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LOGIC AND COHERENCE IN THE LIGHT OF COMPETITIVE GAMES

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Abstract

The class of non-strictly competitive games is commonplace in game theory, but it has not been applied to logic before. In this paper, it is argued that one way of motivating non-coherence in logic is by means of the class of non-strictly competitive games, applied to the framework of semantic games. It is shown that just as partial logics are generated by games of imperfect information, formulas with over-defined truth-values arise either by having non-strictly competitive semantic games or by adding a weak negation to partial logic. Finally, a couple of implications to games and logic are discussed.

1. *Introduction*

How does the notion of non-coherence arise in logic? The received suggestions often fail to provide proper motivations, because either the unary connective of the system does not really function as a genuine negation, or else the semantics of the system as such is not foundationally motivated. This problem is reminiscent of the similar one in partial logics, for they have also lacked sufficient theoretical backing and an underlying mechanism that would explain reasons for having it in the first place.

However, comparable motivations can be seen to work for non-coherent formulas as for partial ones. Just as the phenomenon of partiality can be viewed as a game-theoretic reality arising in situations where there is imperfect information flow between players playing a semantic game on formulas, it will be shown that non-coherence arises in situations where the assumption that semantic games be strictly competitive is relaxed. Alternatively, even

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without this change in the defining characteristics of the semantic games, one can keep the strictness and affix, in addition to a strong game negation, a weak contradictory negation to the formulas of underlying logic of imperfect information. In both cases, the motivational foundation nonetheless lies in the theory of semantic games.

This paper proceeds by illustrating first the way in which the phenomenon of partiality is related to the failure of perfect information in game-theoretic semantics. Non-coherence is then derived by dropping the property of strict competition from the corresponding games. Finally, some implications to the questions of coherence and rationality in logic are discussed, as well as some aspects of negation in natural language. Further investigation into aspects of partiality and games in various propositional, first-order, and modal logics is reported in Pietarinen & Sandu 1999, Pietarinen 2002, Sandu & Pietarinen 2001.

2. *Aspects of partiality in logic*

2.1. *Motivation*

One way of generating partiality in logic is at the level of atomic sentences, that is, at the level of the signature σ. In this case sentences $S \in \sigma$ may be neither true nor false, which would be equivalent to the case where the models M of a language L are partial. A partial model M is a pair $M = \langle M^+, M^- \rangle$, where M^+ and M^- are disjoint subsets of σ, such that there might be sentences in $\sigma \notin M$, that is, $M^+ \cup M^- \neq \sigma$. The meaning of the superscripts is that M^+ denotes the set of true atomic sentences of $L(\sigma)$ and M^- denotes the set of false atomic sentences of $L(\sigma)$. If M^- is the complement of M^+, that is, $M^+ \cup M^- = \sigma$ and $M^+ \cap M^- = \emptyset$, the model M is a complete partial model. Consequently, a complete partial model is a classical one.

An alternative way of introducing partiality into propositional logic is by having a lack of truth-values at the level of complex sentences. In this case, the models can be complete but the standard language L is enriched with some new propositional connectives. This can be implemented by augmenting the set of classical connectives with a zero place connective $\circ$, and defining that for any M, $\circ$ is neither true nor false. Likewise, a zero-place connective $\Diamond$ could be introduced, defining that it is both true and false in any M. The resulting logic $L(\circ, \Diamond)(\sigma)$ can be shown to be functionally complete for all partial functions, a property that does not generally hold in partial models without these new connectives. The alternative is to introduce connectives that are more complex. The resulting logic turns out to have the properties of persistence and coherence, but it does not have the property of determinacy.

The third option is to have partiality as a result of combining the first and the second strategies. As expected, the logic will be persistent and coherent, but not determined.

Falsity of a sentence and it being not true are two different things, as are the truth of a sentence and its non-falsity. However, false sentences are also not true and true sentences are also not false, but non-true sentences do not imply that they are false and non-false sentences do not imply that they are true.

However, games make available an alternative semantics for a variety of logics. In games, the phenomenon of partiality is related to regulations on information flow within formulas, arising from an imperfect transmission of information between players. By introducing a non-determined propositional logic where in the associated semantic game the information can be lost with respect to connective choices, one can generate complex sentences corresponding to those lacking a definite truth-value in partial semantics. This is how the game-theoretic semantics accounts for the phenomenon of partiality: it is a consequence of certain natural informational constraints between players in a game.

Not only can partiality be endorsed from this game-theoretic viewpoint, but also the close relative of partiality, the property of non-coherence receives a game-theoretic motivation in terms of the class of non-strictly competitive games. This class is liable for the conception of non-coherent sentences having over-defined truth-values.

2.2. *Extensions of propositional logic*

Let the usual propositional language $L(\sigma)$ extended with a four-place connective $W(\varphi, \psi, \theta, \chi)$ be $L(W)(\sigma)$, the smallest set closed with respect to the sentences in σ, the familiar rules for the connectives in $\{\neg, \vee, \top\}$, and the following rule for W:

- If φ, ψ, θ, and χ are $L(W)(\sigma)$-sentences, then so is $W(\varphi, \psi, \theta, \chi)$.

Alternatively, the connective $W(\varphi, \psi, \theta, \chi)$ may be written as $W(\varphi_{ij})_{i,j \in \{1,2\}}$, with $\varphi_{11} = \varphi, \varphi_{12} = \psi, \varphi_{21} = \theta$, and $\varphi_{22} = \chi$. The sentences $\varphi_{11}, \varphi_{12}, \varphi_{21}$, and φ_{22} are expressions of the metalanguage.

Models for $L(W)$ are pairs $M = \langle M^+, M^- \rangle$ with $M^+, M^- \subseteq \sigma$, $M^+ \cap M^- = \emptyset$ (disjointedness), and $M^+ \cup M^- = \sigma$ (completeness). The semantics involves the notions $M \models^+ \varphi$ (the sentence φ is true in M) and $M \models^- \varphi$ (the sentence φ is false in M), and is defined by a double induction on the length of φ.

