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CORRECTIONS AND ADDITIONS TO MY PAPER "A NOTE ON UNPROVABILITY-PRESERVING SOUND TRANSLATIONS", MORE GENERAL CONSTRUCTIONS*

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Abstract

In this paper, we shall be concerned with the embedding-construction problem from a given proof-theoretic embedding and with some criterion to distinguish one embedding with another.

We shall first point out some false argument in the second section of the paper [12] of the above title. However, thanks to certain constructions introduced in the present paper, the main results of the section, i.e. the affirmative answers to the problem posed in it (= Question 2 in [12]) remain valid and so does the rest of the paper [12]. Further we shall generalize the obtained results on constructions with atomic formulas to those with more general unprovable formulas. In addition, we shall show that a certain set obtained by such a construction forms a lattice isomorphic to a certain sublattice of the Nishimura-Rieger lattice of formulas of one variable p of intuitionistic propositional logic for any p.

For all the logics treated in this paper, i.e. classical and intuitionistic logics and modal logics **S4** and **S5**, some of the constructions give even an affirmative answer to Question 1 of [12]: i.e., "Let X and Y be given consistent formal systems. For any embedding τ of X in Y, is there a construction $\mathscr{C}$ such that by the construction $\mathscr{C}$, we can obtain a set with cardinality $\kappa \geq \aleph_0$ of mutually non-C-equivalent embeddings from τ?".

Further we shall give a necessary and sufficient condition for an affirmative answer to Question 1. We shall also show that a certain set of such constructions forms a lattice for certain formal systems.

This paper is the overture to my forthcoming paper [14] (on a generalized Gödel's theorem on embedding) and our further work on the theme of this paper.

*This paper is dedicated to the late Dr. Diana Raykova. This paper is the revised and enlarged version of Part I of [13].

1. *Introduction*

On December 31, 1993, about two hours before the New Year, so before new year's fireworks began, it was still quiet outside. I was then in my room. I took a book [34] in mathematical logic from one of my bookselves and comfortably sat to browse it through as a pastime. Soon after I began, I noticed an interesting intuitionistically valid formula $((((p \supset q) \supset p) \supset p) \supset q) \supset q$ on page 104 of it. *That* was the Mints's formula, a counterexample of Komori's conjecture (see [32]) for intuitionistic logic and it *is* that of my Theorem 2.1 (and thus Corollary 2.2) in [12], too.

That is, it is not correct that for any atomic formula $p \neq \bot$ of intuitionistic predicate (propositional) logic **IQC** (**IPC**) without equality, $\vdash_{\mathbf{IQC(IPC)}} A \supset p. \supset p \Rightarrow \vdash_{\mathbf{IQC(IPC)}} A$. However, thanks to three constructions introduced in this paper, the main results of the section, i.e. the affirmative answers to the problem posed in it (see §2 of this paper), to be precise, Theorems 2.3, 2.5 and Corollaries 2.4, 2.6 in [12], remain valid and so does the rest of the paper [12].[1]

We shall see the first construction in §2 of the present paper, using the original argument of [12] as much as possible. The second construction is useful for a case involved in Gödel translation. The idea of the third construction is based on the disjunction property of intuitionistic systems. The second and the third ones will be introduced in §4. In the same section, we shall extend the first construction to more general unprovable formulas, incorporating with falsum. And we shall prove general theorems for the extended construction, in particular Theorem 4.6. The third and the fourth sections of this paper, therefore, mean the substitute of the second one of [12].

In §5, making use of the developed techniques in §4, we shall show as a generalization of the affirmative answer that a certain set obtained by such a construction forms a lattice isomorphic to a certain sublattice of the Nishimura-Rieger lattice of formulas of one variable p of intuitionistic propositional logic for any p.

In §6, we shall introduce a natural notion, *non-U-equivalence* as another criterion to distinguish one embedding with another. Under certain weak conditions, non-U-equivalence and non-C-equivalence are equivalent.

In §7, we shall apply the idea for the third construction to classical propositional logic and modal logic **S4**.

In §8, we shall show that some of constructions introduced in the previous sections eventually give an affirmative answer to Question 1 for all the

[1] Properly saying, the rest of it does not depend on the results of the previous section.

logics treated in this paper, i.e. classical and intuitionistic logics and modal logic **S4** and **S5**.

In §9, we shall give a necessary and sufficient condition for an affirmative answer to Question 1.

In §10, we shall show that a certain set of constructions forms a lattice for certain formal systems.

In §11, that is, in the last section, we shall close this paper with some general comments.

In the following section, we shall review the definitions and the problems of [12] in a slightly more general setting. So one can read the present paper independently of [12].

2. *Preliminaries*

Definition 2.1 Let X be a formal system. By F_X we denote the set of all well-formed formulas of X. By F_X^{at} we denote the set of all atomic formulas of X. If $\perp \in F_X$, we understand F_X^{at-} as F_X^{at} - $\{\perp\}$, where $\perp$ is falsum, otherwise $F_X^{at-} = F_X^{at}$.

Definition 2.2 Let X and Y be formal systems. Let D be a subset of F_X. By *a sound translation τ from X to Y with respect to D*, we mean a mapping from D to F_Y such that τ is *sound with respect to D*: i.e. for any $A \in D$, $\vdash_X A \Rightarrow \vdash_Y \tau(A)$, where $\vdash_X A$ means that A is a theorem of X. We shall call a sound translation τ from X to Y with respect to D to be a *sound translation τ from X to Y*, if $D = F_X$ holds.

By an *unprovability-preserving translation τ from X to Y with respect to D*, we mean a mapping from D to F_Y such that τ *preserves unprovability with respect to D*: i.e. for any $A \in D$, $\dashv_X A \Rightarrow \dashv_Y \tau(A)$, where $\dashv_X A$ means that A is not a theorem of X. We shall call an unprovability-preserving translation τ from X to Y with respect to D to be an *unprovability-preserving translation τ from X to Y*, if $D = F_X$ holds.

A mapping from F_X to F_X is called to be an *unprovability-preserving sound translation with respect to D*, if it is a sound with respect to D and it also preserves unprovability with respect to D.

In the ordinary terminology, an unprovability-preserving translation is called a *faithful* one. We shall call an unprovability-preserving sound translation from X to Y with respect to a subset D of F_X to be an *embedding of X in Y with respect to D*. As usual, we shall call an unprovability-preserving sound translation from X to Y to be an *embedding of X in Y*, if $D = F_X$ holds.

Definition 2.3 We call an embedding of a formal system X in X to be an *autoembedding of* X.

We understand "auto" similarly when we apply it to other translations.

We shall recall the definition of *non-C-equivalence.*[2] In this paper, we do not restrict the notion only to unprovability-preserving translations on the contrary to that of [12].

Definition 2.4 Let X and Y be consistent formal systems.[3] Suppose that truth-functional[4] equivalence $\equiv$ is one of the logical symbols of Y, or it is definable in Y. For any translations τ_1 and τ_2 from X to Y, they are said to be *non-C-equivalent* if there is a formula A of X (called a *C-ditcher of* τ_1 *and* τ_2) such that neither $\vdash_X A$ nor $\vdash_X \tau_1(A) \equiv \tau_2(A)$ holds. Otherwise we say that they are *C-equivalent.*

We soon intuitively understand that if τ_1 and τ_2 are non-C-equivalent, then they are different translations.

The following questions with respect to cardinality $\kappa = \aleph_0$ were posed in [12].

Question 1: Let X and Y be given consistent formal systems. For any embedding τ of X in Y, is there a construction $\mathscr{C}$ such that by the construction $\mathscr{C}$, we can obtain a set with cardinality $\kappa \geq \aleph_0$ of mutually non-C-equivalent embeddings from τ?

[2] This definition contains some possible ambiguities: for example, "what does a formal system precisely mean?" and so on. But we do not here pursuit to make all the possible ambiguities in it clear, because we can understand the content of it with common sense of logicians, and because even this rather ambiguious definition will be useful for a general consideration as will be seen in §9.

It is very difficult to say what logic (or a formal system) is, for example. I think that we are able to do something general and theoretical about logic before we know the precise answer to such a question. But in some situation, in order to develop rigorous treatments, it is necessary to define what a logic is precisely. For example, in the study of intermediate propositional logic, a *logic* is defined as a subset of the set of all theorems of **CPC** which includes the set of all theorems of **IPC** and is closed under the rule of substitution and modus ponens. I think that the rigor required for definitions depends on what we study and how we approach to it as we can learn from the history of mathematics.

[3] To make sure, a formal system X is said to be *consistent* if there is a formula A of X such that $\dashv_X A$ holds.

[4] The word "truth-functional" is intended to indicate that the equivalence is not strict one, for example in modal logic. But it is an interesting problem to ask whether strict equivalence can effectively play a role for (non-)C-equivalence.

Question 2: Let X and Y be given consistent formal systems. Suppose that an embedding τ of X in Y is given. Is there a construction $\mathscr{C}$ such that by the construction $\mathscr{C}$, we can obtain a set with cardinality $\kappa \geq \aleph_0$ of mutually non-C-equivalent embeddings from τ?

Remark: For Questions 1 and 2, we assume that every embedding obtained by the construction is defined using at least one value $\tau(A)$ for some $A \in F_X$.

In §3, we shall give an affirmative answer to Question 2, when given X, Y, τ and κ are the following:[5] (all the predicate logics treated below are formulated without equality)

(P1) $X = Y =$ intuitionistic first-order predicate (propositional) logic **IQC** (**IPC**); $\tau =$ the identity translation *id* (*idem*); $\kappa = \aleph_0$,

(P2) $X =$ classical first-order predicate (propositional) logic **CQC** (**CPC**); $Y =$ **IQC** (**IPC**); $\tau =$ Gödel translation (*idem*) from **CQC** (**CPC**) to **IQC** (**IPC**) (see [8] and refer e.g. to [17]); $\kappa = \aleph_0$.

In the seventh section, we shall also give an affirmative answer to Question 2, when given X, Y, τ and κ are the following:

(P3) $X = Y =$ **CPC**; $\tau = id$; $\kappa = \aleph_0$,

(P4) $X = Y =$ modal logic **S4**; $\tau = id$; $\kappa = \aleph_0$,

(P5) $X =$ **IPC**; $Y =$ modal logic **S4**; $\tau =$ Gödel translation from **IPC** to **S4** (see [9]); $\kappa = \aleph_0$,

(P6) $X =$ modal logic **S5**; $Y =$ modal logic **S4**; $\tau =$ Matsumoto's translation from **S5** to **S4** (see [23]); $\kappa = \aleph_0$.

In §8, we shall give an affirmative answer to Question 1 for (P1)–(P6), making use of the constructions used for Question 2.[6]

We follow [42, vol. I, p. 36] for the formulation for **IQC** (**IPC**) without equality, although we take different notations for them, e.g. for metavariables and logical symbol $\supset$ in place of $\rightarrow$. So we shall employ $\wedge$, $\vee$, $\supset$, $\perp$,

[5] These cases are only examples.

