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A GAME-THEORETIC LOGIC OF NORMS AND ACTIONS

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Abstract

Both rational choice theorists and logicians have made important contributions to the formal analysis of (legal) norms. However, the results from the logic of norms —deontic logic— did not have much impact on the rational choice approach to the analysis of individual rights nor did the work of rational choice theorists have much influence on the development of deontic logic. This paper presents the basic outlines of the models presented in (Van Hees 1995) in which it is shown that a fruitful synthesis of the two lines of research is possible. The semantics of deontic logic and action logic is usually given in terms of a Kripke model of possible worlds with a primitive binary relation R between the worlds. We show how such a relation R can be defined game-theoretically, hence achieving a synthesis between logic and game theory.

1. *Introduction*

In the more than twenty-five years since Amartya Sen first introduced the concept of individual rights into the theory of rational choice, there has been much discussion among rational choice theorists about the proper way of modelling such rights. Two approaches can be distinguished: one in which rights are defined in terms of individual *preferences*, and one in which they are defined by the *strategies* of individuals. The first approach originates in Sen's seminal work (Sen 1970), whereas the second approach constitutes the game-theoretic analysis of rights (Gärdenfors 1981; Deb 1990; Gaertner *et al.* 1992; Fleurbaey and Gaertner 1996). In this paper we adopt a game-theoretic framework. In the game-theoretic approach, game

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forms are used to model the rights of individuals. A game form is a specification of a set of outcomes, a set of admissible strategies for each individual and an outcome mapping from the set of all possible strategy combinations to the set of outcomes. The idea is that an individual's rights are determined by the freedom to choose any of his or her admissible strategies and/or by a concomitant obligation of others not to interfere (Gaertner *et al.* 1992, p. 173; Suzumura 1991, p. 229).

The development of the rational choice analysis of rights has taken place virtually without any reference to the study of deontic logic or to the logic of action, nor has there been a systematic application of game theory to problems of deontic logic.[1] The model presented here was developed in (Van Hees 1995) and integrates the two approaches. We show that the game-theoretic analysis can be used to give a better understanding of the Kripkean models used in deontic logic. In our approach the relation between the various 'possible worlds' of a Kripkean model is not a primitive but is defined game-theoretically. Conversely, we argue that the concepts and tools derived from deontic logic and from the logic of action can be used to provide a systematic foundation for the game-theoretic analysis of rights. For instance, our model clearly distinguishes between *admissible* and *feasible* strategies and thus permits a distinction between the things an individual 'may' or 'shall' do on the one hand, and the things an individual 'can' do or 'cannot avoid' doing on the other. Except for the important recent work of Fleurbaey and Gaertner (1996), this distinction has played virtually no role in the game-theoretic models of rights.

The structure of our presentation is as follows. In Section 2 we present the syntax of the language of our *Deontic Logic of Action (DLA)*. The language contains expressions about the permissions and obligations of individuals. Sections 3 and 4 describe the semantic machinery which is defined game-theoretically. Section 5 contains some concluding remarks.

2. *Deontic Logic of Action: Syntax*

The language of DLA contains the following signs:

Basic propositions:	$x_1, x_2, \ldots$
Temporal symbols:	$t_i, t_j, \ldots$ (countably many)
Temporal predicate:	$<$
Individual symbols:	$i, j, \ldots$ (countably many)

[1] For some important exceptions, see (Apostel 1960; Åqvist 1974; Pörn 1977; Belnap and Perloff 1989; Åqvist and Mullock 1989).

Action operators:	Do, Dó
Alethic operators:	Can, Unav
Deontic operators:	May, Shall
Identity sign and non-identity sign:	$=$, $\neq$
Sentential connectives:	$\&$, $\vee$, $\sim$, $\rightarrow$, $\leftrightarrow$

The construction of the language of DLA is given by the following two definitions.

2.1 *Definition*
(1) For all temporal symbols t_i, t_j, $(t_i < t_j)$ and $(t_i = t_j)$ are atomic formulas of DLA;
(2) For all individual symbols i, j, $(i = j)$ is an atomic formula of DLA;
(3) DLA contains no atomic formulas other than those defined by (1) and (2).