Definition 2.1: Let S be an atomic $L(W)(\sigma)$-sentence.

$$
\begin{array}{lll}
M \models^+ S & \text{iff} & S \in M^+ \\
M \models^- S & \text{iff} & S \in M^- \\
M \models^+ \top; \ \text{not } M \models^- \top & & \\
M \models^+ \neg\varphi & \text{iff} & M \models^- \varphi \\
M \models^- \neg\varphi & \text{iff} & M \models^+ \varphi \\
M \models^+ \varphi \vee \psi & \text{iff} & M \models^+ \varphi \ \text{ or } \ M \models^+ \psi \\
M \models^- \varphi \vee \psi & \text{iff} & M \models^- \varphi \ \text{ and } \ M \models^- \psi \\
M \models^+ W(\varphi, \psi, \theta, \chi) & \text{iff} & (M \models^+ \varphi \text{ and } M \models^+ \theta) \\
 & & \text{or } (M \models^+ \psi \text{ and } M \models^+ \chi) \\
M \models^- W(\varphi, \psi, \theta, \chi) & \text{iff} & (M \models^- \varphi \text{ and } M \models^- \psi) \\
 & & \text{or } (M \models^- \theta \text{ and } M \models^- \chi).
\end{array}
$$

The definition of truth and falsity for $W(\varphi, \psi, \theta, \chi)$ can equivalently be written as:

$$M \models^+ W(\varphi, \psi, \theta, \chi) \quad \text{iff} \quad \exists j \in \{1,2\} \forall i \in \{1,2\} M \models^+ \varphi_{ij}$$

$$M \models^- W(\varphi, \psi, \theta, \chi) \quad \text{iff} \quad \exists i \in \{1,2\} \forall j \in \{1,2\} M \models^- \varphi_{ij},$$

where $\varphi_{11} = \varphi, \varphi_{12} = \psi, \varphi_{21} = \theta$, and $\varphi_{22} = \chi$.

Let us fix a propositional logic L and an arbitrary signature σ. Let φ be an $L(\sigma)$-sentence and let the models be partial (not necessarily complete ones). By $M \subseteq N$ we mean that $M^+ \subseteq N^+$ and $M^- \subseteq N^-$. The sentence φ is truth-persistent, if for all models M, N in σ such that $M \subseteq N$, $M \models_L^+ \varphi$ implies $N \models_L^+ \varphi$, and falsity-persistent, if $M \models_L^- \varphi$ implies $N \models_L^- \varphi$. If φ is truth-persistent and falsity-persistent then φ is persistent. The logic L is persistent, if for any σ, φ is persistent in σ.

The sentence φ is coherent, if for all M not both $M \models_L^+ \varphi$ and $M \models_L^- \varphi$. L is coherent, if all $L(\sigma)$-sentences φ are coherent. The sentence is determined, if for all M either $M \models_L^+ \varphi$ or $M \models_L^- \varphi$. If, for any σ, all the $L(\sigma)$-sentences are determined then L is determined.

The proofs of the following results can be found in Sandu & Pietarinen 2001.

Proposition 2.1: *The logic $L(W)$ is coherent and persistent, but not determined.* □

The truth-value of the $L(W)(\sigma)$-sentence φ in the model M is denoted as $\|\varphi\|^M$. This function can have the following values.

$$\|\varphi\|^M = 1, \text{ if } M \models^+ \varphi \text{ (hence, by the previous proposition, not } M \models^- \varphi)$$

$$\|\varphi\|^M = 0, \text{ if } M \models^- \varphi \text{ (hence, not } M \models^+ \varphi)$$

$$\|\varphi\|^M = ?, \text{ if not } M \models^+ \varphi \text{ and not } M \models^- \varphi.$$

Let $\varphi \wedge \psi$ be defined as a shorthand for $\neg(\neg\varphi \vee \neg\psi)$, $\varphi \rightarrow \psi$ for $\neg\varphi \vee \psi$ and $\varphi \leftrightarrow \psi$ for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. In classical L, every truth-function $f\colon \{0,1\}^n \rightarrow \{0,1\}$ from a class of models K into $\{0,1\}$ is definable by a sentence in L such that for all $M \in \mathsf{K}$, $\|\varphi\|^M = f(M)$. This result extends to $L(W)$.

Theorem 2.1: *Let* K *be a class of classical models in* σ. *For any function* f *from* K *into* $\{0,1,?\}$, *there is an* $L(W)(\sigma)$*-sentence* φ *such that* $\|\varphi\|^M = f(M)$ *for all* $M \in \mathsf{K}$. □

2.3. *Weak negation*

The interpretation of '$\neg$' makes it strong (positive) negation, transforming truths to falsehoods and falsehoods to truths, but not meddling with non-determined values. There is a version of weak (contradictory) negation available, defined in the following way. Let $L(W,\neg_w)$ be $L(W)$ extended with a contradictory negation '$\neg_w$'. Definition 2.1 is now augmented with the following two clauses.

Definition 2.2: For any $L(W,\neg_w)(\sigma)$-sentence φ and a model M:

(i) $M \models^+ \neg_w\varphi$ iff not $M \models^+ \varphi$

(ii) $M \models^- \neg_w\varphi$ iff not $M \models^- \varphi$.

Having a contradictory negation defines $\varphi \rightarrow_w \psi$ as a shorthand for $\neg_w\varphi \vee \psi$ and $\varphi \leftrightarrow_w \psi$ for $(\varphi \rightarrow_w \psi) \wedge (\psi \rightarrow_w \varphi)$. The equivalence $\models^+ \varphi \leftrightarrow_w \psi$ holds when φ and ψ are true in exactly the same models, and $\models^+ \neg\varphi \leftrightarrow_w \neg\psi$ holds when φ and ψ are false in exactly the same models. The former can be called a positive equivalence and the latter a negative equivalence. If both equivalences hold, φ and ψ are strongly equivalent, denoted by $\models \varphi \leftrightsquigarrow \psi$.

If $M = \langle\{S_1, S_4\}, \{S_2, S_3\}\rangle$, then we have neither $M \models^+ W(S_1, S_2, S_3, S_4)$ nor $M \models^- W(S_1, S_2, S_3, S_4)$. But then by Definition 2.2, we have both $M \models^+ \neg_w W(S_1, S_2, S_3, S_4)$ and $M \models^- \neg_w W(S_1, S_2, S_3, S_4)$.

Consequently, the presence of weak negation introduces a fourth truth-value, and so the law of excluded fourth does not hold. The interpretation of an $L(W, \neg_w)(\sigma)$-sentence φ can have the following values.

$$\begin{aligned}
\|\varphi\|^M = &\quad 1, \quad \text{if } M \models^+ \varphi \text{ and not } M \models^- \varphi \\
\|\varphi\|^M = &\quad 0, \quad \text{if } M \models^- \varphi \text{ and not } M \models^+ \varphi \\
\|\varphi\|^M = &\quad ?, \quad \text{if not } M \models^+ \varphi \text{ and not } M \models^- \varphi \\
\|\varphi\|^M = &\quad !, \quad \text{if both } M \models^+ \varphi \text{ and } M \models^- \varphi.
\end{aligned}$$

It is readily seen that $L(W, \neg_w)$ is persistent (all the models are complete), but that it is neither coherent nor determined. Also, $L(W, \neg_w)$ has a functionally complete set of connectives.