[6] Although the cardinality κ of construction in the above cases (P1)–(P6) is $\aleph_0$, I already noticed as an answer to one of questions of Prof. D. van Dalen that there are some constructions with $\kappa = 2^{\aleph_0}$. By a personal communication, Prof. M. Takano also let me know some construction with $\kappa = 2^{\aleph_0}$. However, I do not want to treat those cases with the cardinality of continuum in the present paper, which will be left to another paper.

$\forall$ and $\exists$ as primitive logical symbols of **IQC**. We shall often write $\neg A$ for $A \supset \bot$. For **CQC**, we shall take $\wedge$, $\vee$, $\supset$, $\neg$, $\forall$ and $\exists$ as primitive.

To make sure, we shall recall the definition of Gödel translation from **CQC** (**CPC**) to **IQC** (**IPC**).

Definition 2.5 Let X be **CQC** (**CPC**) and Y **IQC** (**IPC**). Then we define a translation $(\cdot)^\circ$ from X to Y (called *Gödel translation*) as follows: for any formula A of F_X,

(1) $A^\circ = \neg\neg A$ for A atomic,
(2) $(\neg A)^\circ = \neg A^\circ$,
(3) $(A \# B)^\circ = A^\circ \# B^\circ$, $\# \in \{\wedge, \supset\}$,
(4) $(A \vee B)^\circ = \neg(\neg A^\circ \wedge \neg B^\circ)$,
(5) $(\forall x A)^\circ = \forall x A^\circ$,
(6) $(\exists x A)^\circ = \neg\forall x \neg A^\circ$.

The following theorem is very well-known and will be often used in this paper.

Theorem 2.1 (Gödel-Gentzen-Bernays) ([8]) *Gödel translation* $(\cdot)^\circ$ *is an embedding of* **CQC** (**CPC**) *in* **IQC** (**IPC**).

The following well-known lemma will be also useful.

Lemma 2.1 (cf. [17, p. 495]) *Gödel translation* $(\cdot)^\circ$ *and a translation* $\neg\neg((\cdot)^\circ)$ *are C-equivalent embeddings of* **CQC** (**CPC**) *in* **IQC** (**IPC**).

Kleene's Gentzen-style sequent calculus **G3** for **IQC** ([17, p. 481]) will play an important role for this paper. Throughout this paper, we shall use the following theorem for **G3**, usually without mentioning.

Theorem 2.2 (Gentzen) (cf. [17]) *When* $n \geq 1$, *if* $A_1, \ldots, A_n \rightarrow B$ *is provable in* **G3**, *then* $A_1 \wedge \ldots \wedge A_n. \supset B$ *is provable in* **IQC**. *If* $\rightarrow A$ *is provable in* **G3**, *then* A *is provable in* **IQC**.[7]

Definition 2.6 Let X be a formal system. For any formula $A \in F_X$, $\langle A \rangle$ stands for the set of all atomic subformulas occurring in A.[8] For any for-

[7]Note: if $A \rightarrow$ is provable in **G3**, then $\neg A$ is provable in **IQC**.

[8]We shall follow the definition of subformula in [17, p. 449] for the predicate case.

mula $A \in F_X$, if $\perp \in F_X$, then $< A >^{d\perp}$ stands for $<A>$ - $\{\perp\}$, otherwise $< A >^{d\perp} = <A>$.

The following lemma is the intuitionistic analogue of the Ono-Kleene's theorem (see [33] and [18]) (cf. [22] and [17]). The lemma will be seen below as a strong tool to prove the faithfulness of a given translation in a general setting.

Lemma 2.2 Let X be **IQC** (**IPC**). *For any A, B* $\in F_X$, *if* $\vdash_X A \supset B$ *holds with* $< A >^{d\perp} \cap < B >^{d\perp} = \emptyset$, *then* $\vdash_X \neg A$ *or* $\vdash_X B$.

Proof. This lemma can be proved as a corollary of the interpolation theorem for **IQC** (**IPC**). That is, by the interpolation theorem, we have an interpolation formula C with the assumption of Lemma 2.2 such that $\vdash_X A \supset C$ and $\vdash_X C \supset B$ hold. Because of $< A >^{d\perp} \cap < B >^{d\perp} = \emptyset$, the C, which is built up only by $\perp$ and logical symbols, must be equivalent to $\perp$ or to $\neg\perp$, since $\perp$ behaves just like classical falsity even in intuitionistic logic. If $\vdash_X C \equiv \perp$, then $\vdash_X \neg A$ holds, otherwise $\vdash_X B$ holds.[9]
We can also prove this lemma, reducing it to the modal analogue of it with the Gödel translation from intuitionistic to modal logics (for the details, see [15]). □

3. *Embedding of classical in intuitionistic logics, Part I*

This section and the following two sections mean the corrected and extended version of the second section of [12]. That is, we shall, in a correct way, give affirmative answers to Question 2 in the case of (P1) and (P2), respectively.
Only in this section, in order to make use of Kleene's version **G3** *of Gentzen's sequent calculus* **LK**, *we shall take a formulation without* $\perp$ *(falsum) for* **IQC** (**IPC**) *with the rest of the formulation intact.*
We shall only take care of the predicate case of (P1) in great detail. The other cases of them are, *mutatis mutandis*, similarly dealt with.
Let us begin with a proposition. For any set S, by $|S|$ we denote the cardinality of S.

[9] I learned this proof from Prof. A. S. Troelstra.

Proposition 3.1 Let X be **IQC** (**IPC**).[10] *Then we have* $|F_X^{at}| = |F_X^{at-}| = \aleph_0$.

Proof. For **IPC**, it is trivial. For **IQC**, we can easily modify the argument in [43, p. 108] (note that our language for it allows countably many individual variables, countably many predicate symbols, countably many function symbols and countably many constants). (For cardinal arithmetic, see e.g. [21, pp. 28–30, in particular, Lemma 10.21 on p. 30].) □

The following construction $\tau_{\iota\{p\}}$ gives an answer for (P1) and (P2).

Definition 3.1 Let Y be **IQC** (**IPC**). Let τ be an embedding of a consistent formal system X in Y. For any $p \in F_Y^{at-}$, we define a translation $\tau_{\iota\{p\}}$ as follows: for any $A \in F_X$,

$$\tau_{\iota\{p\}}(A) = \begin{cases} \tau(A) \supset p. \supset p & \text{if } p \notin <\tau(A)>, \\ \tau(A) & \text{otherwise} \end{cases}$$

It is obvious that for any atomic p, $id_{\iota\{p\}}$ is sound, since for any formula A of **IQC**, $\vdash_{\mathbf{IQC}} A \supset (A \supset p. \supset p)$ holds. We shall prove that for any atomic p, $id_{\iota\{p\}}$ is unprovablility-preserving. Let A be a formula of X. Suppose $\vdash_{\mathbf{IQC}} id_{\iota\{p\}}(A)$. If $p \in <A>$, then we immediately have $\vdash_X A$ since τ is an embedding. So, suppose $p \notin <A>$. We shall consider the provability of $id_{\iota\{p\}}$ in Kleene's Gentzen-style sequent calculus **G3** for **IQC** (see [17, p. 481] or Appendix of [12]) for which Gentzen's normal form theorem (Hauptsatz or cut elimination theorem) holds. So there is some proof (figure) of $id_{\iota\{p\}}(A)$. Every successful proof of it should look the following:

$$\dfrac{\dfrac{\dfrac{\vdots}{A \supset p \to A} \qquad p, A \supset p \to p}{A \supset p \to p}}{\to A \supset p. \supset p.}$$

Hence, we obtain $\vdash_{\mathbf{IQC}} A \supset p. \supset A$. Since

$$(A \supset p. \supset A) \supset .p \supset A$$

is a theorem of **IQC**, we have $\vdash_{\mathbf{IQC}} p \supset A$. Since $p \notin <A>$ holds, by Lemma 2.2, A should be provable in **IQC**.

[10] For this proposition, we take the original formulation for the logics with $\perp$. The proposition of course holds for ones without $\perp$, too.

Taking the contraposition of the argument considered above, we can conclude that $id_{\iota\{p\}}$ is unprovability-preserving. For any atomic formulas $p \neq q$, $id_{\iota\{p\}}(r) \equiv id_{\iota\{q\}}(r)$ is not a theorem of **IQC** if $r \neq p$ or $r \neq q$ (this atomic formula r is a C-ditcher of $id_{\iota\{p\}}$ and $id_{\iota\{q\}}$). Thus the construction $id_{\iota\{p\}}$ yields countably many mutually non-C-equivalent unprovability-preserving sound translations from **IQC** to **IQC**. The case (P2) can similarly be taken care of. So it is left to the reader.

Summing up the above, we have the following theorems.

Theorem 3.1 Let X be **IQC** (**IPC**). *A set* $\{id_{\iota\{p\}} : p \in F_X^{at-}\}$ *is that of mutually non-C-equivalent autoembeddings of X.*[11]

Corollary 3.1 Let X be a consistent formal system. Let Y be **IQC** (**IPC**). *Suppose that we have an embedding τ of X in Y. Then, for any $p \in F_Y^{at-}$, $\tau_{\iota\{p\}}$ is an embedding of X in Y.*

Proof. Immediate from Theorem 3.1. □

For brevity, we shall write $(\cdot)^{\circ}_{\iota\{p\}}$ for $((\cdot)^{\circ})_{\iota\{p\}}$, where $(\cdot)^{\circ}$ is Gödel translation from **CQC** (**CPC**) to **IQC** (**IPC**).

Theorem 3.2 Let X be **CQC** (**CPC**) *and Y* **IQC** (**IPC**). *A set* $\{(\cdot)^{\circ}_{\iota\{p\}} : p \in F_Y^{at-}\}$ *is that of mutually non-C-equivalent embeddings of X in Y.*

4. *Embedding of classical in intuitionistic logics, Part II*

In this section, we shall take care of (P2) by means of two alternative constructions. Also we shall generalize the results of the previous section, extending the definition of $\tau_{\iota\{p\}}$ to that of $\tau_{\iota\{R\}}$ with a more general unprovable formula R instead of an atomic formula. As in the previous section, we shall only consider the case of predicate logic.

Definition 4.1 Let Y be **IQC** (**IPC**). Let τ be a given embedding of a consistent formal system X in Y. Take $r \in F_Y^{at-}$ and fix it. For any $p \in F_Y^{at}$, we define τ_p as follows: for any $A \in F_X$,

$$\tau_p(A) = \begin{cases} p & \text{if } A = p \neq \bot, \\ r & \text{if } \vdash_Y A \equiv \bot \text{ and } p \neq \bot, \\ \tau(A) & \text{otherwise.} \end{cases}$$

[11] In [3] for example, some elementary properties of this construction $id_{\iota\{p\}}$ are given in the setting of type theory.

