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THE REVOLUTION IN THE PHILOSOPHY OF MATHEMATICS

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Abstract
This paper offers a general survey of the recent development of the philosophy of mathematics, while focusing on the radical changes of the central problems, the methodology, the basic positions for research, and the basic views of mathematics, which as a whole, in the author's view, clearly represents a revolution in this field. Besides, it also includes an analysis about the significance of the philosophy of mathematics to actual mathematical activities.

Since the sixties and seventies of this century, the philosophy of mathematics has in its historical development undergone great changes, which include not only the radical transition of the basic views of mathematics, but big changes in its central problems, methodology, and basic positions for research as well. Using the terminology of the philosophy of science, we can say that there has been a revolution in the philosophy of mathematics.

Specifically, the revolution in the philosophy of mathematics I will be talking about here is defined as opposed to the foundational studies of mathematics.

So, as a necessary basis for the discussion, I shall first make a brief survey of the foundational studies of mathematics; and then offer an analysis about the main features of the recent development of the philosophy of mathematics, which in my view speaks clearly for its revolutionary nature; and finally, I will give some more words on revolutions in the philosophy of mathematics, and on the significance of the philosophy of mathematics to actual mathematical activities as well.

1. *The Foundationalist Philosophy of Mathematics*

The fifty years between 1890 to 1940 has been described as the golden age of the philosophy of mathematics. During that period, G. Frege, B. Russell, L.E.J. Brouwer, D. Hilbert and some others made systematic studies on the foundations of mathematics, thus developing respectively logicism, intuitionism and formalism which all exerted widespread and profound influence on the contemporary and further development of the philosophy of mathematics, and so opened a new epoch for the research of the philosophy of mathematics. (cf. Benacerraf & Putnam [1964])

Although the basic positions adopted by logicists, intuitionists and formalists in their foundational studies are different, these three schools also have some common elements, which in my view can properly be regarded as the main features of the foundationalist philosophy of mathematics.

(1) The foundational studies have a long tradition in mathematics. Because mathematical theories have deductive structure, 'when we start from its most familiar positions', B. Russell pointed out in his *Introduction to Mathematical Philosophy*, 'two opposite directions might be pursued', that is, either by studying 'what can be defined and deduced from what is assumed to begin with', to move gradually towards increasing complexity; or by asking 'what more general ideas and principles can be found, in terms of which what was our starting-point can be defined or deduced', to reach increasingly greater abstractness and logical simplicity. Obviously, the latter proper is the foundational studies in a general sense. A case in point here is the so-called 'Arithmetization of analysis' in the 19th century.

It is to be noted here that, in comparison with the foundational studies in a general sense, the research work carried out by the above-mentioned three schools had a special background which, generally speaking, had to do with the so-called '(the third) foundational crisis of mathematics'. Especially, as set theory had been taken as a firm foundation for the whole mathematics, the discovery of paradoxes in set theory was in fact a great threat to the soundness of the whole of mathematics, and indeed aroused great confusion and unsettlement among mathematicians at that time, and so showed clearly the necessity and urgency for the foudational studies of mathematics.

The great concern about the soundness of mathematical theories and methods thus had constituted the common starting-point for all the three schools' foundational studies. In other words, the foundationalist philosophy of mathematics was focusing on the following question: how could we resolve the problem of the soundness of mathematics?

(2) Deeply concerned with the soundness of the extant mathematics, the three schools all took critical and reformative positions in their foundational studies. That is to say, according to the three schools, all the mathematical theories and methods built before should be strictly criticized or carefully examined, and further, the corresponding reform or rebuilding work should be carried out, and only by such work, could the problem of the soundness of mathematics be solved once for all. So, just as Benacerraf and Putnam [1964] said, what attracted the schools was mainly 'the question of what an acceptable mathematics should be like: what methods, practices, proofs, and so on, are legitimate and therefore justifiably used.'

Therefore, the foundationalist philosophy of mathematics was mainly prescriptive rather than descriptive in nature.

