater than any cardinal — the collection of all cardinals for example. Likewise the ordered collection of ordinals is longer than any ordinal. However, we can define a collection of well behaved collections which the cardinals and ordinals are sufficient to measure — for example, the collection of sets and the collection of well ordered sets respectively. What is different in the case of the Infinitärcalcül is that we have no obvious collection of functions to play the role that the collection of sets plays with respect to cardinality. As a result, I won't try to argue for any canonical measure here. It will be enough to take orders of infinity to give a relative measure rather than an absolute one.

3.4. *The Race Unresolved*

Du Bois-Reymond's Infinitärcalcül allows us to compare any two regular functions with respect to which one has the faster rate of increase. Thus it seems reasonable to compare the functions A and T representing Achilles' and Tortoise's progress through the race. If A turned out to be a higher order of infinity than T according to this comparison, then we would have provided a way of underwriting Achilles' claim to have won the race.

Unfortunately, the attentive reader will have noticed that thus far we have no new resolution to the race. On Du Bois-Reymond's theory one function represents a higher order infinity than another just in case the ratio of the two tends to infinity. But if we consider the two functions that represent the progress of Achilles and the Tortoise, their ratio tends to 10, not to ∞. Thus we get $A \asymp_{DBR} T$ rather than the $A \succ_{DBR} T$ that pretheoretic intuition might suggest. In order for $A \succ_{DBR} T$ to hold, Achilles would not just have to run faster than Tortoise, he would have to accelerate more rapidly than Tortoise. In order to resolve the race, and understand how the Du Bois-Reymond's theory can provide the basis for our earlier definitions, we will need a theorem due jointly to Du Bois Reymond and Pincherle.

4. *Pincherle's Result*

The key to is to see that while du Bois-Reymond compared functions with respect to the ratio of their values, we can just as easily compare them by some other relation between their values. The Infinitärcalcül is a specific case of a more general result.

We have defined $f \succ_{DBR} g$ as $f/g \to \infty$. It should be obvious that all the results would have been the same if instead we had defined it as $\log f - \log g \to \infty$ and gotten the same result. But this highlights the fact that we could have chosen some other function rather than $\log$ on the basis of which to compare functions. We can now define $f \succ_F g$ as $F(f) - F(g) \to \infty$ and the other relations similarly.

In general, the greater the rate of increase of F, the more able it will be to distinguish functions on the basis of $\succ_F$. Specifically, if $f \succ_F g$ and G is any increasing function, then $f \succ_{FG} g$. Furthermore,

Theorem 2: (Pincherle and Du Bois-Reymond) (1) *However rapid the increase of f, as compared with that of g, we can so choose F that $f \asymp_F g$.*

(2) *If $f - g$ is positive for $x > x_0$, we can so choose F that $f \succ_F g$.*

(3) *If $f - g$ is monotonic and not negative for $x > x_0$, and $f \asymp_F g$, however great be the increase of F, then $f = g$ from a certain value of x onwards.*

Finally, it is important to note that the results of 3.3.2 apply generally to $\succ_F$ and the other relativized relations. Different relativations have more or less discriminating power, but they generate similar orderings of functions.

5. *Modifying the Infinitärcalcül*

With the relativized versions of Du Bois-Reymond's relations, we are now in position to resolve the problem presented by Achilles and Tortoise's second race. The main issue is choosing a function F that both gives the desired result and is philosophically defensible.

The key to choosing such a function is to bear clearly in mind what we are comparing when we ask who has won the race. Du Bois-Reymond's relations essentially compare the acceleration of the contestants. The ratio of laps run will tend to a limit based upon the acceleration of the contestants. So for example, if Zeus tripled Achilles' speed every time he doubled Tortoise's, then $A \succ_{DBR} T$ would hold. However since both accelerate equally in the race as described, du Bois-Reymond's relations make no distinction. But the problem with deciding the winner of the race isn't a problem about acceleration. Instead it's a problem about the number of laps run. Achilles does not claim that he accelerates faster than the Tortoise. He just claims that he is always ahead of the Tortoise.