It is obvious that for any such a p, τ_p is an embedding of X in Y. Let $(\cdot)^\circ$ be Gödel's negative translation from **CQC** (**CPC**) to Y. We write $(\cdot)^\circ_p$ for $((\cdot)^\circ)_p$. Then, a set $\{(\cdot)^\circ_p : p \in F_Y^{at-}\}$ with cardinality $\aleph_0$ gives an affirmative answer to (P2). For any $p \neq q \in F_Y^{at-}$, $(\cdot)^\circ_p$ and $(\cdot)^\circ_q$ are non-C-equivalent, since $p \equiv \neg\neg p$ is not a theorem of Y: i.e. in this case, p is a C-ditcher of them. For any $p \in F_Y^{at-}$, $(\cdot)^\circ_\perp$ and $(\cdot)^\circ_p$ are non-C-equivalent, since $\neg\neg r \wedge \neg\neg\neg r. \equiv r$ is not provable in Y: i.e. in this case, $r \wedge \neg r$ is a C-ditcher of them.

Theorem 4.1 Let X be **CQC** (**CPC**) *and Y* **IQC** (**IPC**). *A set* $\{(\cdot)^\circ_p : p \in F_Y^{at}\}$ *is that of mutually non-C-equivalent embeddings of X in Y.*

Before we introduce the third construction, we shall review the disjunction property of **IQC** (**IPC**).

Theorem 4.2 ([10]) (*The disjunction property*) *Let X be* **IQC** (**IPC**). *For any* $A, B \in F_X$, *we have:* $\vdash_X A \vee B \Rightarrow (\vdash_X A$ *or* $\vdash_X B)$.

The third construction is the following.

Definition 4.2 Let Y be **IQC** (**IPC**). Let τ be an embedding of a consistent formal system X in Y. For any $R \in F_Y$ with $\dashv_Y R$, we define $\tau_{\delta\{R\}}$ as follows: for any $A \in F_X$,

$$\tau_{\delta\{R\}}(A) = \tau(A) \vee R.$$

Let X be a consistent formula system. Let Y be **IQC** (**IPC**). For any unprovable formula R of Y, $\tau_{\delta\{R\}}$ is an embedding of **CQC** (**CPC**) in X. Indeed, $\tau_{\delta\{R\}}$ is sound, since $\tau(A) \supset \tau_{\delta\{R\}}(A)$ is a theorem of Y. It is faithful, too. Suppose that $\vdash_Y \tau(A) \vee R$ holds. Then, by the disjuction property, we have $\vdash_Y \tau(A)$ or $\vdash_Y R$. Then, we obtain $\vdash_Y \tau(A)$, since R is not provable in Y. So we have $\vdash_X A$, since τ is faithful.

Let $R, Q \in F_Y$ with $\dashv_Y R$, $\dashv_Y Q$ and $\dashv_Y R \equiv Q$, arbitrarily. Suppose that for a given τ, there is a formula $A \in F_X$ such that $\dashv_X A$, $\dashv_Y \neg\neg\tau(A)$ and $<\tau(A)>^{d\perp} \cap <R \equiv Q>^{d\perp} = \emptyset$. We may assume without loss of generality that $\dashv_Y R \supset Q$ holds. By Lemma 2.2 and the above assumptions, we get

$$\dashv_Y \neg\tau(A) \supset .R \supset Q. \ (*)$$

We know

$$\vdash_Y (\tau(A) \vee R) \supset (\tau(A) \vee Q). \supset .\neg\tau(A) \supset (R \supset Q). \ (**)$$

So we obtain, from the above (*) and (**),

$$\dashv_Y \tau(A) \vee R. \supset .\tau(A) \vee Q.$$

This means that A is a C-ditcher of $\tau_{\delta\{R\}}$ and $\tau_{\delta\{Q\}}$. So, summing up the above, we have:

Theorem 4.3 Let Y be **IQC (IPC)**. *Let τ be an embedding of a consistent formal system X in Y. For any $R \in F_Y$ with $\dashv_Y R$, a translation $\tau_{\delta\{R\}}$ is an embedding of X in Y. For any $R, Q \in F_Y$ with $\dashv_Y R$ and $\dashv_Y Q$, if $\dashv_Y R \equiv Q$ holds, and if there is a formula $A \in F_X$ such that $\dashv_X A$, $\dashv_Y \neg\neg\tau(A)$ and $<\tau(A)>^{d\perp} \cap <R \equiv Q>^{d\perp} = \emptyset$, then $\tau_{\delta\{R\}}$ and $\tau_{\delta\{Q\}}$ are non-C-equivalent.*

Corollary 4.1 Let X be **IQC (IPC)**. *A set $\{id_{\delta\{p\}} : p \in F_X^{at}\}$ is that of mutually non-C-equivalent autoembeddings of X.*

Let τ be an embedding of a formal system X in **IQC (IPC)**. If τ preserves provability relation $\vdash$ (that is, for all $A, B \in F_X$, $A \vdash B \Rightarrow \tau(A) \vdash \tau(B)$), then for any unprovable formula R of **IQC (IPC)**, $\tau_{\delta\{R\}}$ preserves $\vdash$, too.

For brevity, we shall write $(\cdot)^{\circ}_{\delta\{R\}}$ for $((\cdot)^{\circ})_{\delta\{R\}}$.

Let X = **IQC (IPC)**. A set $\{(\cdot)^{\circ}_{\delta\{p\}} : p \in F_X^{at}\}$ with cardinality $\aleph_0$ then gives an affirmative answer to (P1) and (P2). For any $p \neq q \in F_X^{at}$, $(\cdot)^{\circ}_{\delta\{p\}}$ and $(\cdot)^{\circ}_{\delta\{q\}}$ are non-C-equivalent. Indeed, $r \in F_X^{at-} - \{p, q\}$ is a C-ditcher of them with $r \neq p$ and $r \neq q$. For any $p \in F_X^{at-}$, $(\cdot)^{\circ}_{\delta\{p\}}$ and $(\cdot)^{\circ}$ are non-C-equivalent since for any $p \neq q \in F_X^{at}$, q is a C-ditcher of them. So, we have:

Theorem 4.4 Let X be **CQC (CPC)** *and Y* **IQC (IPC)**. *A set $\{(\cdot)^{\circ}_{\delta\{p\}} : p \in F_Y^{at}\}$ is that of mutually non-C-equivalent embeddings of X in Y.*

We shall also see the following easily.

Theorem 4.5 For any $p, q \in F^{at}_{\mathbf{IQC(IPC)}}$, unless $p = q = \perp$, then $(\cdot)^{\circ}_{\iota\{p\}}$ and $(\cdot)^{\circ}_{\delta\{q\}}$ are non-C-equivalent.

We shall extend the definition of $\tau_{\iota\{p\}}$ as follows.

Definition 4.3 Let Y be **IQC (IPC)**. Let τ be an embedding of a consistent formal system X in Y. For any $R \in F_Y$ with $\dashv_Y R$, we define a translation $\tau_{\iota\{R\}}$ from X to Y as follows: for any $A \in F_X$,

$$\tau_{\iota\{R\}}(A) = \begin{cases} \tau(A) \supset R. \supset R & \text{if } Cond(\tau, A, R) \text{ holds,} \\ \neg\neg\tau(A) & \text{if } \vdash_Y R \equiv \perp, \\ \tau(A) & \text{otherwise.} \end{cases}$$

where

$$Cond(\tau, A, R) \Leftrightarrow (<\tau(A)>^{d\perp} \cap <R>^{d\perp} = \varnothing \wedge \dashv_Y \neg R).$$

We can extend Theorem 3.1 for the new $\tau_{\iota\{R\}}$. Before we carry it out, we need some definitions.

Definition 4.4 Let X be **IQC** (**IPC**). Let A be a formula of X. Suppose that A contains at least one $\perp$. A falsum $\perp$ contained in A is said to be a *negation-component of* A if $B \supset \perp$ is a subformula of A for the falsum $\perp$ and some formula B of X, otherwise we call it a *non-negation-component of* A.

Definition 4.5 Let X be **IQC** (**IPC**). Suppose that A has n falsums as the non-negation-components of it. Take an ordered sequence $p_1, p_2, ..., p_n$ ($n \geq 1$) of mutually distinct proposition letters of X such that $p_i \notin <A>$ holds for any $1 \leq i \leq n$. Then we define a formula $A^{d\perp}[p_1, p_2, ..., p_n]$ of X as a result obtained by the following procedures (1) and (2): (1) in A, replace all non-negation-components of A by $p_1 \wedge \neg p_1, p_2 \wedge \neg p_2, ..., p_n \wedge \neg p_n$ in such an order as from the leftmost occurrence of the $\perp$'s to the rightmost; (2) in the result obtained by applying (1) to A, replace all subformulas of the form $B \supset \perp$ for some formula B by $\neg B$. We call $A^{d\perp}[p_1, p_2, ..., p_n]$ *the non-$\perp$-formula of* A *with respect to* $p_1, p_2, ..., p_n$ and we shall write $A^{d\perp}$ for $A^{d\perp}[p_1, p_2, ..., p_n]$, if no ambiguity arises.

Example: Let $p_1, p_2, p_3, q, r, h, g$ be mutually distinct proposition letters of **IPC**. Let A be $[\perp \supset \{(((p_1 \supset \perp. \vee \perp) \supset \perp. \wedge p_2) \supset .\neg p_1 \wedge \perp) \supset .\perp \supset \perp\}] \supset \neg p_3$. Then, as $A^{d\perp}[q, r, h, g]$, we have $[q \wedge \neg q. \supset \{(\neg(\neg p_1 \vee (r \wedge \neg r)) \wedge p_2. \supset .\neg p_1 \wedge (h \wedge \neg h)) \supset .\neg(g \wedge \neg g)\}] \supset \neg p_3$.

Proposition 4.1 With the same notations and assumptions of Definition 4.5, we have $\vdash_X A \equiv A^{d\perp}[p_1, p_2, ..., p_n]$.

Remark: Given a formula A, the formula $A^{d\perp}$ is not uniquely determined. It depends on the chosen proposition letters for the operation, of course. Below we shall often check whether $<A^{d\perp}> \cap <B^{d\perp}> = \varnothing$ holds or not for given A and B. If we do not specially mention, we shall understand that the operation $(\cdot)^{d\perp}$ used in the expression $<A^{d\perp}> \cap <B^{d\perp}>$ is applied to $A \supset B$.

Let us see some examples for the just above remark. Let p_1, p_2, p_3, q, h, g be mutually distinct proposition letters of **IPC**. Let A be $p_1 \supset \neg(\perp \supset p_2)$ and $B = (p_3 \supset \perp) \vee \neg \perp$. Then, we see

$$\begin{aligned}&<A^{d\perp}> \cap <B^{d\perp}>\\ &= <p_1 \supset \neg(q \wedge \neg q. \supset p_2)> \cap <(p_3 \supset h \wedge \neg h) \vee \neg(g \wedge \neg g)>\\ &= \emptyset.\end{aligned}$$

We shall see another example. Let A be $\perp \supset .\neg \perp$ and $B = \neg \perp$. Then, we see

$$\begin{aligned}&<A^{d\perp}> \cap <B^{d\perp}>\\ &= <q \wedge \neg q. \supset \neg(h \wedge \neg h)> \cap <\neg(g \wedge \neg g)> = \emptyset.\end{aligned}$$

In both cases, we note $<A^{d\perp}> \cap <B^{d\perp}> = \emptyset$. In general, we easily see the following proposition.