Theorem 2.2: *Let* K *be a class of classical models in* σ. *For any function* $f\colon \mathsf{K} \to \{0, 1, ?, !\}$, *there is an* $L(W, \neg_w)(\sigma)$*-sentence* φ *such that* $\|\varphi\|^M = f(M)$ *for all* $M \in \mathsf{K}$. □

Given σ, a classical model M is a subset of σ, while a partial model N is a pair $\langle N^+, N^-\rangle$ with $N^+, N^- \subseteq \sigma$ and $N^+ \cap N^- = \emptyset$. If $N^+ \cup N^- = \sigma$, N can be turned into a classical model $N_C = N^+$, and similarly any M can be turned into a complete partial model $M_P = \langle M, \sigma \setminus M\rangle$. Thus there is a one-one correspondence between the class of classical models and the class of complete partial models.

Let us define by induction two mappings ‘*’ (the truth-preserving mapping) and ‘$^\sharp$’ (the falsity-preserving mapping), which map $L(W, \neg_w)(\sigma)$-sentences into $L(\wedge, \bot)(\sigma)$-sentences (where $L(\sigma)$ has the connectives $\vee, \neg$ and $\top$).

$$\begin{aligned}
S^\star &= S \\
\top^\star &= \top \\
(\neg\varphi)^\star &= \varphi^\sharp \\
(\neg_w\varphi)^\star &= \neg(\varphi^\star) \\
(\varphi \vee \psi)^\star &= \varphi^\star \vee \psi^\star \\
(\varphi \wedge \psi)^\star &= \varphi^\star \wedge \psi^\star \\
(W(\varphi, \psi, \theta, \chi))^\star &= (\varphi^\star \wedge \theta^\star) \vee (\psi^\star \wedge \chi^\star)
\end{aligned}$$

$$\begin{array}{lcl} S^\sharp & = & \neg S \\ \top^\sharp & = & \bot \\ (\neg\varphi)^\sharp & = & \varphi^\star \\ (\neg_w\varphi)^\sharp & = & \neg(\varphi^\sharp) \\ (\varphi \vee \psi)^\sharp & = & \varphi^\sharp \wedge \psi^\sharp \\ (\varphi \wedge \psi)^\sharp & = & \varphi^\sharp \vee \psi^\sharp \\ (W(\varphi,\psi,\theta,\chi))^\sharp & = & (\varphi^\sharp \wedge \psi^\sharp) \vee (\theta^\sharp \wedge \chi^\sharp). \end{array}$$

Theorem 2.3: *For any $L(W, \neg_w)(\sigma)$-sentence φ and a complete partial model M:*

(i) $M \models^+ \varphi$ *iff* $M_C \models \varphi^\star$

(ii) $M \models^- \varphi$ *iff* $M_C \models \varphi^\sharp$. $\square$

The language $L(W)$ is an extension of L interpreted on complete partial models, and it is persistent and coherent, but not determined. This extension does not add to the set of determined sentences of L, however. Likewise, $L(W, \neg_w)$ is persistent, non-coherent and non-determined, and it does not add to the set of coherent and determined sentences of L. To see this, fix an arbitrary $L(W, \neg_w)(\sigma)$-sentence ψ. Its image under the *-mapping is an $L(\wedge, \bot)(\sigma)$-sentence $\psi^\star$. If $\bot$ is replaced by $\neg\top$, $\psi^\star$ becomes an $L(\wedge)(\sigma)$-sentence, and since $\wedge$ can be defined by $\vee$ and $\neg$, $\psi^\star$ becomes an $L(\sigma)$-sentence.

Theorem 2.4: *For any $L(W, \neg_w)(\sigma)$-sentence φ, one can find an $L(\sigma)$-sentence ψ such that φ and ψ are positively equivalent, that is, $\models^+ \varphi \leftrightarrow_w \psi$.* $\square$

Theorem 2.5: *For any $L(W)(\sigma)$-sentence φ, φ is determined if and only if one can find an $L(\sigma)$-sentence ψ such that φ and ψ are strongly equivalent, that is, $\models \varphi \leftrightsquigarrow \psi$.* $\square$

Theorem 2.6: *For any $L(W, \neg_w)(\sigma)$-sentence φ, φ is coherent and determined if and only if one can find an $L(\sigma)$-sentence ψ such that $\models \varphi \leftrightsquigarrow \psi$.* $\square$

3. *Independence-friendly propositional language*

Let us consider a language which is a variant of an IF (independence-friendly) first-order language studied, for example, in Hintikka 1996, Hintikka & Sandu 1997, Sandu & Pietarinen 2001 and Pietarinen & Sandu 1999. It consists of a set Φ of propositional symbols, each having its own arity, and a finite set $i_1 \dots i_n$ of indices ranging over a set of two elements.

The well-formed formulas of L_{IF} are defined by the following clauses:

- If $p \in \Phi$, the arity of p is n, and $i_1 \dots i_n$ are indices, then $p_{i_1 \dots i_n}$ and $\neg p_{i_1 \dots i_n}$ are L_{IF}-formulas. Let us write $p_{i_1 \dots i_n}$ also as $p(i_1 \dots i_n)$.
- If φ and ψ are L_{IF}-formulas then $\varphi \vee \psi$ and $\varphi \wedge \psi$ are L_{IF}-formulas.
- If φ is an L_{IF}-formula then $\forall i_n \varphi$ and $\exists i_n \varphi$ are L_{IF}-formulas.
- If φ is an L_{IF}-formula then $(\exists i_n/U)\varphi$ is an L_{IF}-formula (U is a finite set of indices, $i_n \notin U$.)

The notions of free and bound variables are the same as in first-order logic. In $(\exists i_n/U)\varphi$ the indices on the right-hand side of the slash are free. For simplicity, the clauses for dual prefixes such as $(\forall i_n/U)$ are omitted.

The models for the language are of the form $M = \langle I^M, (p^M)_{p \in \Phi} \rangle$, where I^M is any set with two elements, and each p^M is a set of finite sequences of indices from I^M.

The sentences of L_{IF} are interpreted by semantic games. With every L_{IF}-sentence φ and a model $M = \langle I^M, (p^M)_{p \in \Phi} \rangle$ an extensive semantic game of imperfect information $\mathcal{G}^*(\varphi, M)$ is associated (see Appendix).

The game ends with an atomic formula or its negation $p(i_1 \dots i_n)$, and a sequence of elements $\langle a_1 \dots a_n \rangle$, where each $a_n \in I^M$. Let us stipulate that:

- If $\langle a_1 \dots a_n \rangle \in p^M$, then $\exists$ wins.
- If $\langle a_1 \dots a_n \rangle \notin p^M$, then $\forall$ wins.

Let $M \models^+_{\text{GTS}} \varphi$ mean truth of φ in M under the game-theoretic evaluation, and $M \models^-_{\text{GTS}} \varphi$ mean falsity of φ in M.