Definition 2.1 defines the atomic formulas of DLA. Next we define the well-formed formulas (wffs) of DLA.

2.2 *Definition*
(1) For all basic propositions x and all temporal symbols t, (t, x) is a wff of DLA;
(2) Each atomic formula of DLA is a wff of DLA;
(3) For all wffs φ of DLA, $\sim\varphi$ is a wff of DLA;
(4) For all wffs φ and ψ of DLA, $(\varphi \& \psi)$ is a wff of DLA;
(5) For all individual symbols i and all temporal symbols t_j:
  (a) if φ is a non-atomic wff of DLA, the expression $\mathrm{Do}_i(t_j, \varphi)$ is called a *weak action statement of type* i/t_j and $\mathrm{D\acute{o}}_i(t_j, \varphi)$ a *strong action statement of type* i/t_j;[2]
  (b) if φ is a weak (strong) action statement of type i/t_j, then $\sim\varphi$ is a weak (strong) action statement of type i/t_j;
  (c) if φ and ψ are weak (strong) action statements of type i/t_j then $(\varphi \& \psi)$ is also a weak (strong) action statement of type i/t_j;
  (d) there are no other action statements of type i/t_j than those defined by (a)-(c);

[2]We call the formulas $\mathrm{D\acute{o}}_i(t_j, \varphi)$ and $\mathrm{Do}_i(t_j, \varphi)$ a *strong* action statement and a *weak* action statement, respectively, because the first statement is logically stronger; as we shall below, the truth of the strong action statement logically entails the truth of the weak action statement. In ordinary English, however, we sometimes speak about an action that is unavoidable —and which can thus only be described by a weak action statement— as being 'stronger' than an action that could have been omitted (say the compulsive drug addict's action of taking drugs versus his action of eating candy).

(e) any action statement of type i/t_j is a wff of DLA;
(6) For all individual symbols i and all temporal symbols t_j: if φ is an action statement of type i/t_j, then
(a) Canφ is a wff of DLA;
(b) Mayφ is a wff of DLA;
(7) the clauses (1)-(6) define all wffs of DLA.

The usual conventions are adopted with respect to the sentential connectives. Moreover, for all action statements φ we use Unavφ for ~Can~φ and Shallφ for ~May~φ.[3]

The formulas are read as follows:

$t_i < t_j$	t_i is an earlier point in time than t_j
$t_i = t_j$	t_i is the same point in time as t_j
(t,x)	x is the case at time t
$\mathrm{Do}_i(t_j, \varphi)$	At time t_j, i sees to it that φ
$\mathrm{D\acute{o}}_i(t_j, \varphi)$	At time t_j, i sees to it that φ but he could have performed an action by which he does not see to φ
Canφ	It is possible that φ ('... can see to ...')
Unavφ	It is necessary that φ ('... cannot avoid seeing to ...')
Mayφ	It is permissible that φ ('... may see to ...')
Shallφ	It is obligated that φ ('.. shall see to ...')

3. *Model structures*

The notion of a model structure plays a crucial role in the semantics of DLA. A model structure M of DLA consists of three components: a complex game tree Γ, a play structure $\wp$ and an interpretation $\Im$.

Let $N = \{1, ..., n\}$ be a finite set of individuals and X with elements q, r, s ... a non-empty, denumerable set the elements of which are called *points*.

Definition 3.1 A *Game Form G* is a structure $< A, \sigma, \pi >$ such that
(1) A is a non-empty and finite subset of X. The elements of A are called the *outcomes* of the game form;
(2) σ is a mapping assigning to each $i \in N$ a non-empty set $\sigma(i)$ of which the elements are called *strategies*;

[3] The operator expressing alethic necessity is called 'Must' in (Van Hees 1995). However, since in common English 'must' is often used in a normative sense ('you must not lie'), we here use 'Unav'.

(3) π is a mapping from $\sigma(1) \times \ldots \times \sigma(n)$ onto A; π is called the *outcome mapping* of G.

An action of agent i consists in making a choice from $\sigma(i)$. Every combination of actions $(s_1, \ldots, s_n)$ in $\sigma(1) \times \ldots \times \sigma(n)$, i.e. every *play* p of G, leads to an outcome $\pi(p)$ of G. A play p' of G is called an *N-i-variant* of a play p of G if i adopts the same strategy in p' as in p.