Proposition 4.2 Let X be **IQC** (**IPC**). (1) *For any A, $B \in F_X$, if there is some $(\cdot)^{d\perp}$ such that $<A^{d\perp}> \cap <B^{d\perp}> = \emptyset$, then for any $(\cdot)^{d\perp}$, we have $<A^{d\perp}> \cap <B^{d\perp}> = \emptyset$.* (2) *For any A, $B \in F_X$, for any $(\cdot)^{d\perp}$, we have*

$$<A^{d\perp}> \cap <B^{d\perp}> = \emptyset \Leftrightarrow <A>^{d\perp} \cap <B>^{d\perp} = \emptyset.$$

We need some lemmas for our further arguments below.

Lemma 4.1 Let X be **IQC** (**IPC**). *For any $A \in F_X$, if $\vdash_X \neg A$, then for any B, $C \in F_X$, we have $\vdash_X A \supset B$ and $\vdash_X A \wedge C. \supset B$.*[12]

Proof. Immediate from $\vdash_X \neg A \supset .A \supset B$ and $\vdash_X \neg A \supset (A \wedge C. \supset B)$. □

Lemma 4.2 Let X be **IQC** (**IPC**). *For any A, $B \in F_X$, if $\vdash_X A \supset B$ holds with $<A^{d\perp}> \cap <B^{d\perp}> = \emptyset$ for some operation $(\cdot)^{d\perp}$, then $\vdash_X \neg A$ or $\vdash_X B$.*

Proof. This is the same lemma as Lemma 2.2 in the present setting with Proposition 4.2. □

Lemma 4.3 Let X be **IQC** (**IPC**). *For any A, $B \in F_X$, if $\dashv_X \neg B$ holds, then $\dashv_X \neg(A \supset B)$ holds.*

Proof. It follows from $\vdash_X \neg(A \supset B) \supset \neg B$. □

Definition 4.6 Let X be a formal system. Then we define $F_X^{d\vee}$ as a subset of F_X such that no formula of it contains $\vee$. Also we define $F_X^{d\vee\exists}$ as a subset of F_X such that no formula of it contains $\vee$ or $\exists$.

[12]For any $A \in F_X$, we note: $\vdash_X \neg A \Leftrightarrow \vdash_X A \equiv \perp$.

We shall show the following extensions of Theorems 3.1 and 3.2, and Corollary 3.1, respectively, for our new $\tau_{\iota\{R\}}$.

Theorem 4.6 Let X be **IQC (IPC)**. *For any $R \in F_X$, $id_{\iota\{R\}}$ is a sound translation from X to X. For any $R \in F_X$ with $\dashv_X R \vee \neg R$,*[13] *if $R \in F_X^{at-}$ holds, then $id_{\iota\{R\}}$ is an unprovability-preserving autotranslation of X, otherwise $id_{\iota\{R\}}$ is an unprovability-preserving translation from X to X with respect to $F_X^{d\vee\exists}$ ($F_X^{d\vee}$). For any $R, Q \in F_X$ with $\dashv_X R \vee \neg R$ and $\dashv_X Q \vee \neg Q$, if there is a proposition letter $p \notin$ <R>$\cup$ <Q> such that either $\dashv_X R \wedge \neg p. \supset Q$ or $\dashv_X Q \wedge \neg p. \supset R$ holds, then $id_{\iota\{R\}}$ and $id_{\iota\{Q\}}$ are non-C-equivalent autotranslations of X (in addition, the p is a C-ditcher of them). As a special case, for any $R \in F_X$ with $\dashv_X R \vee \neg R$, $id_{\iota\{R\}}$ and id are non-C-equivalent autotranslations of X, and so are $id_{\iota\{R\}}$ and $id_{\iota\{\perp\}}$ (in this case, every proposition letter $p \notin$ <R> is a C-ditcher of them).*

Proof. For any $R \in F_X$, the soundness of $id_{\iota\{R\}}$ is obvious. Take an arbitrary $R \in F_X$ with $\dashv_X R$ and $\dashv_X \neg R$. We shall show the faithfulness of $id_{\iota\{R\}}$. Suppose that $R \in F_X^{at-}$ holds. This case has already been in essence proved in Theorem 3.1. So we shall not repeat it. Then we shall assume $R \notin F_X^{at-}$. We shall thus show the faithfulness of $id_{\iota\{R\}}$ with respect to $F_X^{d\vee\exists}$ ($F_X^{d\vee}$). Let $A \in F_X^{d\vee\exists}$ ($F_X^{d\vee}$) arbitrarily. Suppose $\vdash_X id_{\iota\{R\}}(A)$. We know $\dashv_X R \equiv \perp$ because of $\dashv_X \neg R$. Unless $Cond(\tau, A, R)$ holds, then $id_{\iota\{R\}}$ is A itself. So we have $\vdash_X A$. Suppose that $Cond(\tau, A, R)$ holds. Then $id_{\iota\{R\}}$ is of the form $A \supset R. \supset R$. With some suitable proposition letters, the sequent

$$\rightarrow A^{d\perp} \supset R^{d\perp}. \supset R^{d\perp}$$

is provable in **G3** and so is

$$A^{d\perp} \supset R^{d\perp} \rightarrow R^{d\perp}. \ (+)$$

Let $\mathcal{P}$ be a proof of (+) in **G3** and fix it.

Because of $\dashv_X R^{d\perp}$, in every possible proof of (+), say $\mathcal{P}$, the succedent of (+) has to ask the antecedent of it some help for its provability. By Lemma 4.3 and $\dashv_X \neg R^{d\perp}$, the antecedent of (+) also needs some help from the succedent of it. Thus, we must find such a sequent

[13] We note that $\dashv_X R \vee \neg R \Leftrightarrow$ ($\dashv_X R$ and $\dashv_X \neg R$). (The disjunction property!)

$$A^{d\perp} \supset R^{d\perp}, C_1, ..., C_n \to A^{d\perp} \ (++)$$

for the introduction of implication for $A^{d\perp} \supset R^{d\perp}$ in the antecedent of some sequent in the proof $\mathcal{P}$, where $C_1, ..., C_n$ ($n \geq 0$) are subformulas of $R^{d\perp}$ such that below the sequent (++), there are no applications of $\supset \to$ to $A^{d\perp} \supset R^{d\perp}$.

Since $\dashv_X R^{d\perp}$ holds, $\dashv_{\mathbf{G3}} C_1, ..., C_n \to$ holds, considering also that there are no applications of $\supset \to$ to $A^{d\perp} \supset R^{d\perp}$. In **G3**, all sequents to be applied by a rule of introducing a logical symbol in the antecedent of a sequent[14] should already have the same principal formula to be introduced by the rule in the antecedent of it. Hence, for the provability of $A^{d\perp} \supset R^{d\perp}. \supset R^{d\perp}$, the formulas $C_1, ..., C_n$ should have all information for the inferences for subformulas of $R^{d\perp}$ in $\mathcal{P}$ (Fact 1).

Further, since $A^{d\perp}$ contains neither disjunctions nor existential quantifiers, $A^{d\perp}$ in the succedent of (++) need to ask no more extra new information (for the provability of $A^{d\perp} \supset R^{d\perp}. \supset R^{d\perp}$) from the formulas of the forms $G \vee H$ or $\exists x G(x)$ in $A^{d\perp}$ of $A^{d\perp} \supset R^{d\perp}$ in the antecedent of it by the application of $\supset \to$ (Fact 2) (see the examples just below this proof).

In view of these facts and $<C_i> \cap <A^{d\perp}> = \emptyset$ ($\forall 1 \leq i \leq n$) and $\dashv_{\mathbf{G3}} C_1, ..., C_n \to$, the sequent

$$C_1, ..., C_n \to A^{d\perp}$$

is provable in **G3**. Thus, from it, we obtain $\vdash_X C_1 \wedge ... \wedge C_n. \supset A^{d\perp}$. By Lemma 4.2 and $\dashv_{\mathbf{G3}} C_1, ..., C_n \to$ (i.e. $\dashv_X \neg(C_1 \wedge ... \wedge C_n)$), we then have $\vdash_X A^{d\perp}$, thus $\vdash_X A$ holds. So we have proved the faithfulness of $id_{\iota\{R\}}$ with respect to $F_X^{d\vee\exists}$ ($F_X^{d\vee}$).

Let $R, Q \in F_X$ with $\dashv_X R \vee \neg R$ and $\dashv_X Q \vee \neg Q$ arbitrarily. Suppose that there is a proposition letter $p \notin <R> \cup <Q>$ such that either $\dashv_X R \wedge \neg p. \supset Q$ or $\dashv_X Q \wedge \neg p. \supset R$ holds. Then, the p is a C-ditcher of $id_{\iota\{R\}}$ and $id_{\iota\{Q\}}$, since we know

$$\vdash_X (p \supset R. \supset R) \supset (p \supset Q. \supset Q). \supset (R \wedge \neg p. \supset Q),$$
$$\vdash_X (p \supset Q. \supset Q) \supset (p \supset R. \supset R). \supset (Q \wedge \neg p. \supset R),$$

that is,

$$\dashv_X R \wedge \neg p. \supset Q \Rightarrow \dashv_X (p \supset R. \supset R) \supset (p \supset Q. \supset Q),$$
$$\dashv_X Q \wedge \neg p. \supset R \Rightarrow \dashv_X (p \supset Q. \supset Q) \supset (p \supset R. \supset R).$$

[14] That is, the rule is one of $\wedge \to$, $\vee \to$, $\supset \to$, $\neg \to$, $\forall \to$ and $\exists \to$.

For the rest of the proof, by a standard model-theoretic technique, we immediately see the unprovability of the universal closure of following formulas

$$(p \supset R. \supset R) \supset p,$$
$$(p \supset R. \supset R) \supset \neg\neg p,$$

for every proposition letter $p \notin \langle R \rangle$. □

The restriction with "with respect to $F_X^{d\vee\exists}$ ($F_X^{d\vee}$)" in Theorem 4.6 may be eventually weaker. That is, instead of $F_X^{d\vee\exists}$ ($F_X^{d\vee}$), we may take a larger set $E \subseteq F_X$ such that $E \neq F_X^{d\vee\exists}$ ($F_X^{d\vee}$), $E \neq F_X$ and $F_X^{d\vee\exists}$ ($F_X^{d\vee}$) $\subseteq E$. But we shall not here carry it out.

Theorem 4.6 is a generalization of Theorem 3.1.

We shall note that with the notation of Theorem 4.6, $id_{\iota\{R\}}$ is not always an unprovability-preserving translation from X to X, if $R \notin F_X^{at-}$ holds. The restriction with "with respect to $F_X^{d\vee\exists}$ ($F_X^{d\vee}$)" is essential in this case. We shall just below give counterexamples of the faithfulness of $id_{\iota\{R\}}$ (with respect to F_X).