- $M \models^+_{\text{GTS}} \varphi$ iff there exists a strategy f that is winning for $\exists$ in $\mathcal{G}(\varphi, M)$;
- $M \models^-_{\text{GTS}} \varphi$ iff there exists a strategy f that is winning for $\forall$ in $\mathcal{G}(\varphi, M)$.

Proposition 3.1: *For any $L(W)(\sigma)$-sentence φ and a complete partial model M:*

(i) $M \models^+_{\text{GTS}} \varphi$ *iff* $M \models^+ \varphi$

(ii) $M \models^-_{\text{GTS}} \varphi$ *iff* $M \models^- \varphi$. □

We do not need any additional rules to the definition of semantic game rules. Instead, what we have is the informational partition of histories of the game, giving rise to strategy functions that are uniform on indistinguishable histories. This is done in order to account for the imperfect information, signalled by the indices in U. The intended meaning is that the player moving at $(\exists i_n/U)$ is not informed about the choices made for the elements in U. The information partition $(\Im_i)_{i\in N}$ will be as described in Appendix. In short, the information sets $S^i_j \in \Im_i$ tell what the players know and what they do not know when making their moves. If a player cannot distinguish between the histories within the same information set, he or she is not allowed to know something that has happened earlier in the game. When there are only singleton information sets, that is, no two histories belong to the same information set, one has perfect information as a special case.

Example 3.1: Let $M = \langle I^M, (p^M)_{p\in\Phi}\rangle$, where $I^M = \{\mathsf{Left}, \mathsf{Right}\}$.

$M \models^+_{\mathrm{GTS}} \forall i_1(\exists i_2/i_1)\, p_{i_1 i_2}$ iff $M \models \exists i_2 \forall i_1\, p_{i_1 i_2}$ iff
$\langle \mathsf{Left}, \mathsf{Left}\rangle \in p^M$ and $\langle \mathsf{Right}, \mathsf{Left}\rangle \in p^M$,
or $\langle \mathsf{Left}, \mathsf{Right}\rangle \in p^M$ and $\langle \mathsf{Right}, \mathsf{Right}\rangle \in p^M$.

$M \models^-_{\mathrm{GTS}} \forall i_1(\exists i_2/i_1)\, p_{i_1 i_2}$ iff $M \models \exists i_1 \forall i_2\, \neg p_{i_1 i_2}$ iff
$\langle \mathsf{Left}, \mathsf{Left}\rangle \in p^M$ and $\langle \mathsf{Left}, \mathsf{Right}\rangle \notin p^M$,
or $\langle \mathsf{Right}, \mathsf{Left}\rangle \in p^M$ and $\langle \mathsf{Right}, \mathsf{Right}\rangle \notin p^M$.

Given $W(\varphi, \psi, \theta, \chi)$, and assuming the obvious connections between $p_{i_1 i_2}$'s and φ, ψ, θ and χ, it follows that:

$$M \models^+ W(\varphi, \psi, \theta, \chi) \text{ iff } \langle I^M, (p^M)_{p\in\Phi}\rangle \models^+_{\mathrm{GTS}} \forall i_1(\exists i_2/i_1)\, p_{i_1 i_2}.$$

$$M \models^- W(\varphi, \psi, \theta, \chi) \text{ iff } \langle I^M, (p^M)_{p\in\Phi}\rangle \models^-_{\mathrm{GTS}} \forall i_1(\exists i_2/i_1)\, p_{i_1 i_2}.$$

Example 3.2: In a game for $W(\varphi, \psi, \theta, \chi)$, the information sets of a player at any non-terminal history will be singletons, except for the two histories h_1 and h_2 for which $L(h_1) = \varphi \vee \psi$ and $L(h_2) = \theta \vee \chi$, which are of course the immediate successors of h for which $L(h) = W(\varphi, \psi, \theta, \chi)$. A common way to indicate this in the extensive game representation is by drawing a dashed oval around the two histories whose labelled formulas are within the same information set, and annotating this information set for the respective player. The game is in Figure 1.

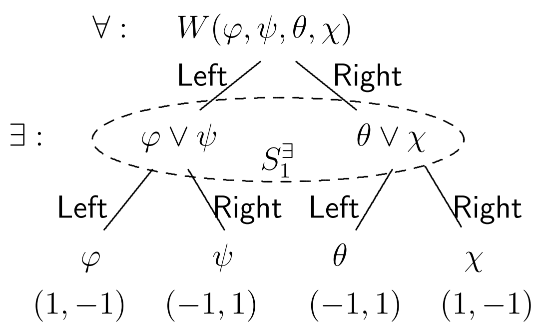


Figure 1. Extensive semantic game of imperfect information $\mathcal{G}^*(W(\varphi, \psi, \theta, \chi), M)$, with one nontrivial information set $S_1^{\exists}$ of $\exists$.

4. *Non-strictly competitive games*

One of the characteristic features of logics with imperfect information is that the negation '$\neg$' as given in the game rules denotes a strong game-theoretic negation, whose meaning is that the roles of the two players are transposed throughout the rest of the formula and the associated game. As noticed in §2, it is possible to introduce a weak contradictory negation '$\neg_w$' into $L(W)$, but as noticed in Hintikka & Sandu 1997, it cannot be quite captured by the game rules. The definition of this classical negation would rather be the familiar one, repeated here:

(i) $M \models^+ \neg_w \varphi$ iff not $M \models^+ \varphi$

(ii) $M \models^- \neg_w \varphi$ iff not $M \models^- \varphi$.

Nevertheless, in (i) the sentence $\neg_w \varphi$ being a truth-consequence of a model M says that φ cannot be verified, and in (ii) $\neg_w \varphi$ being a falsity-consequence of M asserts that φ cannot be falsified. The sentences prefixed with weak negation are thus assertions about games, denoting when a winning verifying or a winning falsifying strategy does not exist. As seen above, by letting $L(W, \neg_w)$ be $L(W)$ extended with this weak negation, it is readily seen that $L(W, \neg_w)$ is neither coherent nor determined, but that it is persistent.

Let us remark here that it is also possible to have just one direction in the definition of contradictory negation. That is, we can have definitions where (i) if not $M \models^+ \varphi$ then $M \models^+ \neg_w \varphi$, and (ii) if not $M \models^- \varphi$ then $M \models^- \neg_w \varphi$. These kinds of conditional non-truth-functional definitions of classical negations have not been studied in the context of partial or imperfect information logics before.