(3) As concrete manifestations of their basic positions, logicists, intuitionists and formalists had all put forward their own programs for the foundational studies. For logicists, it was how to build the whole mathematics on the basis of logic; while intuitionists tried to replace the old 'classical' mathematics with some new mathematics which was developed according to the criterion of 'constructibility'; the basic idea of Hilbert's program was to organize mathematical theories with infinite elements into formal systems and then to prove the consistency of the systems with 'finite method'.

Thus it could be seen that the foundational studies of all the three schools were characterized by logical analysis. In other words, logical analysis was the basic method for the foundationalist philosophy of mathematics.[1]

It was precisely due to their emphasis of logical method, that the foundationalist philosophy of mathematics seemed to dismiss the actual history of mathematics. For example, in their view, the problem of discovery of mathematics belonged wholly to the field of psychology and had nothing to do with the philosophical analysis of mathematics.

(4) The ultimate goal of all the three schools was to found through their own work a final and firm basis for mathematics, and thus to solve the problem of the soundness of mathematics once for all. Therefore, although they had different opinions on what the final basis of mathematics really was, the three schools on the whole all held a static, absolute view of mathematics. That is to say, they all believed that the development of mathematics was chiefly a monotonous increase of indubitable mathematical truths in number.

What is more, in the opinion of the foundationalists, the final basis of mathematical truths was not practice (experience) but pure reason. In this sense, their static, absolute view of mathematics was directly opposed to the empirical view of mathematics. Kitcher [1988] said clearly on this point: 'Foundationalist philosophy of mathematics bear a tacit commitment to apriorist epistemology. ... Subtract the apriorist commitments and there is no motivation for thinking that there must be some first mathematics, some special discipline from which all the rest must be built.'

[1] Of course, as far as the basic philosophical position is concerned, intuitionists would insist that it was intuition rahter than logic which should be taken as the final basis of mathematics; however, what intuitionists aimed at was still the reconstruction of mathematics which obviously relied on the analysis of the logical structure of the whole subject, although the intuitionist logic was delineated only by A. Heyting's later work.

2. *The End of the Foundationalist Epoch*

The foundationalist epoch of the philosophy of mathematics has now passed. This in a certain sense can be described as a negative development: although the three schools had done their best, their programs for foundational studies all failed at last. Thus, after the golden age mentioned above, the development of the philosophy of mathematics passed into 'a pessimistic, stagnant period'. The situation showed clearly in the words of Weyl (cf. Kline, [1972]):

> "The question of the ultimate foundations and the ultimate meaning of mathematics remains open; We do not know in what direction it will find its final solution or even whether a final objective answer can be expected at all."

Yet, the historical development of the philosophy of mathematics of course would not stop. Some main elements which resulted in its new development are summarized as follows:

(1) The reflections on and the criticism of the foundationalist studies in general

For example, there is a deep dissatisfaction among the people concerned with the philosophy of mathematics with all the views put forward by the foundationalist, and also a feeling of the need to find new ways to get the philosophy of mathematics out of distress. For instance, although Robinson [1964] described the fifty years from 1890 to 1940 as 'the golden age of the philosophy of mathematics', he also asserted that 'all points of view that have been put forward as a philosophical basis for Mathematics involve serious gaps and difficulties.' Putnam [1975 a] obviously took a more critical position when he said that 'the various systems of mathematical philosophy, without exception need not be taken seriously.'

Besides, based on deep analysis, some people have even claimed that there were no 'foundational crises' in mathematics at all, and therefore, the foundational studies did not have any special significance, and the problem of foundations should not be regarded as the central problem of the philosophy of mathematics. Obviously, this view in comparison with the aforementioned ideas is more profound. In fact, it indicates that the study of the philosophy of mathematics has broken away from the old foundational tradition and begun to enter a new period in its development.