Let's review Achilles' argument. Achilles says “I started off with a one lap lead, and I increased that lead throughout the race. By the time you had run one lap, I had a lead of 10 laps. When you finished your second lap, I had a lead of 19. All through the race my lead kept getting bigger and bigger. You never decreased my lead in the slightest. In fact, if you consider the infinite series comprised of my lead after each lap you ran (i.e the series 10, 19, 28, ...) you can see that it approaches infinity. So it seems reasonable to conclude, that by the end of the race I was ahead by an infinite number of laps. So not only did you never catch me, you always just got farther behind and ended by being infinitely far behind."

Achilles' argument is grounded on the fact that a certain relationship between A and T approaches ∞. Thus his argument has the appropriate form to ground a claim of $A \succ_F T$ for some F. The relationship that Achilles sees as approaching ∞ is just $f - g$. So F is the identity function. This makes a certain amount of sense when we think about it. The question about who won the race can be rephrased as a question about who had a lead at the end of the race. Achilles won the race if and only if he had a lead at the end, otherwise Tortoise won. But leads are established by subtracting the position of one runner from that of another. Thus we are subtracting the positions of the contestants and not some function of the positions. If we were instead interested in the question of which runner accelerated faster, we could discover this by finding the difference between the logs of their positions, or what is probably easier for most of us, simply looking at the ratio of the positions.

From this perspective we can see that our original definitions are just Du Bois Reymond's relations relativized to the identity function. As a result we have access to most of Du Bois Reymond's Infinitärcalcül as a mathematical backdrop for the intuitions that we formalized in response to the race.

The relativized version of the Infinitärcalcül defines a class of classes of infinite functions. Every differentiating function F divides the class of functions into orders differently. One might choose a different F depending on one's purpose. Du Boise Reymond was interested in comparing rates of growth, so his choice of $\log$ is appropriate. However, our primary interest is in comparing the size of collections yielded by the functions, or in other words the limits of the functions. Because we are primarily interested in comparing the limits of the functions, it makes most sense for us to choose identity as our F.

We thus have a new class of infinite numbers which I shall call the infinitals. Infinitals are limits of functions which tend toward infinity. They are differentiated and ordered according to the growth of the functions which define them. We have a choice of equivalence relations depending on our application. Which notion of equivalence we take as primary will depend on the specific application we are concerned with. For example in some applications finite differences may matter, in others we might wish to ignore such differences. Likewise the issue of whether $f - g$ has a definite limit or is merely bounded may be important in some contexts but not in others. There is more work to be done to spell out a full theory of infinitals. But even so, the portion of a theory presented here will already allow us to provide a nice resolution to some puzzles involving infinity.

5.1. *What advantages do orders of infinity have?*

5.1.1. *Evens*

A common stumbling block for students learning about equinumerosity is the discovery that there are just as many even numbers as there are even and odd numbers combined. Even after the student clearly sees that the two sets are equinumerous, there often remains a lingering suspicion that something is amiss. To my mind, this lingering doubt points to the thesis that equinumerosity does not fully capture our pretheoretic intuitions about size. Normally we justify this doubt by pointing out that the evens are a proper subset of the naturals. Thus there must be more naturals than evens because the set of naturals contains all of the evens and then some more. One nice result of the theory of infinitals is that we can give some basis to this intuition without needing to undermine the claim that the two sets are equinumerous.

Suppose you have a bag of marbles, only some of which are red, and you wish to demonstrate that there are more marbles than there are red marbles. You might lay the marbles out in a row and begin counting through them keeping a simultaneous tally of how many marbles there were as you went along versus how many red marbles there were. It's easy to see that in the finite case, this procedure will always give the right answer. As long as some of the marbles aren't red, then no matter how you count through the marbles, eventually the tally of marbles will become and remain greater than the tally of red marbles. It turns out that this is also true in the infinite case. Even if you have an infinite bag of marbles with at least one that isn't red, any way of counting through the bag will yield a marble tally that is eventually greater than the red tally. If there are an infinite number of non-red marbles, then the difference between the two tallies will approach infinity.