Let p, q, r be mutually distinct proposition letters in **IPC**. Let Y be **IPC**. Then, we immediately see: $\vdash_Y (p \supset q. \vee p) \supset \neg\neg r. \supset \neg\neg r$, $\dashv_Y p \supset q. \vee p$, $\dashv_Y \neg\neg r$ and $\dashv_Y \neg\neg\neg r$. Note that $\vdash_Y (p \supset q. \vee p) \supset \neg r. \supset \neg r$, $\vdash_Y (p \supset q. \vee p) \supset \bot. \supset \bot$ and $\dashv_Y (p \supset q. \vee p) \supset r. \supset r$! This is a difference between Theorem 3.1 and Theorem 4.6 with $R \notin F_X^{at-}$.

Here are other examples. Let Y be **IQC**. Let p be a proposition letter in Y. Let $G(x)$ be a monadic predicate in Y. Let a and b be individual variables of Y. Then, we see: $\vdash_Y (\neg G(a) \supset \neg G(b). \supset \exists x \neg G(x)) \supset \neg\neg p. \supset \neg\neg p$, $\dashv_Y \neg G(a) \supset \neg G(b). \supset \exists x \neg G(x)$, $\dashv_Y \neg\neg p$ and $\dashv_Y \neg\neg\neg p$. We also note that $\vdash_Y (\neg G(a) \supset \neg G(b). \supset \exists x \neg G(x)) \supset \neg p. \supset \neg p$, $\vdash_Y (\neg G(a) \supset \neg G(b). \supset \exists x \neg G(x)) \supset \bot. \supset \bot$ and $\dashv_Y (\neg G(a) \supset \neg G(b). \supset \exists x \neg G(x)) \supset p. \supset p$.

We shall also give counterexamples of the following meta-implications:

$$\vdash_{\mathbf{IPC}} \neg\neg A \Rightarrow \vdash_{\mathbf{IPC}} A, \ (\dagger)$$
$$\vdash_{\mathbf{IQC}} \neg\neg A \Rightarrow \vdash_{\mathbf{IQC}} A. \ (\dagger\dagger)$$

For A of (†), we have $(p \supset \bot) \supset p. \supset p$, where p is a proposition letter of **IPC**. This is again an application of Mints's formula in Introduction. For a more well-known counterexample of (†) is $p \vee \neg p$ as the A, where p is again a proposition letter of **IPC**. For A of (††), we have $\forall x\,(A \vee B(x)) \supset .A \vee \forall x B(x)$ (see [17, Theorem 58, p. 487]), where A and B are predicate letters of **IQC**. However, for any negative formula[15] A, we have $\vdash_{\mathbf{MQC}} A \equiv \neg\neg A$, where **MQC** is the minimal predicate logic, and in **MQC**, $\perp$ is treated as an arbitrary proposition letter (see [42, p. 57]).

Corollary 4.2 Let X be **IQC** (**IPC**). *For any $R \in F_X$ with $\dashv_X R \vee \neg R$, if $R \in F_X^{at-}$ holds, then $id_{\iota\{R\}}$ is an autoembedding of X, otherwise, $id_{\iota\{R\}}$ is an autoembedding of X with respect to $F_X^{d\vee\exists}$ ($F_X^{d\vee}$).*[16]

Corollary 4.3 Let X be a consistent formal system. Let Y be **IQC** (**IPC**). *Suppose that we have an embedding τ of X in Y. Then, for any $R \in F_Y^{at-}$, $\tau_{\iota\{R\}}$ is an embedding of X in Y. For any $R \in F_Y$ with $\dashv_Y R$ and $\dashv_Y \neg R$, if $\tau(F_X) \subseteq F_Y^{d\vee\exists}$ ($F_Y^{d\vee}$) holds, then $\tau_{\iota\{R\}}$ is an embedding of X in Y.*

Corollary 4.3 is a generalization of Corollary 3.1.

Let X be a formal system and Y **IQC** (**IPC**). Let τ be an embedding of X in Y. If τ preserves provability relation $\vdash$, then we cannot say in general that for any $R \in F_Y$ with $\dashv_Y R$, $\tau_{\iota\{R\}}$ preserves $\vdash$. However, for any $A, B \in F_Y$ if we have $\tau(A) \vdash \tau(B)$ and if $\tau_{\iota\{R\}}(A)$ and $\tau_{\iota\{R\}}(A)$ are either of the form $A \supset R. \supset R$ and $B \supset R. \supset R$ for some $R \in F_Y$, respectively, or of the forms $\tau(A)$ and $\tau(B)$, respectively, then of course $\tau_{\iota\{R\}}(A) \vdash \tau_{\iota\{R\}}(A)$ holds with the R. So it depends on given formulas A and B of X and a given R of Y.

For brevity, we shall, as in the previous section, write $(\cdot)^\circ_{\iota\{R\}}$ for $((\cdot)^\circ)_{\iota\{R\}}$, where $(\cdot)^\circ$ is Gödel translation from **CQC** (**CPC**) to **IQC** (**IPC**).

Theorem 4.7 Let X be **CQC** (**CPC**) *and Y* **IQC** (**IPC**). *For any $R \in F_Y$ with $\dashv_Y R$, $(\cdot)^\circ_{\iota\{R\}}$ is an embedding of X in Y. For any $R, Q \in F_Y$ with $\dashv_X R \vee \neg R$ and $\dashv_X Q \vee \neg Q$, if there is a proposition letter $p \notin \langle R\rangle \cup \langle Q\rangle$ such that either $\dashv_Y R \wedge \neg p. \supset Q$* or *$\dashv_Y Q \wedge \neg p. \supset R$ holds, then $(\cdot)^\circ_{\iota\{R\}}$ and $(\cdot)^\circ_{\iota\{Q\}}$ are non-C-equivalent embeddings of X in Y (in addition, the p is a C-ditcher of them). As a special case, for any $R \in F_X$ with $\dashv_X R \vee \neg R$, $(\cdot)^\circ_{\iota\{R\}}$ and $(\cdot)^\circ$ are non-C-equivalent embeddings of X in Y (in this case, every proposition letter $p \notin \langle R\rangle$ is a C-ditcher of them).*

Proof. Observe $(F_X)^\circ \subseteq F_Y^{d\vee\exists}$ ($F_Y^{d\vee}$). By Theorem 4.6, we can prove this, considering Lemma 2.1. The non-C-equivalence of $(\cdot)^\circ_{\iota\{R\}}$ and $(\cdot)^\circ$ follows from the unprovability of the universal closure of $(\neg\neg p \supset R. \supset R) \supset \neg\neg p$ for every proposition letter $p \notin \langle R\rangle$. □

[15] A *negative* formula A is a formula of **IQC** which does not contain $\vee$ or $\exists$ with every atomic p of A being negated (that is, of the form $\neg p$ in A).

[16] The restriction with "with respect to $F_X^{d\vee\exists}$ ($F_X^{d\vee}$)" in Corollary 4.2 may be eventually weaker.

□

Theorem 4.7 is a generalization of Theorem 3.2.

For some suitable R and Q, we have already seen a criterion for the non-C-equivalence of $id_{\iota\{R\}}$ and $id_{\iota\{Q\}}$ in Theorem 4.6. Here is a similar criterion for the non-C-equivalence of $id_{\delta\{R\}}$ and $id_{\iota\{Q\}}$ for some suitable R and Q.

Theorem 4.8 Let X be **IQC** (**IPC**). *For any* $R, Q \in F_X$ *with* $\dashv_X R$ *and* $\dashv_X Q \vee \neg Q$, *if there is a proposition letter* $p \notin \langle R\rangle \cup \langle Q\rangle$ *such that either* $\dashv_X R \wedge \neg p. \supset Q$ *or* $\dashv_X Q \wedge \neg p. \supset R$ *holds, then* $id_{\delta\{R\}}$ *and* $id_{\iota\{Q\}}$ *are non-C-equivalent autotranslations of X* (*in addition, the p is a C-ditcher of them*). *In particular, for any* $R, Q \in F_X$ *with* $\dashv_X R$ *and* $\dashv_X Q \vee \neg Q$, *if there is a proposition letter* $p \notin \langle R\rangle \cup \langle Q\rangle$ *such that* $\dashv_X Q \wedge \neg p. \supset R$ *holds, then* $id_{\delta\{R\}}$ *and* $id_{\iota\{Q\}}$ *are non-C-equivalent autotranslations of X* (*in addition, the p is a C-ditcher of them*). *As a special case, for any* $R \in F_X$ *with* $\dashv_X R$, $id_{\delta\{R\}}$ *and* $id_{\iota\{R\}}$ *are non-C-equivalent autotranslations of X* (*in this case, every proposition letter* $p \notin \langle R\rangle$ *is a C-ditcher of them, for example*).

Proof. Let $R, Q \in F_X$ *with* $\dashv_X R \vee \neg R$ and $\dashv_X Q \vee \neg Q$ arbitrarily. Suppose that there is a proposition letter $p \notin \langle R\rangle \cup \langle Q\rangle$ such that either $\dashv_X Q \wedge \neg p. \supset R$ or $\dashv_X R \wedge \neg p. \supset Q$ holds. Then, the p is a C-ditcher of $id_{\delta\{R\}}$ and $id_{\iota\{Q\}}$, observing

$$\vdash_X (p \supset Q. \supset Q) \supset (p \vee R). \supset (Q \wedge \neg p. \supset R),$$
$$\vdash_X (p \vee R) \supset (p \supset Q. \supset Q). \supset (R \wedge \neg p. \supset Q),$$

from which we see

$$\dashv_X Q \wedge \neg p. \supset R \Rightarrow \dashv_X (p \supset Q. \supset Q) \supset .p \vee R,$$
$$\dashv_X R \wedge \neg p. \supset Q \Rightarrow \dashv_X p \vee R. \supset (p \supset Q. \supset Q).$$

If we do not know if $\dashv_X \neg R$ holds, take the first meta-implication for the second statement. For the last statement, we can easily prove, by a standard argument with Kripke models, the unprovability of the universal closure of $(p \supset R. \supset R) \supset .p \vee R$ for any proposition letter $p \notin \langle R\rangle$. This completes the proof. □

Since we also observe

$$\vdash_X (\neg\neg p \supset Q. \supset Q) \supset (\neg\neg p \vee R). \supset (Q \wedge \neg p. \supset R),$$
$$\vdash_X (\neg\neg p \vee R) \supset (\neg\neg p \supset Q. \supset Q). \supset (R \wedge \neg p. \supset Q),$$
$$\dashv_X (\neg\neg p \supset R. \supset R) \supset (\neg\neg p \vee R) \text{ (with } \dashv_X \neg R),$$

we have the following theorem for Gödel translation.