(However, while asserting that the problem of foundations is no longer the central problem of the philosophy of mathematics, we at the same time should not deny totally the significance of the foundational studies. As a matter of fact, the latter has now become a special domain of mathematics; and the corresponding philosophical analysis, which can be regarded as the continuation and further development of the earlier foundational philosophy, also has some philosophical significance. In particular, as set theory has occupied a very important position in modern mathematics, deep analysis into the concept of set no doubt should be regarded as an important topic for the modern studies of the philosophy of mathematics. But still it is only a part of the philosophy of mathematics, and should not be taken as the central problem or the mainstream.)

(2) New thoughts on how to carry out the study of the philosophy of mathematics

Such new thoughts are clearly closely connected with the reflections on and the criticism of the foundationalist studies, but there is also an important difference between them: the former has a more direct constructive sense, and compared with the concrete studies in this field, it also reaches a higher level of abstraction. That is to say, it can in a sense be regarded as meta-studies of the philosophy of mathematics, for what it concerns mainly is not the problem of how to develop some concrete philosophical theory of mathematics, but rather the more basic problem of what is (or what should be) the philosophy of mathematics.

It is exactly in such a spirit, that people criticize severely the foundationalist philosophy of mathematics for its deviation from the actual practice of mathematicians. For example, Hersh [1979] wrote, we 'deny the right of any a priori philosophical dogma to tell mathematicians what they should do, or what they really are doing in spite of themselves or without knowing it.' Conversely, people suggest that, the philosophy of mathematics should be 'the working philosophy' of the professional mathematicians. That is to say, 'the philosophy of mathematics should be based on observations of what is actually presented in the subject.' (MacLane [1992]) In other words, the practice of the working mathematicians should be taken as the starting-point and final basis for the studies of the philosophy of mathematics.

Apparently, it reflects a great transition of the basic positions in the research of the philosophy of mathematics, and is also a direct answer to the call from the working mathematicians.

(3) The influence of the philosophy of science

While talking about the recent development of the philosophy of mathematics, we need to acknowledge the great influence of the philosophy of science.

The recent studies of the philosophy of mathematics have taken from the philosophy of science not only instructive ideas but also new research problems and useful methods. For example, by extending K. Popper's fallibilist philosophy of science into the field of mathematics, I. Lakatos developed his quasi-empirical view of mathematics.(cf. Zheng [1990 b]) Besides, recent discussions about the growth of mathematical knowledge and its rationality (cf. Kitcher, [1983]; D. Gillies, [1992]; Breger and Grosholz, [in press]) are to a great extent a direct echo of the corresponding discussions in the philosophy of science. In addition, it is also a good lesson the philosophy of mathematics has learnt from the philosophy of science in methodology that great attention is being paid to the history of mathematics. As Hersh [1979] pointed out:

> "The famous work of Thomas Kuhn is a paradigm of the kind of insight in the philosophy of science that is possible only on the basis of historical studies. Such work has yet to be done in the philosophy and history of mathematics."

So, from the above discussion we can see that the study of the philosophy of mathematics has already stepped out of the foundationalist epoch and entered a new era of development. It is what Tymoczko [1985] observed, 'the time is right for the post-foundationalists to move into the mainstream of the philosophy of mathematics'.

3. *The Revolution of the Philosophy of Mathematics*

In order to clarify the revolutionary nature of the recent development of the philosophy of mathematics, I shall compare it with the foundational studies in terms of the afore-mentioned four aspects respectively: basic positions for research, methodology, central problems and basic views of mathematics.

(1) Transition of the basic positions: from grave deviation from actual mathematical activities to close connection with them

Although there are big differences in the basic positions adopted by logicists, intuitionists and formalists, they all did primarily prescriptive work, offering definite prescriptions for actual activities of mathematicians, and aimed at reforming or rebuilding mathematics in accordance with such prescriptions. As a natural result of this, their foundational studies deviated far from the actual mathematical activities. By contrast, people take a different position in the recent studies of the philosophy of mathematics, insisting that the philosophy of mathematics should be 'the working philosophy' of the professional mathematician, i.e., it should 'reflect honestly on what we do when we use, teach, invent or discover mathematics'.