So as long as B is a proper subset of enumerable A, any way of counting through A while keeping a simultaneous tally of B will yield the result that the number of $a \in A \succcurlyeq$ the number of $b \in B$

In general we'll say that a function f tallies an enumerable set A inside an ordered set O just in case for all $o \in O$, $f(o) =$ the number of $a \in A$ such that $a \leq o$.

It is not too difficult to prove the following theorem:

Theorem 3: *Let F and G be enumerable sets such that $G \subset F$, let O be any enumeration of F, and let f and g be functions that tally F and G in O respectively. In such a case $f \succcurlyeq g$ and if $F - G$ is infinite, then also $f \succ g$.*

Obviously neither f nor g ever decrease. But f has the further property that it always increases, *i.e.* that if $x > y$ in O, then $f(x) > f(y)$ while for

g it is merely true that $g(x) \geq g(y)$. So once the first non-G is counted, f will become greater than g and remain so ever after.

The important point here is that the tally function for F outstrips the tally function for G no matter what O you pick. This is achieved by requiring that O be an enumeration of F and not just one that includes F. We can extend the concepts by relaxing the relationships between F, G, and O so that F and G are both subsets of of some enumerated set O but not necessarily of each other. In this more general case we will still find that if G is a proper subset of F then f will outstrip g no matter what enumeration is picked for O. However, when neither F nor G is a proper subset of the other, then the ordering of O makes a difference. There are, for example, enumerations of the natural numbers in which the tally functions for the primes and the evens keep pace with each other.

In cases where neither F nor G is a subset of the other, our intuition about which is bigger will depend upon whether there is some ordering which is more "natural" than the others. In the case of numbers, it may often be argued that the standard orderings are the most natural. So we may want to claim that there are more even numbers than primes because the tally function of the evens outstrips the tally function of the primes in the standard enumeration of the natural numbers. Of course there are orderings for which this is not true. Technically we can only say that the number of evens in the standard ordering is a greater infinital than the number of primes in that ordering. Still, the "naturalness" of the standard ordering is sufficiently strong that we may in general take it as the supposed background ordering and simply say that there are more evens than primes.

Sometimes, however, there is an ordering that is sufficiently more natural than any other that it can be taken as the only relevant ordering. This may happen, for example, when we are concerned with events that unfold over time such as the race between Achilles and Tortoise. There are orderings of the laps in which the tally functions for Tortoise and Achilles have the same limit, and even orderings on which Tortoise's tally function outstrips that of Achilles. However, these orderings are not plausible bases for answering the question of who won the race as run.

Infinitals are not without limitations however. For example, intuitively we might want to say that there are fewer real numbers whose first digit (in standard decimal notation) is 1 than there are Reals whose first digit is not 1. After all for every Real that begins with 1 there are nine other reals that are exactly like it except that they begin with another numeral. However, our theory cannot underwrite this intuition using the standard ordering of the reals. This is because any function which counts through the two sets will have an infinite value for every argument. Furthermore, if we take the infinite values to be cardinal numbers, then the values are the same. In general, the theory of infinitals thus far developed can only compare countable collections, for it's arguments and values must be finite.

5.1.2. *Time-Share Hell*

Imagine that you and I find ourselves standing before the Pearly Gates awaiting our assignment to Heaven or Hell. St. Peter looks at us and says "Well neither of you have been good enough to get into Heaven, but you haven't been bad enough to spend the rest of eternity in Hell either. So I'll let you share one spot in Heaven and one in Hell. When one of you is in Heaven, the other will be in Hell and you can switch back and forth. You two work out a schedule for switching, just make sure it's fair in the end." While we are trying to agree on a schedule, I propose the following to you. "You can spend your birthday in Heaven, I'll spend it in Hell. The other 364 days per year, I'll be in Heaven and you'll be in Hell. And just to show you how fair I'm being, you can be in Heaven on Leap Day too."