Theorem 4.9 Let X be **CQC** (**CPC**) *and Y* **IQC** (**IPC**). *For any $R, Q \in F_Y$ with $\dashv_Y R$ and $\dashv_Y Q \vee \neg Q$, if there is a proposition letter $p \notin \langle R\rangle \cup \langle Q\rangle$ such that either $\dashv_Y R \wedge \neg p. \supset Q$ or $\dashv_Y Q \wedge \neg p. \supset R$ holds, then $(\cdot)^{\circ}_{\delta\{R\}}$ and $(\cdot)^{\circ}_{\iota\{Q\}}$ are non-C-equivalent embeddings of X in Y (in addition, the p is a C-ditcher of them). In particular, for any $R, Q \in F_Y$ with $\dashv_Y R$ and $\dashv_Y Q \vee \neg Q$, if there is a proposition letter $p \notin \langle R\rangle \cup \langle Q\rangle$ such that $\dashv_Y R \wedge \neg p. \supset Q$ holds, then $(\cdot)^{\circ}_{\delta\{R\}}$ and $(\cdot)^{\circ}_{\iota\{Q\}}$ are non-C-equivalent embeddings of X in Y (in addition, the p is a C-ditcher of them). As a special case, for any $R \in F_X$ with $\dashv_Y R \vee \neg R$, $(\cdot)^{\circ}_{\delta\{R\}}$ and $(\cdot)^{\circ}_{\iota\{R\}}$ are non-C-equivalent embeddings of X in Y (in this case, every proposition letter $p \notin \langle R\rangle$ is a C-ditcher of them, for example).*

Theorem 4.9 is a genearalization of Theorem 4.5.

The following lemma, which is a special case of Lemma 2.2, will be in particular useful in the next section.

Lemma 4.4 Let X be **IQC** (**IPC**). *For any $A, B \in F_X$ and any $p \in F_X^{at-}$ with $p \notin \langle A\rangle \cup \langle B\rangle$, if $\dashv_X A \supset B$ holds, then $\dashv_X A \wedge \neg p. \supset B$ holds.*

Proof. Let $A, B \in F_X$ and $p \in F_X^{at-}$ arbitrarily. Suppose $p \notin \langle A\rangle \cup \langle B\rangle$. It is sufficient to show that

$$\vdash_X A \wedge \neg p. \supset B \Rightarrow \vdash_X A \supset B.$$

Suppose $\vdash_X A \wedge \neg p. \supset B$. Since $(A \wedge \neg p. \supset B) \supset (\neg p \supset .A \supset B)$ is a theorem of X, we get $\vdash_X \neg p \supset .A \supset B$. From it, by Lemma 2.2, we obtain $\vdash_X A \supset B$, since $\langle\neg p\rangle^{d\perp} \cap \langle A \supset B\rangle^{d\perp} = \emptyset$ holds. □

In addition, by Lemma 4.4, we can simplify the description of Theorems 4.6, 4.7, 4.8 and 4.9.

5. *The Nishimura-Rieger lattice and embedding of classical in intuitionistic logics*

In this section, we shall show that a certain set obtained by the introduced construction forms a lattice isomorphic to a sublattice of the Nishimura-Rieger lattice of formulas of one propositional variable p of **IPC** for any p. First we shall recall the Nishimura-Rieger lattice of formulas of one variable of **IPC**.

Definition 5.1 ([30]) For any $p \in F_{\mathbf{IPC}}^{at-}$, *basic formulas* (*with respect to p*) $N_\infty(p)$, $N_i(p)$ ($i \geq 0$) are recursively defined as follows: $N_\infty(p) = p \supset p$, $N_0(p) = p \wedge \neg p$, $N_1(p) = p$, $N_2(p) = \neg p$, $N_{2n+3}(p) = N_{2n+1}(p) \vee N_{2n+2}(p)$, $N_{2n+4}(p) = N_{2n+2}(p) \supset N_{2n+1}(p)$, for $n \geq 0$.

Theorem 5.1 ([30]) *Let X be* **IPC**. *For any $p \in F_X^{at-}$, any formula A with $<A> \subseteq \{p, \bot\}$, if $\dashv_X A$, then we have $\vdash_X A \equiv N_i(p)$ for some $i \geq 0$, otherwise $\vdash_X A \equiv N_\infty(p)$ holds.*

Theorem 5.2 ([30]) *Let X be* **IPC**. *For any $p \in F_X^{at-}$, any $i \neq j \geq 0$, we have: $\dashv_X N_i(p)$ and $\dashv_X N_i(p) \equiv N_j(p)$.*

Corollary 5.1 Let X be **IPC**. *For any $p \in F_X^{at-}$, any $i \neq j \geq 0$, we have: (1) If $i \neq 0$, then $\dashv_X \neg N_i(p)$; (2) $\dashv_X \neg N_i(p) \equiv \neg N_j(p)$.*

Proof. This follows from Theorems 5.1 and 5.2. □

Corollary 5.2 For any $p \neq q \in F_{\mathbf{IPC}}^{at-}$, any $i, j \geq 0$, unless $i = j = 0$, we have: $\dashv_{\mathbf{IPC}} N_i(p) \equiv N_j(q)$.

Proof. Immediate from Theorems 5.1 and 5.2, and Lemma 2.2. □

Definition 5.2 Let X be **IPC**. We define an order relation $\geq$ on every subset F of the equivalence class $F_X/_\equiv$ with $\equiv$ as an equivalence relation as follows: for any $[A]$ and $[B] \in F$, $[A] \geq [B] \Leftrightarrow \vdash_X B \supset A$ (By $[a]$, we denote an element of an equivalence class with a representative a.)

Theorem 5.3 ([30]) *For any $p \in F_{\mathbf{IPC}}^{at-}$, a set $\{[N_i(p)] : i \geq 0\} \cup \{[N_\infty(p)]\}$ forms a lattice with respect to $\geq$. The lattice is called the Nishimura-Rieger lattice with respect to p.*

Theorem 5.4 ([30]) *For any $p \in F_{\mathbf{IPC}}^{at-}$, a set $\{[N_i(p)] : i \geq 0\}$ is a sublattice of the Nishimura-Rieger lattice with respect to p. We call the sublattice the pure Nishimura-Rieger lattice with respect to p.*

We need some definitions for further arguments below.

Definition 5.3 Let X and Y be formal systems. Then, we define $\mathrm{Hom}_{em}(X, Y)$ as a class of all embeddings of X in Y. Also, for any subset of D of F_X, we define $\mathrm{Hom}_{em}(X|D, Y)$ as a class of all embeddings of X in Y with respect to D.

Definition 5.4 Let X be **CPC** and Y **IPC**. Then, we define an order relation $\geq_C$ on every subset H of the equivalence class $\mathrm{Hom}_{em}(X, Y)/\equiv_C$ with C-equivalence $\equiv_C$ as an equivalence relation as follows: for any $[\tau]$ and $[\sigma] \in H$, $[\tau] \geq_C [\sigma] \Leftrightarrow$ (for any $A \in F_X$, $\vdash_X \sigma(A) \supset \tau(A)$).

From the above theorems in this section, the following theorems hold.

Lemma 5.1 Let X be **CPC** *and Y* **IPC**. *For any* $p \in F_Y^{at-}$, *any* $i \neq j \geq 0$, *we have:* (1) $[id_{\delta\{N_i(p)\}}] \in \mathrm{Hom}_{em}(Y, Y)/\equiv_C$; (2) $[id_{\iota\{N_i(p)\}}] \in \mathrm{Hom}_{em}(Y \restriction F_Y^{d\vee}, Y)/\equiv_C$; (3) $[(\cdot)^\circ_{\delta\{N_i(p)\}}] \in \mathrm{Hom}_{em}(X, Y)/\equiv_C$; (4) $[(\cdot)^\circ_{\iota\{N_i(p)\}}] \in \mathrm{Hom}_{em}(X, Y)/\equiv_{C_0}$; (5) If $\vdash_Y N_i(p) \supset N_j(p)$ *holds, then we have:* (i) $[(\cdot)^\circ_{\delta\{N_j(p)\}}] \geq_C [(\cdot)^\circ_{\iota\{N_i(p)\}}]$; (ii) $[(\cdot)^\circ_{\delta\{N_j(p)\}}] \geq_C [(\cdot)^\circ_{\delta\{N_i(p)\}}]$.

Proof. The statements (1), (2), (3) and (4) immediately follow from Theorems 4.3, 4.6, 4.7 and 5.2. To prove (5) is easy. □

Theorem 5.5 Let X be **IPC**. *For any* $p \neq q \in F_X^{at-}$, *any* $i, j \geq 0$, *any* $x, y \in \{\delta, \iota\}$, *we have:*

(1) $(\cdot)^\circ_{x\{N_i(p)\}}$ *and* $(\cdot)^\circ_{y\{N_j(p)\}}$ *are non-C-equivalent, if neither* $i = j = 0$ *nor* ($i = j \neq 0$ *and* $x = y$) *holds.*
(2) $(\cdot)^\circ_{x\{N_i(p)\}}$ *and* $(\cdot)^\circ_{y\{N_i(q)\}}$ *are non-C-equivalent, unless* $i = 0$ *holds.*
(3) $(\cdot)^\circ_{x\{N_i(p)\}}$ *and* $(\cdot)^\circ_{y\{N_j(q)\}}$ *are non-C-equivalent, unless* $i = j = 0$ *holds.*

Proof of (1). We have the following four cases to prove.
(Case 1): The case of $i = j = 0$. Easy with Lemma 2.1.
(Case 2): The case of $i = j \neq 0$. Suppose $x \neq y$. Then apply Theorem 4.9.
(Case 3): The case of ($i > j = 0$ or $j > i = 0$). Use Theorems 4.3 and 4.7, and Lemma 2.1.
(Case 4): The case of ($i \neq j$ and $min\{i, j\} > 0$). In view of Theorem 5.2 and Lemma 4.4, we can make use of Theorems 4.3 and 4.7 for the proof.

Proof of (2). By Corollary 5.2, we know $\dashv\vdash_X N_i(p) \equiv N_i(q)$. Making use of it, we can similarly prove it as the proof of (1).

Proof of (3). Similar to the proof of (2). □

Note that Theorem 5.5 is a generalization of Theorems 3.2 and 4.5 for the propositional case.

Definition 5.5 Let X be a consistent formal system. For any embeddings τ and σ of X in **IPC**, we define embeddings $\tau \wedge \sigma$ and $\tau \vee \sigma$ of X in **IPC** as follows: for any formula $A \in F_X$, $(\tau \wedge \sigma)(A) = \tau(A) \wedge \sigma(A)$, $(\tau \vee \sigma)(A) = \tau(A) \vee \sigma(A)$.

Theorem 5.6 For any $p \in F^{at-}_{\mathbf{IPC}}$, a set

$$\{[(\cdot)^{\circ}_{\delta\{N_i(p)\}}] : i \geq 0\} \subseteq \mathrm{Hom}_{em}(\mathbf{CPC}, \mathbf{IPC})/\equiv_C$$

forms a lattice with respect to $\geq_C$, which is isomorphic to the pure Nishimura-Rieger lattice with respect to p.