(2) Great emphasis on the history of mathematics

As one concrete manifestation of the new basic position, people now working in this field generally pay more attention to the history of mathematics. Kant's celebrated dictum is paraphrased: 'philosophy of mathematics without history of mathematics is empty; history of mathematics without philosophy of mathematics is blind'. And much work is done in this direction. Especially, some people are trying to construct a historically based philosophy of mathematics. Thus, the historical method has in fact become one of the most important methods in the modern studies of the philosophy of mathematics. (cf. Lakatos, [1976]; Kitcher, [1983]; Koetsier, [1991])

In contrast to the foundationalists' sole emphasis on the logical analysis, wide use of the historical method now obviously indicates a great change in methodology, and it in turn has also opened some new directions for the studies of the philosophy of mathematics, such as the social-cultural approach to mathematics. (cf. Kline, [1954]; Wilder, [1980])

(3) Transition of the research problems

Generally speaking, people interested in the philosophy of mathematics nowadays attach little importance to the problem of the soundness of mathematics. This in fact is a reflection of the real attitude of the working mathematicians. For example, it has been emphasized by Steiner [1975], Lehman [1979], Kitcher [1983] and many others that, it should be taken as an obvious and indubitable starting-point that modern man is in possession of a great deal of mathematical knowledge and such knowledge is reliable, or in other words, well justified.

To people who aimed at developing a working philosophy for the professional mathematicians, the central problem of the philosophy of mathematics is no doubt the nature of mathematics. For example, Ernest [1992] asserted that

> "philosophy of mathematics should be accounting for the nature of mathematics, including the practice of mathematicians, the application of mathematics, the place of mathematics in human culture..."

Specifically, directly corresponding to the modern studies of the philosophy of science, people now working in the field of the philosophy of mathematics also take great interest in the methodology of mathematics, such as: How do mathematical definitions get revised? How are methods of proof modified? Or more generally: How does mathematical knowledge grow? What makes some mathematical ideas (or theories) better than others? Are there methodological rules that mathematicians follow in their work? Obviously, the above problems also show that the significance of the recent studies of the philosophy of mathematics to both practicing mathematicians and historians of mathematics is mainly heuristic or illuminating by nature. Aspray and Kitcher [1988] made this point quite clear:

> "If we had such a canon (i.e., the methodology of mathematics), then historians could use it to investigate the match between actual history and the ideal, perhaps finding intriguing cases in which some extrinsic factor prompted a departure from the recommendations of methodology. Moreover, mathematicians might find it illuminating both to see how their chosen field of investigation had emerged from the mathematics of the past and how certain kinds of methodological considerations were paramount in fashioning its central concepts. It is not even out of question that the answers ... might help illuminate disputes among mathematicians about the legitimacy of various approaches or the significance of certain ideas."

(4) The replacement of the static, absolute view of mathematics with the dynamic, empirical and quasi-empirical view of mathematics

If the static, absolute view of mathematics can be said to dominate the foundationalist philosophy of mathematics, then, as the focus of study has shifted to actual mathematical activities, the development of mathematics is now no longer understood as a monotonous increase of the number of indubitable mathematical truths but rather as creative activities of human beings which are under the influence of the various elements of society and culture. The development of mathematics is thereby a very complicated process including conjectures, trial and error, proofs and refutations, and depends on the joint effort of individuals and community.

As an explication of the modern view of mathematics, I think the following theories are of great importance:

Firstly, there is 'the theory of mathematical activity' (cf. Kitcher, [1983]), which asserts that, since we should take into consideration not only the final results of mathematical activities but also the whole process, mathematics should be regarded as a composite consisting of multiple elements such as 'language', 'methods', 'problems', and 'propositions'. What is more, as mathematicians in modern society all work in some social environment, they are in fact members of certain social (mathematical) communities and are all (consciously or unconsciously) under the influence of some 'mathematical traditions'. It follows that the conceptions or views shared by the members of mathematical communities, especially the common views about what is mathematics and how we should do mathematics, should also be taken as important elements of mathematics (or mathematical activities).