The way of creating the fourth truth-value by a contradictory negation is somewhat limited, however. The alternative approach would be to relax the assumption that games be strictly competitive altogether. One way to implement this idea is to go back to the definition of partial models as pairs $M = \langle M^+, M^- \rangle$ and to relax the assumption of disjointedness, namely that $M^+ \cap M^- = \emptyset$. In this case it would be obvious that if $M^+ \cap M^- \neq \emptyset$, there will have to be some terminal nodes in our extensive games that can be winning for both $\exists$ and $\forall$. In such games it may then happen that both players have a winning strategy, and one such example is provided by $\mathcal{G}^*(\varphi, M)$, where φ is $(p \vee q) \wedge (q \vee p)$, and $M = \langle \{p, q\}, \{p, q\} \rangle$. The difference to the previous case is that we get the fourth truth-value already in the logic $L(W)$ without any need of extending it to $L(W, \neg_w)$.

However, in order to get a different class of game to those of the received ones, we can drop the competitiveness:

Definition 4.1: The game $\mathcal{G}(\varphi, M)$ or $\mathcal{G}^*(\varphi, M), N = \{\exists, \forall\}$ is strictly competitive, iff for any $\varphi \in L$:

- if there exists a strategy f that is winning for $\exists$ then there does not exist a strategy g that is winning for $\forall$, and
- if there exists a strategy g that is winning for $\forall$ then there does not exist a strategy f that is winning for $\exists$.

If the game is not strictly competitive, call it non-strictly competitive. In non-strictly competitive games, it may happen that both players have a winning strategy in $\mathcal{G}$ or $\mathcal{G}^*$. One can for instance stipulate that there are some terminal histories $K \subseteq Z$ that are winning for both $\exists$ and $\forall$. This would be interpreted in a straightforward game-theoretic way by using the payoff function $u_i(h)$ that gives the matrix $(1, 1)$ for those histories in K, in addition to the zero-sum ones attached to those histories in $H \setminus K$. Consequently, given a literal ψ it will be interpreted as $\|\psi\|^M = !$, that is, is has both the truth-value True and the truth-value False, and hence, has a truth-value $\mathsf{Overdefined}$.

Example 4.1: An example of a non-strictly competitive game for the formula

$$\phi = \forall i_1 (\exists i_2 / i_1)\, \varphi_{i_1 i_2}, \tag{1}$$

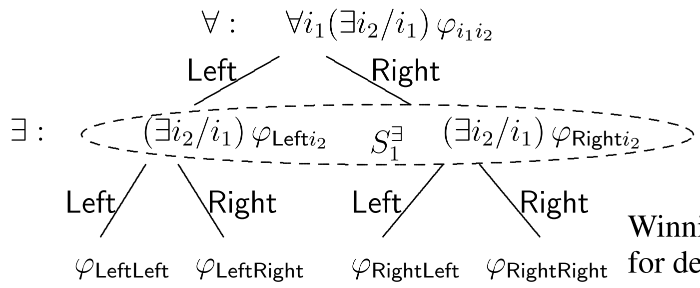
with $\varphi_{i_1 i_2}$ as atomic, is given in Figure 2. As to the game $\mathcal{G}_1^*(\phi, M)$, the first line in the payoff distribution says that $M \models^+ \varphi_{\mathsf{LeftLeft}}, M \models^+ \neg\varphi_{\mathsf{LeftLeft}}$, $M \models^+ \varphi_{\mathsf{LeftRight}}, M \models^- \neg\varphi_{\mathsf{LeftRight}}$ and so on. That is, $u_\exists((\mathsf{Left}, \mathsf{Left})) = u_\forall((\mathsf{Left}, \mathsf{Left})) = 1, u_\exists((\mathsf{Left}, \mathsf{Right})) = 1, u_\forall((\mathsf{Left}, \mathsf{Right})) = -1$, and so on. Hence $\varphi_{i_1 i_2}$ can receive both the value True and the value False, namely at $L(h) = \varphi_{\mathsf{LeftLeft}}$. This does not mean that the whole formula in interpreted similarly, since for that to be the case there would have to be suitable winning strategies for both players.

Let us call the pairs of strategies that are winning for both players those *for non-coherence*. In the previous example, winning strategies for non-coherence do not exist in ϕ, because there is no winning strategy that would lead $\forall$ to $L(h) = \varphi_{\mathsf{LeftLeft}}$. The same holds for $\exists$, because she has to use a uniform strategy that gives her either Left or Right at both histories $\forall$ has chosen, leading her to $L(h) = \varphi_{\mathsf{LeftLeft}}$ and $L(h') = \varphi_{\mathsf{RightLeft}}$, or $L(h) = \varphi_{\mathsf{RightLeft}}$ and $L(h') = \varphi_{\mathsf{RightRight}}$, while only $\varphi_{\mathsf{LeftLeft}}$ receives the variable-sum payoff $(1, 1)$.

With respect to determinacy, the presence of non-zero-sum payoffs can cancel the effect of imperfect information, which otherwise would have turned a strictly competitive game into a non-determined one. To see this, let the strategies that change a non-determined game into a determined one be termed *winning strategies for determinacy*. Assume that $u_\exists(h) = u_\forall(h) = 1$ for some $h \in Z$, and that the rest of the payoffs are strict. Then all $h' \in Z$ reached from an immediate predecessor $pr(h) \in H$ have to get $u_\exists(h') = -1$ or $u_\forall(h') = -1$, because otherwise a player would have a winning move at $pr(h)$. Suppose that a strategy that a player applies at $pr(h)$ is uniform. Then every action from $pr(h)$ has to have a corresponding action at k, such that $pr(h), k \in S^i_j, i \in \{\exists, \forall\}$. But any action a corresponding to the action that would leading to h can lead only to $k' = k \frown a$ that has either $u_\exists(k') = 1$ or $u_\forall(k') = 1$, which hence constitutes a winning step for either $\exists$ or $\forall$.

This situation can be illustrated by concerning all the possible payoff distributions and winning strategies for determinacy for the game $\mathcal{G}^*_i(\phi, M), i = 1 \dots 8$ of Example 4.1 (see Figure 2). (It is assumed, for simplicity, that non-coherent payoffs obtain only as $u_\exists((\mathsf{Left}, \mathsf{Left})) = u_\forall((\mathsf{Left}, \mathsf{Left})) = 1$.) What are now needed in order to restore non-determinacy are in this case partially interpreted models. Namely, we would need to have a partial logic where partiality arises at the level of atomic formulas. In terms of this example, setting $L(h) = \varphi_{\mathsf{RightLeft}}, h \in Z, u_\exists(h) = -1, u_\forall(h) = -1$ would turn any $\mathcal{G}^*_i(\phi, M), i = 1 \dots 8$ into a non-determined game.