Proof. Take $p \in F^{at-}_{\mathbf{IPC}}$ arbitrarily. Let

$$E = \{[(\cdot)^{\circ}_{\delta\{N_i(p)\}}] : i \geq 0\}.$$

We shall show that E is a lattice with respect to $\geq_C$. In view of Theorem 5.4, this immediately follows from the following: for any $A \in F_{\mathbf{IPC}}$,

$$\vdash_{\mathbf{IPC}} (A^{\circ} \vee N_i(p). \wedge .A^{\circ} \vee N_j(p)) \equiv (A^{\circ} \vee .N_i(p) \wedge N_j(p)),$$
$$\vdash_{\mathbf{IPC}} (A^{\circ} \vee N_i(p). \vee .A^{\circ} \vee N_j(p)) \equiv (A^{\circ} \vee .N_i(p) \vee N_j(p)),$$

Let N_p be the pure Nishimura-Rieger lattice with respect to p. Define a mapping $f : N_p \to E$ as follows: for any $[N_i(p)] \in E$,

$$f([N_i(p)]) = [(\cdot)^{\circ}_{\delta\{N_i(p)\}}].$$

Then, f is surjective. By Lemma 5.1.(5), f is order preserving, that is,

$$[N_i(p)] \geq [N_j(p)] \Rightarrow f([N_i(p)]) \geq_C f([N_j(p)]).$$

Further, we see

$$[(\cdot)^{\circ}_{\delta\{N_i(p)\}}] \geq_C [(\cdot)^{\circ}_{\delta\{N_j(p)\}}] \Rightarrow [N_i(p)] \geq [N_j(p)]$$

Indeed, $[(\cdot)^{\circ}_{\delta\{N_i(p)\}}] \geq_C [(\cdot)^{\circ}_{\delta\{N_j(p)\}}]$ implies

$$\vdash_{\mathbf{IPC}} (\bot)^{\circ} \vee N_j(p). \supset (\bot)^{\circ} \vee N_i(p),$$

that is, $\vdash_{\mathbf{IPC}} N_j(p) \supset N_i(p)$, which is nothing but $[N_i(p)] \geq [N_j(p)]$. Thus f is an order-isomorphism. Hence, f is a lattice-isomorphism (cf. [2, p. 114]). $\square$

6. *Some remarks on non-C-equivalence*

First we shall introduce the notion of U-equivalence, which will be below understood as an equivalent one of C-equivalence under a certain weak condition, so it holds in many logics[17] which include all logics dealt with in this paper.

Definition 6.1 Let X and Y be consistent formal systems. Suppose that truth-functional equivalence $\equiv$ is one of the logical symbols of Y, or it is definable in Y. For any translations τ_1 and τ_2 from X to Y, they are said to be *non-U-equivalent* if there is a formula A of X (called a *U-ditcher of* τ_1 *and* τ_2) such that $\tau_1(A) \equiv \tau_2(A)$ is not a theorem of Y. Otherwise we say that they are *U-equivalent*.

The following proposition says that under a very weak condition, those notions are equivalent. So usually, they are the same.

Proposition 6.1 Let X and Y be consistent formal systems. Suppose that Y satisfies the following conditions (i)–(iv): (i) *truth-functional equivalence $\equiv$ and implication $\supset$ are logical symbols of Y, or they are definable in Y;* (ii) *Y is closed under modus ponens;* (iii) *for any $A, B \in F_Y$, $\vdash_Y A \equiv B$ is equivalent to* ($\vdash_Y A \supset B$ *and* $\vdash_Y B \supset A$); (iv) *Y satisfies one of the following conditions* (a) *and* (b): (a) *the deduction theorem*[18] *holds in Y;* (b) $\vdash_Y A \supset (B \supset A)$ *holds for any $A, B \in F_Y$. Let τ_1 and τ_2 be embeddings of X in Y. Then, we have:*

τ_1 and τ_2 are non-C-equivalent if and only if they are non-U-equivalent.[19]

Proof. Straightforward from Definitions 2.5 and 6.1. □

[17] For instance, the cardinality of a set of such logics as intermediate (or superintuitionistic) propositional (predicate) logics is at least $2^{\aleph_0}$ (see e.g. [44], [45] and [46]).

[18] It is well-known that intuitionistic implication is the weakest possible one fulfilling the deduction theorem (see [5]).

[19] In this case, if we have a C-ditcher of τ_1 and τ_2, then it is a U-ditcher of them, and vice versa.

7. *Embedding for classical and intuitionistic logics and modal logics* **S4** *and* **S5**

First we shall recall an elementary fact: for any formulas A and C of **CPC** (**CQC**),

$$\vdash_{\mathbf{CPC(CQC)}} (A \supset C. \supset C) \equiv .A \vee C.^{20}$$

A simple semantic consideration and the above equivalence suggest a possibility to have embeddings for **CPC**.

Definition 7.1 Let Y be **CPC**. Let τ be an embedding of a consistent formal system X in Y. Then, for any $R \in F_Y$ with $\dashv_Y R$, we define a translation $\tau_{\delta C\{R\}}$ from X to Y as follows: for any $A \in F_X$,

$$\tau_{\delta C\{R\}}(A) = \begin{cases} \tau(A) \vee R & \text{if } < \tau(A) > \cap < R > = \varnothing, \\ \tau(A) & \text{otherwise.} \end{cases}$$

Theorem 7.1 Let Y be **CPC**. *Let τ be an embedding of a consistent formal system X in Y. Then, for any $R \in F_Y$ with $\dashv_Y R$, $\tau_{\delta C\{R\}}$ is an embedding X in Y.*

Proof. The soundness of it is obvious. We shall semantically prove its faithfulness. Suppose $\dashv_X A$. Since τ is an embedding of X in Y, there is a valuation v such that $v(\tau(A)) = f$ (false). Since $<\tau(A)> \cap <R> = \varnothing$ holds, we can easily construct a valuation $\tilde{v}$ from v such that $\tilde{v}(R) = f$ and $\tilde{v}(\tau(A)) = v(\tau(A))$ hold. This proves the faithfulness. □

Definition 7.2 Let Y be **S4**. Let τ be an embedding of a consistent formal system X in Y. Then, for any $R \in F_Y$ with $\dashv_Y R$, we define a translation $\tau_{\delta\Box\{R\}}$ from X to Y as follows: for any $A \in F_X$,

$$\tau_{\delta\Box\{R\}}(A) = \Box\tau(A) \vee \Box R.$$

We shall recall the following theorem.

[20] This formula is not intuitionistically valid! We can only say $\vdash_{\mathbf{IPC(IQC)}} A \vee C. \supset (A \supset C. \supset C)$. It is well-known that none of the logical symbols for **IPC** can be expressed in terms of the remaining ones (see e.g. [24]).

Theorem 7.2 ([9] and [26]) (*The modal disjunction property for* **S4**) *Let X be* **S4**. *For any* $A, B \in F_X$, *we have:* $\vdash_X \Box A \vee \Box B \Rightarrow (\vdash_X \Box A$ *or* $\vdash_X \Box B)$.

Theorem 7.3 Let Y be **S4**. *Let* τ *be an embedding of a consistent formal system X in Y. Then, for any* $R \in F_Y$ *with* $\dashv_Y R$, $\tau_{\delta\Box\{R\}}$ *is an embedding X in Y.*

Proof. Obvious from Theorem 7.2 and a fact that for any $A \in F_Y$, $\vdash_Y \Box A \Rightarrow \vdash_Y A$. $\Box$

Before we give an affirmative answer to Question 2 for (P3)–(P6), we shall recall some well-known theorems below. Let $(\cdot)^{\Box}$ be the Gödel translation from **IPC** to **S4** (see [9] and [26]) (cf. [4] and [37])).

Theorem 7.4 ([9] and [26]) *For any* $A \in F_{\mathbf{IPC}}$, *we have:* $\vdash_{\mathbf{IPC}} A \Leftrightarrow \vdash_{\mathbf{S4}} A^{\Box}$.

Theorem 7.5 ([23]) *For any* $A \in F_{\mathbf{S5}}$, *we have:* $\vdash_{\mathbf{S5}} A \Leftrightarrow \vdash_{\mathbf{S4}} \Diamond\Box A$.

Here is an affirmative answer for (P3)–(P6). (The proof of it is easy.[21])

Theorem 7.6 (1) *A set* $\{id_{\delta C\{p\}} : p \in F^{at}_{\mathbf{CPC}}\}$ *is that of mutually non-C-equivalent autoembeddings of* **CPC**; (2) *A set* $\{id_{\delta\Box\{p\}} : p \in F^{at}_{\mathbf{S4}}\}$ *is that of mutually non-C-equivalent autoembeddings of* **S4**; (3) *A set* $\{((\cdot)^{\Box})_{\delta\Box\{p\}} : p \in F^{at}_{\mathbf{S4}}\}$ *is that of mutually non-C-equivalent embeddings of* **IPC** *in* **S4**; (4) *A set* $\{(\Diamond\Box(\cdot))_{\delta\Box\{p\}} : p \in F^{at}_{\mathbf{S4}}\}$ *is that of mutually non-C-equivalent embeddings of* **S5** *in* **S4**.[22]

8. *An affirmative answer to Question 1 for all the cases treated in this paper*

Some of the constructions introduced above, that is, $\tau_{\delta\{R\}}$, $\tau_{\delta C\{R\}}$ and $\tau_{\delta\Box\{R\}}$ eventually give an affirmative answer to Question 1 for classical and intuitionistic logics, and modal logics **S4** and **S5**. Because the strategy to prove is common, as its representative, we shall only show the case of (P2) for an arbitrary embedding. For an affirmative answer to Question 1, we can similarly treat the cases of (P1) and (P3)–(P6) for an arbitrary embedding.

[21] To make sure, the equivalence used for non-C-equivalence in **S4** is material equivalence.

[22] For another approach to the affirmative answer, we may take the strategy of the first construction of the fourth section.

The following theorem[23] gives an affirmative answer to Question 1 for (P2) for an arbitrary embedding.

Theorem 8.1 Let X be **CQC** (**CPC**) *and Y* **IQC** (**IPC**). *Let* τ *be an embedding of X in Y. Then, for any* $A \in F_X$ *with* $\dashv_X A$, *a set* $\{\tau_{\delta\{p\}} : p \in F_Y^{at} - <\tau(A)>^{d\perp}\}$ *with cardinality* $\aleph_0$ *is that of mutually non-C-equivalent embeddings of X in Y.*

Proof. X and Y are consistent. Thus there is at least one A such that $\dashv_X A$ and $\dashv_Y \tau(A)$ hold with $\tau(A) \in F_Y$. Let A be such a formula of X. Let

$$E = \{\tau_{\delta\{p\}} : p \in F_Y^{at} - <\tau(A)>^{d\perp}\}.$$

It is obvious that every element of E is an embedding of X in Y. Take $\tau_{\delta\{p\}} \neq \tau_{\delta\{q\}} \in E$ arbitrarily. Without loss of generality, we may assume $p \neq \perp$. We wish to prove that $\tau_{\delta\{p\}}$ and $\tau_{\delta\{q\}}$ are non-C-equivalent. For that, it is sufficient to find a formula $B \in F_X$ such that $\dashv_Y p \supset \tau(B)$ holds.