Secondly, apart from the 'renaissance of empiricism in the philosophy of mathematics' (cf. Kitcher, [1983]; Lehman, [1979]; Kalmar, [1967]; Lakatos, [1978 b]), the so-called 'quasi-empirical view of mathematics' should also be counted as an important content of the modern view of mathematics. The latter asserts that (cf. Lakatos, [1976], [1978 a]; Putnam, [1979]), besides the empirical criterion, i.e., the success of its application in other fields, especially in science, mathematics also has its own special criterion for judging mathematical work, that is, the significance of the new work to mathematics itself, such as whether and how new research can deepen our understanding or increase our capacity in problem solving. In my view, this quasi-empirical view is in fact a direct confirmation of the peculiarity of mathematics. (cf. Zheng [1990a])

Finally, as a direct reflection of the modern development of mathematics, some researchers (cf. Steen, [1989]; Hsu and Zheng, [1990]; and MacLane [1992]) have proposed that mathematics is the science of patterns. For example, MacLane [1992] said clearly on this point: 'mathematics is not about this or that thing, but about a pattern or form suggested by various things and previous patterns. Therefore, mathematical study is not study of the thing, but of patterns.' Obviously, this answers both what is mathematics and how we should do mathematics.

To sum up, in comparison with the foundationalist philosophy of mathematics, there are radical changes in the recent studies of the philosophy of mathematics with regard to the central problems, the methodology, the basic positions for research and the basic views of mathematics. Now, we can clearly draw the conclusion: the recent development of the philosophy of mathematics represents a revolution.

4. *More Words on Revolutions in the Philosophy of Mathematics*

What follows are some more words on revolutions in the philosophy of mathematics:

(1) The foundationalist philosophy of mathematics was also described as a revolution of the philosophy of mathematics by T. Tymoczko [1986]:

> "As a discipline, the philosophy of mathematics underwent an enormous change over a period centering on the turn of the century. If we analogize mathematics to science then, following Kuhn, we can characterize this change as revolutionary or the creation of a new paradigm..."

While asserting the revolutionary nature of the foundationalist philosophy of mathematics, Tymoczko also made quite clear its central problem and research method:

> "The dominant question in the new philosophy of mathematics became: what is the foundation of mathematics? And the answer to this question, it was assumed, was to be found in the newly emerging discipline of mathematical logic."

So, although Tymoczko did not mention the basic position for research and the basic view of mathematics, his words still offer a strong support to the above analysis about the main features of the foundationalist philosophy of mathematics.

Besides, and more important, if we accept Tymoczko's viewpoint, the whole history of the philosophy of mathematics could be divided roughly into the following three periods:

Table 1.

The old tradition of the philosophy of mathematics	around 1900 →	the foundationalist philosophy of mathematics	around 1965 →	the recent development of the philosophy of mathematics

(2) The revolutionary changes in the recent development of the philosophy of mathematics should not be regarded as the only candidate for replacing

the foundationalist paradigm. On the contrary, there are also some other paradigms, or to say, other directions or programs in this field. Especially, there is the one which can be described as 'the renaissance of the older tradition of the philosophy of mathematics'. That is, some people in this field seems have gone back to those questions which had attracted philosophers ever since the time of Plato and Aristotle, i.e., the ontology and epistemology of mathematics. Such new researches of course have also got some new features; especially, they are closely linked with the mainstream of analytic philosophy in general (as a brief survey of contemporary researches in this direction, cf. Aspray & Kitcher [1988]).

The difference between the two different paradigms or different directions in the recent development of the philosophy of mathematics can be seen very clearly by comparing some particularly authorized books in this field. For example, it is quite interesting to compare the contents of the following two books: (1) *New Directions in the Philosophy of Mathematics*, edited by T. Tymoczko [1986], which, by its title, can be recognized immediately as belonging to the revolutionary side; and (2) *The Philosophy of Mathematics*, edited by W. Hart [1996], which I think should be classified as belonging to the other side. As a matter of fact, although both the editors claimed that their book included the most interesting and important work from the recent years in the philosophy of mathematics, or at least 'get (got) together some of the most exciting essays published recently in this field', there is not even one common paper in these two books. It proves that they are really two different paradigms, or we might say, these two editors did seem to live in two different worlds.