In general, non-strictly competitive games are useful in distinguishing between different notions of consistency: even if a version of *ex falso sequitur quodlibet* could be tolerable as $\varphi \wedge \neg\varphi$, it would never be the case that $\varphi \wedge \neg_w \varphi$, which is absurd, for it does not make sense to assert that 'there exists a winning strategy for $\exists$ in φ, but there does not exist a winning strategy for $\exists$ in φ'. Such statements denote a strong version of inconsistency. Should we assume that there is no principled reason to prevent that the two players cannot both have winning strategies in the same game, some sentences could



	$\varphi_{\mathsf{LeftLeft}}$	$\varphi_{\mathsf{LeftRight}}$	$\varphi_{\mathsf{RightLeft}}$	$\varphi_{\mathsf{RightRight}}$	Winning strategies for determinacy:
$\mathcal{G}_1^*$:	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,1)$	$\forall$: Right
$\mathcal{G}_2^*$:	$(1,1)$	$(1,-1)$	$(-1,1)$	$(1,-1)$	$\exists$: Right
$\mathcal{G}_3^*$:	$(1,1)$	$(1,-1)$	$(1,-1)$	$(-1,1)$	$\exists$: Left
$\mathcal{G}_4^*$:	$(1,1)$	$(1,-1)$	$(1,-1)$	$(1,-1)$	$\exists$: Left or $\exists$: Right
$\mathcal{G}_{5-8}^*$:	$(1,1)$	$(-1,1)$			$\forall$: Left

Figure 2. Eight non-strictly competitive but determined games $\mathcal{G}_i^*(\phi, M), i = 1 \ldots 8$.

from a game-theoretic perspective be weakly inconsistent or non-coherent. The possibility of having $\varphi \wedge \neg\varphi$ in fact follows directly from the definition of non-strictly competitive games.

The presence of imperfect information has nonetheless a noticeable effect on winning strategies, in the sense that whereas non-strict payoffs may override it and make a non-determined game a determined one, non-strict winning strategies for non-coherence may nonetheless not exist.

Given a zero-sum attribute of a strictly competitive game, the partial truth-values impose a loss of verifying as well as a loss of falsifying capability on players. Even more importantly, we could have a more expressive game-theoretic framework at hand that would allow us to speak about players' preferences. In that case we would see that in strictly competitive games, players' preferences would be inverses of each other, but yet if the preferences are not assumed to be strictly opposed, a definite truth-value for a sentence does not need to mean a serious deprivation of the purposes and the motivation of the adversary. Just to mention one thing, as known in the theory of games, in non-strictly competitive games the strategy functions may be exposed to the opponent, at least in part and at times, and it sometimes is even strategically advantageous to do so.

5. *Games and competition in a logical perspective*

Motivation for studying non-strictly competitive games in logic is by and large derived from the theory of games, where the strictness is merely a historical remnant of the early developments of the theory, and most of the contemporary work focussed on the non-strict ones.[1] The abundance of non-strict games strongly suggests that in logic, where game-theoretic concepts are quite customarily being applied, one should not rule such games out offhand. Just as physical instances of games can be used as evidence that games in logic often encompass imperfect information, we ought to examine the other basic class of games in logic too.

In the class of strictly competitive games, players' interests are entirely in conflict, the fact that often emerges when players are dividing some fixed amount of gain. From the logical viewpoint this means that players aim at ending up with either true or false atomic sentences. However, when there is some surplus to be divided, games usually do not operate on strictly competitive environments. This suggests that a division of surplus may be an illustrative reflection or metaphor of what goes on in such semantic games where players try to agree on some propositions that are to be distributed among the participants. In this manner we obtain a game-theoretic explication of non-strict games, as a division of excess. One way of actually implementing this proposal is that either of the two players can re-select the already attained atomic sentence, after which the atomic sentence receives a renewed valuation. For example, if $\exists$ chooses an atomic sentence it becomes true (if it already wasn't true), and likewise, $\forall$'s selection renders the sentence false.

A reasonable question to be asked at this point is whether this way of modelling the games has some concrete effect on players' strategies, especially on the winning ones. Obviously one should not insist that any atomic sentence counts as a surplus to be invariably re-distributable among players, since such a manoeuvre would make games to have trivial winning strategies. One can easily see, however, that some new deals, or divisions of surplus, can be done such that they benefit all contestants. New deals may hold out new winning strategies, once the correct locations have been detected and reached in any play of the game.

Yet another important thing is that in non-strictly competitive semantic games, models can be either complete or partial. Every atomic proposition receives a definite valuation, or then there are partial models. But as

[1] Vibrant stories concerning these kinds of games in various social contexts are found in Wright 2000.

seen above, partial models turn the logic in question non-persistent, independently of whether the games are strictly or non-strictly competitive.

Likewise, we can give non-standard definitions of truth and falsity for non-strictly competitive games as well as for strict ones without changing the persistence or coherence properties of a logic. The two alternative non-standard definitions are as follows (φ is atomic, M is a partial model):

- If $L(h) = \varphi$, and not $M \models^- \varphi$ or $M \models^+ \varphi$, then $u_\exists(h) = 1$ (and $u_\forall(h) = -1$, by strictness).
- If $L(h) = \varphi$, and $M \models^- \varphi$, then $u_\forall(h) = 1$ (and $u_\exists(h) = -1$, by strictness).

Or, alternatively:

- If $L(h) = \varphi$, and $M \models^+ \varphi$, then $u_\exists(h) = 1$ (and $u_\forall(h) = -1$, by strictness).
- If $L(h) = \varphi$, and not $M \models^+ \varphi$ or $M \models^- \varphi$, then $u_\forall(h) = 1$ (and $u_\exists(h) = -1$, by strictness).

Similar things happen with respect to perfect versus imperfect information. The distinction of perfect versus imperfect information does not affect the property of persistence, although it affects the property of determinacy. The difference between games for logics with partial models and games for logics with complete models is that in the former, not all terminal positions count as winning ones for any player, and therefore winning strategies cannot be based on such vacuous payoffs.

The other effect is that as seen in the previous sections, non-strict games may be determined even if their strict counterparts are not, and non-determinacy can in that case be restored only by having partial models. It should be mentioned that some systems of paraconsistent logic admit that non-atomic but inconsistent formulas are trivial, which means that anything can be derived from them, and thus non-triviality holds only for atomic inconsistencies (Sette 1973). The game-theoretic perspective does not endorse this, as games can transmit inconsistencies from non-constant-sum payoffs of atomic formulas to complex ones via suitable winning strategies. In this case the existence of imperfect information may nonetheless affect this transmission.

What is being incoherent in non-strictly competitive semantic games is the existence of some mutually beneficiary plays. These plays may consist of just a tiny fragment of the totality of plays. Besides, some minimal inconsistencies are likely to remain hidden when the game is played. Since no winning strategy for one player can reach them as such, and because even the process of establishing winning strategies is a non-trivial (epistemic) matter, one way of looking at them is as a reflection of human ineptitude in detecting minimal hidden inconsistencies. This can be particularly understandable in situations involving massive amount of information, such as in knowledge-based systems.