For the desired B, we can take the A. Since $p \notin <\tau(A)>$ and $\dashv_Y \tau(A)$ hold, we immediately obtain $\dashv_Y p \supset \tau(A)$ by Lemma 2.2. For the propositional case, $|E|$ is $\aleph_0$, since $<\tau(A)>^{d\perp}$ is finite. For the predicate-case, although $|<\tau(A)>^{d\perp}|$ may be $\aleph_0$, it is easy to see that $|F_Y^{at} - <\tau(A)>^{d\perp}|$ is $\aleph_0$. $\square$

9. *A necessary and sufficient condition for an affirmative answer to Question 1*

In this section, we shall give a necessary and sufficient condition for an affirmative answer to Question 1.

Let X and Y be consistent formal systems. For the notational convenience for the proof below, let us first exactly formulate the positive statement (Q1) for Question 1 as follows:

(Q1) For any embedding τ of X in Y, there is a construction $\mathscr{C}$ and an index set I such that[24] (Q1.1) $\mathscr{C}(\tau) = \{\tau_\alpha\}_{\alpha \in I}$ and $|I| \geq \aleph_0$; (Q1.2) For any $\alpha \in I$, τ_α is an embedding of X in Y; (Q1.3) For any $\alpha \in I$, τ_α is defined by using at least one value $\tau(A)$ for some $A \in F_X$; (Q1.4) For any $\alpha \neq \beta \in I$, τ_α and τ_β are non-C-equivalent.

[23] The reader will enjoy this strong result with a simple proof of it.

[24] We also say that a pair (C, I) is a construction.

Here is the necessary and sufficient condition (GC) for (Q1).

(GC) There is a nonempty set $B \subseteq \mathcal{P}(F_Y)$ and an index set J such that (GC1) $B = \{B_\alpha\}_{\alpha \in J}$ and $|J| \geq \aleph_0$; (GC2) For any $A \in B_\alpha$, $\dashv_Y A$ holds; (GC3) For any $\alpha \in I$, there is a surjective mapping φ_α from $F_X^{\dashv}$ to B_α, where $F_X^{\dashv}$ stands for $\{A \in F_X : \dashv_X A\}$; (GC4) For any $\alpha \neq \beta \in J$, there are $C_\alpha \in B_\alpha$ and $C_\beta \in B_\beta$ such that $\dashv_Y C_\alpha \equiv C_\beta$ holds with $\varphi_\alpha^{-1}(C_\alpha) \cap \varphi_\beta^{-1}(C_\beta) \neq \emptyset$.

Theorem 9.1 Let X and Y be consistent formal systems. Then we have: (Q1) $\Leftrightarrow$ (GC).

Proof. ($\Rightarrow$): Let $J = I$. For any $\alpha \in J$, we take $B_\alpha = \{\tau_\alpha(A) : A \in F_X^{\dashv}\}$. Since τ_α is an embedding of X in Y, B_α satisfies (GC2). For (GC3), define φ_α as $\tau_\alpha | F_X^{\dashv}$ for any $\alpha \in J$. Let $\alpha \neq \beta \in J$. Because of (Q1.4), there is a C-ditcher, say $D \in F_X^{\dashv}$ of τ_α and τ_β. Thus, $\dashv_Y C_\alpha \equiv C_\beta$ and $D \in \varphi_\alpha^{-1}(C_\alpha) \cap \varphi_\beta^{-1}(C_\beta)$, where $C_\alpha = \tau_\alpha(D) \in B_\alpha$ and $C_\beta = \tau_\beta(D) \in B_\beta$.

($\Leftarrow$): Let $I = J$. For any $\alpha \in I$, define τ_α as follows: for any $A \in F_X$,

$$\tau_\alpha(A) = \begin{cases} \varphi_\alpha(A) & \text{if } \dashv_X A \\ \tau(A) & \text{otherwise.} \end{cases}$$

A set $\{\tau_\alpha\}_{\alpha \in I}$ is a desired one of mutually non-C-equivalent embeddings of X in Y. We easily see that for any $\alpha \in I$, τ_α is an embedding of X in Y and τ_α satisfies (Q1.3). Take $\alpha \neq \beta \in I$ arbitrarily. By (GC4), there are $C_\alpha \in B_\alpha$ and $C_\beta \in B_\beta$ such that $\dashv_Y C_\alpha \equiv C_\beta$ holds with $\varphi_\alpha^{-1}(C_\alpha) \cap \varphi_\beta^{-1}(C_\beta) \neq \emptyset$. Any element of $\varphi_\alpha^{-1}(C_\alpha) \cap \varphi_\beta^{-1}(C_\beta)$ is a C-ditcher of τ_α and τ_β. $\square$

10. *Constructions and lattices*

In this section, we shall show that a certain set of constructions forms a lattice for certain formal systems. Let us begin with some definition.

Definition 10.1 Let X be a consistent formal system. Let τ be an embedding of X in **IPC**. For any constructions $(\mathcal{C}_\alpha, J_\alpha)$ and $(\mathcal{C}_\beta, J_\beta)$ with respect to τ both of which satisfy the conditions (Q1.1)–(Q1.4) in §9, we define constructions $(\mathcal{C}_\alpha, J_\alpha) \wedge (\mathcal{C}_\beta, J_\beta)$ and $(\mathcal{C}_\alpha, J_\alpha) \vee (\mathcal{C}_\beta, J_\beta)$ (we write $\mathcal{C}_\alpha \wedge C_\beta$ and $\mathcal{C}_\alpha \vee C_\beta$ for them, respectively, if no ambiguity arises) as follows:

$$(\mathcal{C}_\alpha, J_\alpha) \wedge (\mathcal{C}_\beta, J_\beta) = (\mathcal{C}_\alpha \wedge C_\beta, J_\alpha \times J_\beta),$$
$$(\mathcal{C}_\alpha, J_\alpha) \vee (\mathcal{C}_\beta, J_\beta) = (\mathcal{C}_\alpha \vee C_\beta, J_\alpha \times J_\beta),$$

with

$$(\mathscr{C}_\alpha \wedge C_\beta)(\tau) = \{\tau_{\alpha\lambda} \wedge \tau_{\beta\mu}\}_{(\lambda,\mu)\in J_\alpha \times J_\beta / \equiv_C},$$
$$(\mathscr{C}_\alpha \vee C_\beta)(\tau) = \{\tau_{\alpha\lambda} \vee \tau_{\beta\mu}\}_{(\lambda,\mu)\in J_\alpha \times J_\beta / \equiv_C}$$

where $\mathscr{C}_\alpha(\tau) = \{\tau_{\alpha\lambda}\}_{\lambda \in J_\alpha}$ and $\mathscr{C}_\beta(\tau) = \{\tau_{\beta\mu}\}_{\mu \in J_\beta}$.

As understood from the above definition, in this section, we shall think of embeddings as elements of an equivalence class $\mathrm{Hom}_{em}(X, \mathbf{IPC})/\equiv_C$ for any X. Now we shall show our theorem as follows.

Theorem 10.1 Let X be **IPC**. *Let $(\cdot)^\circ$ be Gödel translation from* **CPC** *to X. Define $J_p = \omega - \{0\}$ for any $p \in F_X^{at-}$. Let $\mathscr{C}_p((\cdot)^\circ) = \{(\cdot)^\circ_{\delta\{N_i(p)\}}\}_{i \in J_p}$ for any $p \in F_X^{at-}$. Then, a set $\{\mathscr{C}_p((\cdot)^\circ)) : p \in F_X^{at-}\}$ is a lattice with respect to $\wedge$ and $\vee$.*

Proof. By Theorem 5.6, we know that for any $p \in F_X^{at-}$, $(\mathscr{C}_p((\cdot)^\circ), J_p)$ satisfies the conditions (Q1.1)–(Q1.4) in §9. For any $p \neq q \in F_X^{at-}$,

$$(\mathscr{C}_p((\cdot)^\circ), J_p) \wedge (\mathscr{C}_q((\cdot)^\circ), J_q),$$
$$(\mathscr{C}_p((\cdot)^\circ), J_p) \vee (\mathscr{C}_q((\cdot)^\circ), J_q),$$

are again constructions satisfying (Q1.1)–(Q1.4). It is easy to see that $\wedge$ and $\vee$ satisfy the required properties to make the set a lattice. □

11. *Some general comments*

The reader would have already noticed that the class of all the consistent formal systems (as objects) with the class of all the embeddings between systems (as arrows) is a category (see [27]). That is, we can take the usual composition of mappings as composite $\circ$ of arrows for the category. For the composite, it is easy to see that the associative law holds. For any consistent formal system X, there always is identity $id_X \in \mathrm{Hom}_{em}(X, X)$ ([12, Theorem 1.1, p. 244]). For any $\tau : X \to Y$, $\tau \circ id_X = \tau$ and $id_Y \circ \tau = \tau$ hold obviously. We shall call this category *the category of embeddings*, denoted by **Emb**. So we can expect to have many algebraic results related to **Emb**.

I believe that finding embeddings between formal system has been so far motivated (A): by interpreting one system by another (e.g. see [39] for avoiding a myth created by the problem of incommensurability of theories) or more specifically (B): by translating (or reductioning) metamathematical properties (e.g. consistency) of one system to those of another in order to obtain metamathematical results. In relation to (A), interpretations between theories, which are not always proof-theoretic embeddings, has been of interest among logicians as means of classifying theories and of understanding the algebraic structures (in particular, lattices) of theories, since such interpretations induce partial orders and equivalence relations which will be used for the purposes (see e.g. [28] and [29, pp. 1–10]). A number of papers have been written in this direction and there are still many open problems concerning interpretability.

On the contrary to the better situation for interpretability, I observe that there has been so far no general theory concerning proof-theoretic embeddings and it seems to me that logicians are satisfied with even *one* embedding between systems! I cannot accept such a situation on embeddings. That is the reason why I set forth in the embedding-construction problem treated in this paper and [12] and in the investigation over the relationship among embeddings obtained by constructions. What I intend in this paper and [12] is to begin studying embeddings for their own sake (as mathematical objects).

In [12], I emphasized unprovability and its syntax. But in this paper, my attitude toward the study is not so coherent to the original point of view and its methodology. However, in this paper we have been always concerned with *unprovable formulas*, more generally saying, *unprovability*, which is the consistent theme of both of the papers.

Needless to say, we need both the abstract and the concrete for further investigation. But for the concrete in particular, I wish to propose the following slogan: To have a good knowledge about embeddings of X in Y is to know much about unprovability in Y, more explicitely saying, many concrete examples of unprovable formulas of Y.

In this paper, we have not yet dealt with the iteration of the introduced constructions. However, the iteration of constructions creates real interesting problems, since they do not always give embeddings and we do not know whether given embeddings obtained by them are non-C-equivalent. In my forthcoming paper [14], we shall deal with the simplest case of the iteration, that is, that of elements of

$$\{id_{\delta\{p\}} : p \in F_X^{at}\} \cup \{id_{\iota\{p\}} : p \in F_X^{at}\},$$

where $X =$ **IQC** (**IPC**), as a generalization of Gödel's theorem on embedding of classical in intuitionistic systems. The reader will see there that even the simplest case is *not* simple. So this paper is the overture to [14] and our further work on the theme of this paper. The curtain has already risen.

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