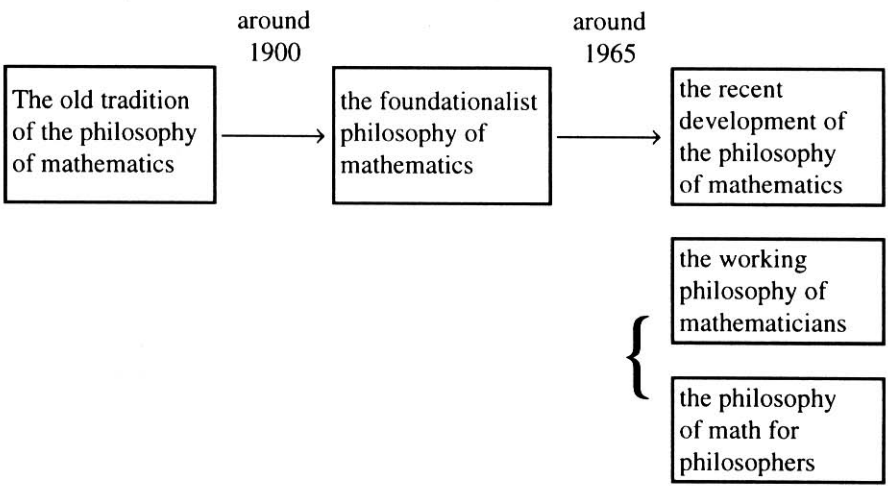
(To extend, we can even make comparisons in a wide range. For example, while Körner's book *The Philosophy of Mathematics* [1960] is obviously still under the influence of the foundationalist philosophy of mathematics; the book *Philosophy of Mathematics, Selected Readings* co-edited by Benacerraf and Putnam [1964] seems to have already turned a little away from the foundational studies. What is more, by comparing the two editions of the last book, i.e., the 1964 edition and the revised edition of 1983, we can even infer that the two editors had changed their conceptions of the philosophy of mathematics to some extent during the period from 1964 to 1983.)

(3) While the revolutionary paradigm in the recent development of the philosophy of mathematics aims at being 'the working philosophy' of the professional mathematicians, the other one I think might be properly described as 'philosophy of mathematics for philosophers', because it focuses on the questions which interest philosophers rather than mathematicians. It is also the point made by Aspray and Kitcher [1988]: 'The problem ... are

continuous with the central problem of epistemology and metaphysics in the twentieth century.'[2]

So, in order to trace the modern development of the philosophy of mathematics, we should make the table made above a little more complicated:

Table 2. The Historical Development of the Philosophy of Mathematics



Using the basic categories of this paper, it will be not too difficult to make a further analysis about the main features of the later paradigm. As a matter of fact, its central problems and basic methodology are already clear enough. Besides, what follows might be regarded as a picture of the real attitude of the philosophers working in this direction: 'We won't disturb you any more; but you as mathematicians please also leave us alone. That is to say, you do your mathematical work, we do our philosophical work, and there is not any relationship between these two different fields.' So just as Aspray and Kitcher [1988] pointed out: 'the distance between the philosophical mainstream and the practice of mathematics seems to grow throughout the twentieth century.'

Therefore, although the latter has already got rid of the influence of the foundationalist philosophy of mathematics, it still had the defect of deviating greatly from actual mathematical activities. And it is precisely for this

[2]It should be mentioned that, Grattan-Guinness in his [1992] also made a differentiation between what he called 'philosophers' philosophy of mathematics' and 'mathematicians' philosophy of mathematics'. But, by careful reading, it is easy to find that these two words was used in a quite different sense there.

reason, I would like to say that, even though the philosophical branch was described by Aspray and Kitcher as the mainstream of the philosophy of mathematics, the revolutionary one will finally become the dominant paradigm in the near future.