The possibility of controlling logic by simply altering some of the basic characteristics of semantic games has some far-reaching implications, as can readily be seen from the fundamental differences between perfect information and imperfect information. Some of the further possibilities in manipulating games can be found in the class of games of incomplete information, where players do not have complete information about the mathematical structure of the game. The structure can be taken to codify things such as the roles of the players or some other random deals based on chance moves by Nature, among other things.

These findings also suggest that an account of mitigating logical omniscience — the problem in epistemic logics of how to prevent an indefinite production of logical consequences about what is known — is forthcoming: by means of game-theoretic semantics for epistemic logics where the games are non-strict, one can readily see how there can be inconsistent worlds (impossible possible worlds in Hintikka 1975), namely worlds that can be epistemically possible in the sense that a player can pick them in a game, but which nevertheless are not logically possible. This idea then becomes just another manifestation of the game-theoretically important notion of bounded rationality.

Yet another implication is the need of taking seriously the possibility of players in a semantic game to have at least some cooperation. This follows because in non-constant-sum games, cooperative solutions are the only truly rational solutions, because of the known fact in game theory that in the presence of cooperation, any non-constant sum game can be converted to a game that profits all participants. A further examination of this result would invariably yield to analyses of players' preferences. In general, cooperative games display at least some compatibility among agents, in order for the efficient outcomes to be legitimately attained.

One caveat is in order here, however. Because of the game-theoretic behaviour of negation, it is not to be expected that the ensuing systems would yield to something known as preservationist treatments of paraconsistent logic. Yet no downright dialetheic approach is being suggested by these games either, since the negation has a non-trivial game-theoretic explication in the activity of a role-switch. No comparable explanation is to be expected from preservationist or dialetheic systems.

6. *Conclusion*

Non-coherence arises when the assumption of games being strictly competitive is relaxed. How viable is this assumption? A number of real life games are not strictly competitive, as witnessed by the prisoner's dilemma, differential games, bargaining games, and so on. In this light, non-coherence is just a logical reflection of non-constant-sum games. When we apply these games to logics, we see how a 'non-classical' logic suddenly emerges — like for partial ones, we have here further evidence that as far as these non-classical logics are concerned, games are a superior semantic theory for them.

It is not the purpose here to provide any methodical treatise of paraconsistent logics, or their relations to the suggestions of this paper. Suffice it to mention that in paraconsistent logics, a continuing problem is that valuations of logical constants are rather arbitrarily chosen. For example, by varying the interpretation of negation one can generate minimal definitions of a paraconsistent system. Such manoeuvres are void of theoretical impetus, however, unless some independent insight into the inconsistencies thereby provoked can be provided.

Related to this point is the so-called Jaśkowski's problem (Jaśkowski, 1948). In brief, it asks for a logic claiming the name of paraconsistency to fulfil three conditions: First, it needs to have a negation leading to a paraconsistent system (that is, to an inconsistent but non-explosive system). Second, its negation must be strong enough to be called negation. Thirdly, its semantics needs to be well motivated. To date, this problem has remained unsolved. The insights of this paper are calculated to answer to this problem, however. Game-theoretic semantics for 'paraconsistent logic' is a well-motivated and systematic method, in contrast to the previous attempts in the literature that remain to be based on some negative criteria — for example, they describe principles that must be rejected, such as ex falso, consistency, or triviality. Yet the game-theoretic negation is a genuine negation, as can be observed from its relation to negative constructions in natural language.

It is thus possible now to see why the question of whether the negating operator in non-coherent systems really denotes 'real' negation is somewhat ill defined (Brown 1999, Slater 1995). In any sufficiently expressive language, there will inevitably be more than one negation present, contributing to various forms of coherence with differing effects.

To see just one example of a possible division of labour between negative expressions in natural language, several distinctions can be made with respect to the type of the negation operator and the properties of its linguistic environment. For example, there can be (i) subminimal negative expressions (e.g., *few* N, *only a few* N, *not all* N, *at most*), (ii) minimal negations (e.g., *none of the* N, *neither* N, *no one*, *not a single*, *not a*), and (iii) classical (weak) negations (e.g., *none of the* N, *no* N, or a negative adverb *not* as in do*n't*). These classes have different characteristics, and the three types of negations can be distinguished from each other by the underlying hierarchy of their functional behaviour.[2]

For example, the quantifier *few* does not respect the classical de Morgan laws, since from 'Few men laughed and jumped' it does not follow that 'Few men laughed and few men jumped', given a model where most of the men jumped but not laughed. The interesting point here is that the negative expressions, apart from just the classical one, do not usually form genuine complements that could be used as a basis for generating inconsistent statements. This can be seen from the example where the complement of the sentence 'Few men laughed', for instance, is not 'Few men didn't laugh', because it introduces an auxiliary classical negative adverbial. The negated expression would rather be along the most salient reading of the sentence 'Many men laughed'.

It would nonetheless be misleading simply to assimilate the game negation with any of these classes of negation. For the role-switch does not express any syntactically marked negative element of language. Rather, it pertains to the strategic resources from which the sentence meaning is derived. This can be illustrated by saying that the interchange of the players in any part of the game can be taken to not mean that something does not happen or does not hold, but that the opponent is given an opportunity of achieving something useful.

Appendix A. Games in extensive forms

A.1. *Perfect information*

Let us fix a family of actions A, where a finite sequence $\langle a^i \rangle_{i=1}^n, n \in \omega$ represents the consecutive actions of players in N (no chance moves), $a^i \in A$.

Definition A.1: An extensive-form game $\mathcal{G}$ of perfect information is a five-tuple

$$\mathcal{G}_A = \langle H, Z, P, N, (u_i)_{i \in N} \rangle$$

such that

[2] Briefly, (i) occur in downward entailing environments $f(X \cup Y) \subseteq f(X) \cap f(Y)$, $f(X) \cup f(Y) \subseteq f(X \cap Y)$; (ii) cover anti-additive expressions satisfying $f(X \cup Y) = f(X) \cap f(Y)$, and (iii) describe anti-morphic expressions $f(X \cup Y) = f(X) \cap f(Y)$ plus the classical $f(X) \cap f(Y) = f(X \cap Y)$, $f(\neg X) = \neg f(X)$, corresponding to classical negation.