Now I take it as obvious that no one would agree to my proposal. But notice that since Heaven and Hell stretch for all eternity, the total number of days we each spend in Heaven and Hell ends up being equal, at least as far as cardinality is concerned. We each spend $\aleph_0$ days in heaven. But clearly what is important here is that at any point prior to eternity, I am ahead of you. We are, in a sense, back to the situation of the Race. I start out ahead and then keep getting farther and farther ahead. Here the progression of time provides the natural ordering within which our respective time in Heaven is tallied. Despite the fact that the number of days we spend in heaven will have the same cardinality, you can object to my proposal on the grounds that my days have a greater infinitality. Clearly this is the result our intuitions demand. If we had the power to reorder the days we could intersperse your birthdays with the other days so that you spent every other day in heaven. You would probably find such a reordering preferable to the original.

One way of thinking about this scenario is in terms of utility functions. For simplicity's sake, let's assume that each day in heaven adds 1 to our store of utility, while each day in hell subtracts one. On an alternating schedule, our utility stores would always be between $+1$ and -1. But on my proposal your utility store would tend to $-\infty$ while mine would tend to $+\infty$. Cardinality alone cannot capture or explain the fact that my proposal would make you increasingly miserable and me increasingly happy. But infinitals can make perfect sense of it. What matters in this case is not so much how many days are spent in heaven, but how those days are arranged.

5.1.3. *Arithmetic*

Another nice feature of infinitals is that they have a very intuitive arithmetic. Since infinitals are limits of functions, their arithmetic is simply the arithmetic of functions. To carry out an arithmetic operation on infinitals one simply carries out the corresponding operation on the underlying functions. Because of this, the arithmetic is very natural.

For example, if F and G are the infinital limits of the functions f and g respectively, then $\mathsf{F} + \mathsf{G} = \lim(f + g)$. Because of this the sum of any two infinitals is greater than either of them. This result does not hold for cardinal arithmetic. The sum of two infinite cardinals is just the larger of the two. So infinital addition acts more like finite arithmetic than does cardinal addition.

Subtraction deserves special mention because the contrast with cardinal arithmetic is especially acute. Specifically, $\aleph_\alpha - \aleph_\alpha$ is not well defined. Removing an infinite number of members from an infinite set may yield a zero membered set, a finitely membered set, or a set with infinite members depending on which members are removed. In order to know the result, one needs to know which members are removed, not merely the cardinality of the members removed. But this is just to say that infinite cardinals do not capture all of the relevant information. In this way, infinite cardinals do not behave very much like finite cardinals. Finite subtraction depends only on the number of things removed, not on their identity. Infinital subtraction, however, does behave like finite subtraction. For infinitals F and G, $\mathsf{F} - \mathsf{G}$ has a determinate value. Just what that value is, will of course depends on what F and G are, but each pair of infinitals has a determinate difference. Furthermore, we get the nice results that $\mathsf{F} - \mathsf{F} = 0$ and that $\mathsf{F} - \mathsf{G} = -(\mathsf{G} - \mathsf{F})$ as with finite subtraction.

As my mention of negative infinitals suggests, operations that are well defined on finite numbers typically will be well defined on infinitals. This nice parallel with finite numbers makes infinitals very intuitive to work with.

6. *Conclusion*

I have argued that the common notions of cardinality and ordinality do not exhaust our pretheoretic conceptions of size. In certain cases we have a strong intuition that one collection is larger than another despite the fact that the two collections are equinumerous and have the same ordinality. I have presented the basics of a theory of infinitals, based on the work of Paul du Bois-Reymond, and argued that it is able to underwrite these intuitions. I have pointed out some remaining problems for the theory, but leave these and a more complete presentation to a later paper.

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REFERENCES

[1] G. H. Hardy. *Orders of Infinity: The 'Infinitärcalcül' of Paul Du Bois-Reymond*, volume 12 of *Cambridge Tracts in Mathematics and Mathematical Physics*. Cambridge University Press, London, 1910.