5. *The Significance of the Philosophy of Mathematics to Actual Mathematical Activities*

I hope the above discussion will not lead to such a misconception that, while the philosophers of mathematics should pay more attention to actual mathematical activities, the philosophy of mathematics on the other hand seems have nothing to do with practicing mathematicians except as a relaxing topic for tea time. To make it sure, it is then appropriate to do as a final section of this paper some further analysis about the relationship between the philosophy of mathematics and the actual mathematical activities, and especially, about the significance of the philosophy of mathematics to the actual mathematical activities.

(1) Mathematicians and philosophers

As already mentioned above, one major defect of the foundationalist philosophy of mathematics is that it deviates greatly from actual mathematical activities; therefore, in our further work, we should guard against this tendency. To do this, we should reflect fully the views of the working mathematicians (especially the famous mathematicians) on their own subject.

But at the same time, we should also recognize that philosophical sayings of the working mathematicians, including their self-conscious reflections of their own works, cannot be taken as the main body of the philosophy of mathematics. It is so not only because intelligence in mathematics cannot be identified with wisdom in philosophy, but also as the philosophy of mathematics must go beyond the naive level and be based on more systematic studies.

To convince this conclusion, I think it is really enough just to mention the 'philosophical strait' of the working mathematicians, to which Hersh [1979] gave a vivid description:

> "the typical 'working mathematician' is a platonist on weekdays and a formalist on Sundays. That is, while he is doing mathematics, he is convinced that he is dealing with an objective reality ...But then, when challenged to give a philosophical account of this reality, he finds it easiest to pretend that he does not believe in it at all."

Generally speaking, it is also one of the lessons we should learn from the historical development of the philosophy of mathematics that, different people and schools often take different perspectives and emphasize different aspects of mathematics; and so, in order to avoid one-sidedness in our views and to catch the real essence of mathematics, we must not stop with the philosophical sayings of the working mathematicians but move forward to make further systematic studies from a philosophical point of view.

The above discussion makes clear not only the importance of strengthening the relationship between mathematicians and philosophers, but also the significance of the philosophy of mathematics to the working mathematicians. That is, although the philosophy of mathematics is not some apriori dogmas to which practicing mathematicians must obey, it can still have positive effect on actual mathematical activities, especially, it can promote the self-conscious reflection of the working mathematicians so that the possible one-sidedness in their views of mathematics or misconceptions about mathematics could be overcome and corrected, this in turn will of course exert positive influence on their mathematical work.[3] For example, MacLane in his [1992] emphasized that 'Understanding the nature of mathematics is an effective guide to productive direction of research. ... Good understanding of the nature of mathematics helps us to realize when an apparent part of mathematics is in fact a dead end.'

(2) The philosophy of mathematics and mathematics education

With regard to the significance of the philosophy of mathematics to actual mathematical activities, I think we should also emphasize its great importance to mathematics education. As all mathematics teachers are doing their work (consciously or unconsciously) under the influence of some views of mathematics, and their teaching in turn will greatly influence the young generation in their forming of the views of mathematics, it in fact relates directly to the future development of mathematics whether mathematics teachers could through the study of the philosophy of mathematics form an advanced view of mathematics.

To make things clearer, we can here use morality as an analogue, because the position of the philosophy of mathematics to actual mathematical activities is almost the same as morality to general people: firstly, as people always (consciously or unconsciously) behave under the influence of some

[3]The above analysis thus also offers an explanation of the above-mentioned heuristic or illuminating nature of the philosophy of mathematics. More general, I think this conclusion is also correct for philosophy in general.

basic principle of morality, we should confirm clearly the great importance of morality to everybody; secondly, adult's moralities are usually firmly formed so that what morality really could do with them is chiefly to promote their self-conscious reflection of their own moral principles; thirdly, we should emphasize the great importance of morality to the young generation, because the latter have not formed their own views of morality and the forming of their moralities relates directly to the future of human society.