- H is a set of finite sequences of actions $h = \langle a^i \rangle_{i=1}^n$ from A, called histories of the game. It is required that:
  - the empty sequence $\langle\rangle$ is in H;
  - if $h \in H$, then any initial segment of h is in H too, that is, if $h = \langle a^i \rangle_{i=1}^n \in H$ then $pr(h) = \langle a^i \rangle_{i=1}^{n-1} \in H$ for all n, where $pr(h)$ is the immediate predecessor of h ($= \emptyset$ for $h = \emptyset$).
- Z is a set of maximal histories (complete plays) of the game. If a history $h = \langle a^i \rangle_{i=1}^n \in H$ can continue as $h' = \langle a^i \rangle_{i=1}^{n+1} \in H$, h is a non-terminal history and $a^n \in A$ is a non-terminal element. Otherwise they are terminal. Any $h \in Z$ is terminal.
- $P : H \backslash Z \to N$ is the player function which assigns to every non-terminal history a player in N whose turn is to move.
- each $u_i, i \in N$ is the payoff function, that is, a function which specifies for each maximal history the payoff for player i.

For any non-terminal history $h \in H$ define

$$A(h) = \{x \in A \mid h \frown x \in H\}.$$

A (pure) strategy for a player i is any function

$$f_i : P^{-1}(\{i\}) \to A$$

such that $f_i(h) \in A(h)$, where $P^{-1}(\{i\})$ is the set of all histories where player i is to move. A strategy specifies an action also for histories that may never be reached.

In strictly competitive game, $N = \{\exists, \forall\}$ and in addition:

- $u_\exists(h) = -u_\forall(h)$;
- either $u_\exists(h) = 1$ or $u_\exists(h) = -1$ (that is, $\exists$ either wins or loses);

for all terminal histories $h \in Z$.

A.2. *Imperfect information*

Definition A.2: Let $\mathcal{G}_A$ be a perfect information game. To represent imperfect information, let us extend $\mathcal{G}_A$ to a six-tuple

$$\mathcal{G}_A^* = \langle H, Z, P, N, (u_i)_{i \in N}, (\Im_i)_{i \in N} \rangle$$

where $\Im_i$ is an information partition of $P^{-1}(\{i\})$ (the set of histories where i moves) such that for all $h, h' \in S_j^i, h \frown x \in H$ if and only if $h' \frown x \in H, x \in A, j = 1 \ldots m, i = 1 \ldots k, m \leq k$. S_j^i is called an information set.

The games are exactly as before, except that now players might not have all the information about the past features of the game. This is achieved by an information partition, which partitions histories into information sets (equivalence classes). Those histories that belong to the same information set are indistinguishable to the players, and thus a player takes no notice of the history that has been played.

In imperfect information games, the strategy function is required to be uniform on indistinguishable histories:

$$\text{If } h, h' \in S^i_j \text{ then } f_i(h) = f_i(h'), \text{ for all } i \in N.$$

A.3. *Semantic games in extensive forms*

Let Sub(φ) denote a set of subformulas of φ.

Definition A.3: An extensive-form semantic game $\mathcal{G}^*(\varphi, M)$ associated with an $\mathcal{L}_{IF}$−formula φ is like an extensive game $\mathcal{G}^*_A$ defined above, except that it has one extra-element: a labelling function $L : H \to \text{Sub}(\varphi)$ such that

- $L(\langle\rangle) = \varphi$ (the root);
- for every terminal history $h \in Z, L(h)$ is an atomic formula or its negation.

In addition, the components H, L, P, u_V and u_F jointly satisfy the following:

- if $L(h) = \neg\varphi$ and $P(h) = \exists$, then $h \in H, L(h) = \varphi, P(h) = \forall$;
- if $L(h) = \neg\varphi$ and $P(h) = \forall$, then $h \in H, L(h) = \varphi, P(h) = \exists$;
- if $L(h) = \psi \vee \varphi$ or $L(h) = \psi \wedge \varphi$, then $h \frown \mathsf{Left} \in H, h \frown \mathsf{Right} \in H, L(h \frown \mathsf{Left}) = \psi$, and $L(h \frown \mathsf{Right}) = \varphi$;
- if $L(h) = \psi \vee \varphi$, then $P(h) = \exists$;
- if $L(h) = \psi \wedge \varphi$, then $P(h) = \forall$;
- for every terminal history $h \in Z$:
  - if $L(h) = p_{i_1 \dots i_n}$ and $\langle a_1 \dots a_n\rangle \in p^M$, then $u_\exists(h) = 1$ and $u_\forall(h) = -1$;
  - if $L(h) = p_{i_1 \dots i_n}$ and $\langle a_1 \dots a_n\rangle \notin p^M$, then $u_\exists(h) = -1$ and $u_\forall(h) = 1$.

The notion of strategy is defined in the same way as before. A winning strategy for $i \in \{\exists, \forall\}$ is a set of strategies f_i that leads i to $u_i(h) = 1$ no matter how the player $-i$ (the player other than i) decides to act.

If there is imperfect information, players may not be able to distinguish between some of the game histories. This is indicated by the information partition $(\mathfrak{I}_i)_{i \in N}$ of Definition A.2, the information sets S^i_j spelling out the information available to the players when making their moves. When there are only singleton information sets, that is, no two history belongs to the same information set, then the game is one of perfect information. Otherwise the game is one of imperfect information.

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REFERENCES

Brown, B.: (1999). Yes, Virginia, there really are paraconsistent logics, *Journal of Philosophical Logic* 28, 489–500.

Hintikka, J.: (1975). Impossible possible worlds vindicated, *Journal of Philosophical Logic* 4, 475–484.

Hintikka, J.: (1996). *The Principles of Mathematics Revisited*, New York: Cambridge University Press.

Hintikka, J. and Sandu, G.: (1997). Game-theoretical semantics, in van Benthem, J. and ter Meulen, A., (eds), *Handbook of Logic and Language*, Amsterdam: Elsevier, 361–410.

Jaśkowski, S.: (1948). Rachunek zdań dla systemów dedukcyjnych sprzecznych, *Studia Societatis Scientiarum Toruensis* 5, 55–77. (Translated in English as: Propositional Calculus for Contradictory Deductive Systems, *Studia Logica* 24, 1969, 143–157.)

Pietarinen, A.: (2002). Knowledge constructions for artificial intelligence, in Hacid, M.-S., Ras, Z.W., Zighed, D.A., Kodratoff, Y., (eds), *Foundations of Intelligent Systems*, LNAI 2366, Springer-Verlag, 303–311.

Pietarinen, A. and Sandu, G.: (1999). Games in philosophical logic, *Nordic Journal of Philosophical Logic* 4, 143–173.

Sandu, G. and Pietarinen, A.: (2001). Partiality and games: Propositional logic, *Logic Journal of the IGPL* 9, 107–127.

Sette, A.M.: (1973). On the propositional calculus P1, *Mathematica Japonae* 16, 173–180.

Slater, B.H.: (1995). Paraconsistent logics?, *Journal of Philosophical Logic* 24, 451–454.

Wright, R.: (2000). *Nonzero: The Logic of Human Destiny*, New York: Pantheon Books.