It is exactly for this reason that modern educators of mathematics have become more and more interested in the philosophy of mathematics. For example, T. Romberg, one of the most influential educators of mathematics in the USA, said [1992]:

> "For over two thousand years, mathematics has been viewed as a body of infallible truth far removed from the affair and values of humanity. These views are being challenged by a growing number of philosophers of mathematics. They argue that mathematics is 'fallible, changing and like any other body of knowledge, the product of human inventiveness ... Such a dynamic view of mathematics has powerful educational consequences."

To strengthen this point of view, I am going to give a brief analysis of constructivism.

It is a well-known fact that constructivism in recent years has become very popular in the field of mathematics education so that some people even asserted that almost every mathematics educator nowadays was a constructivist in one or some other sense.(cf. Davis, Maher & Noddings [1990]). The basic idea of constructivism is that human cognition is not passive reflection of the outer world by human mind, but an active process of construction based on previous experience and knowledge.[4] To be sure, as a direct confirmation of the active nature of human cognition, constructivism has its point. Nevertheless, it should also be recognized that cognition is not a purely personal activity of any individual, and it is the objective reality that offers the source and the final criterion for human cognition. Therefore, while joining the banner of 'constructivism', we at the same time should also keep away from those radical views such as radical

[4] So, although there are some common points between constructivism which has appeared recently in the field of (mathematics) education and intuitionism in the foundational studies of mathematics, they at the same time also have some important differences. In particular, while intuitionists chiefly concerned with the final basis of mathematics; the constructivists in the field of education are clearly focusing on the essence of human cognition including learning.

constructivism, and see clearly the theoretical defect of constructivism as a whole.

Furthermore, as far as mathematical constructions are concerned, I think more attention should be paid to the peculiarity of mathematics. To explicate, as mathematical entities do not exist in the physical world, they are actually constructions of the mind. But, as an account of the objectivity of mathematics, we should recognize that not only the social nature and the objective basis of mathematical construction activities, but also the formal nature of such activities. That is to say, in strict researches, no matter whether the entities involved have or do not have direct empirical background, we can only deduce in the light of relative (implicit or explicit) definitions and rules rather than wholly relying on intuition. Thus, even for the creators of mathematical concepts, there is already a transition from 'mental constructions' to 'mental entities', and the latter should be differentiated clearly from what can be regarded as 'objective entities'. Therefore, mathematical constructions are not simple processes of direct transitions from 'mental constructions' to 'objective entities' but rather more complicated ones consisting of the transitions from 'mental constructions' to 'mental entities' and then from 'mental entities' to 'objective entities'.[5] (cf. Zheng [1995])

Generally speaking, I think it is in fact the recent development of the philosophy of mathematics which had laid the necessary ideological basis for the recent reform movement in mathematics education all over the world. (cf. Zheng [1994]) So, to follow Tymoczko, I would like to say here that, 'The time is right for the philosophers of mathematics to move into the mainstream of the reform movement of mathematics education.'

Finally, I would like to mention that, through close cooperation between mathematicians and philosophers, we in China have developed a systematic theory of the philosophy of mathematics, which is based on the recognition that mathematics is the science of patterns (cf. Hsu and Zheng, [1990], [1993]). And we have also done some work in the methodology of mathematics and the philosophy of mathematics education, which are honestly our conscious effort to give full play to the positive role of the philosophy

[5]What should be emphasized is that, the clear differentiation of 'mental constructions','mental entities' and 'objective entities' seems to have a much wider sense for philosophical studies. For example, in the author's view, a substantial improvment of the conception 'the world 3' could be made on the basis of a clear differentiation between 'mental entities' and 'objective entities'. That is to say, only the latter could be regarded as entities of the world 3, and the transition from 'mental entities' to 'objective entities' depends on their acceptance within the corresponding community.

of mathematics in actual mathematical activities. (cf. Hsu, [1983]; and Zheng, [1992], [1995])

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