Next we define a Complex Game Tree.

Definition 3.2 A *Complex Game Tree (CGT)* Γ is a quadruple $<X, \Sigma, \Xi, \tau>$ such that

(1) Σ is a mapping which assigns to each element of X a game form;
(2) Ξ is a mapping which assigns to each element of X a *game form allocation*, that is, an n-tuple of game forms;
(3) If R is the binary relation over X defined as: qRs if and only if there is (a) a play of at least one game form in $\Xi(q)$ that has r as its outcome, or (b) a play of $\Sigma(q)$ which has r as its outcome, then (X,R) is a rooted tree of infinite length;
(4) τ is the mapping from X to the set of positive integers defined as:
    (a) $\tau(q^*) = 1$ where q^* designates the root of the tree (X,R);
    (b) for all $q, r \in X$, if qRr then $\tau(r) = \tau(q) + 1$.

A Complex Game Tree is a tree structure in which $n + 1$ game forms are assigned to each point, and in which the branches of the tree represent the various plays of those game forms. There is one game form, the so-called *feasible game form*, that is used to ascertain the things an individual can and cannot do. It is assigned by Σ. The remaining n game forms, one for each individual, are the *admissible game forms* and are assigned by Ξ. An individual's admissible game form is used to establish what that individual may and may not do at that point in the tree.

Individual strategies (admissible or feasible) describe only part of the world. We also need to describe the other characteristics of a possible world. The second component of a model structure, the interpretation, provides such a description. It not only assigns to each individual symbol an element of N and to each temporal symbol a positive integer, but it also assigns a set of points to each basic proposition. We say that the proposition is *true* in each of those points.

Definition 3.3 An interpretation is a mapping $\Im$ assigning

(a) to each individual symbol an element of N;
(b) to each temporal symbol a positive integer;
(c) to each basic proposition a subset of X.

The third component of a model structure is a play structure. It represents a path in the tree, that is, an uninterrupted sequence of lines starting at the initial point. The path can be seen as a history of the world. It describes how the world changes from each point in time to the next. Formally, it is defined as a sequence of pairs ('rounds'), each consisting of a point and a play of a game form.

Definition 3.4 A *play structure* of Γ is a sequence $\wp = [(q_1, p_1), (q_2, p_2), \ldots]$ where $q_1 = q^*$, and for all $t \geq 1$:
(1) p_t is a play of the feasible game form $\Sigma(q_t)$ or a play of one of the admissible game forms in $\Xi(q_t)$;
(2) q_{t+1} is the outcome of the play p_t.

A point q_t is called the *q_t-point* or simply the *t-point* of $\wp$. We shall refer to a play p_t as the *t-play* of $\wp$. A play structure which consists of plays of feasible game forms only is called a *feasible play structure.*

4. *Truth in a model structure*

In this section we describe how the notion of a model structure is used to determine which formulas of DLA are true and which are not. In order to present the truth conditions we first have to introduce two more definitions. First, we generalize the notion of an *N-i*-variant of a play of a game form (see Definition 3.1) to that of a play structure of a complex game tree.

Definition 4.1 Let $\wp = [(q_1, p_1), (q_2, p_2), \ldots]$ be a play structure of a CGT Γ, i an element of N, and k a positive integer. An *[N-i, k]-variant* of $\wp$ is any play structure $\wp' = [(q'_1, p'_1), (q'_2, p'_2), \ldots]$ of Γ with the following characteristics:
(1) for all $t < k$: $(q'_t, p'_t) = (q_t, p_t)$;
(2) p'_k is an *N-i*-variant of p_k;
(3) for all $t > k$: p'_k is a play of the feasible game form $\Sigma(q'_k)$.

In other words, an *[N-i, k]-variant* $\wp'$ of $\wp$ is a play structure which is exactly the same as $\wp$ up until the k-round. In the k-round the same game form is played as in $\wp$ and i also adopts the same strategy; the others have adopted a different strategy. At all later points the feasible game form is played.

Next we introduce the notion of (feasible and admissible) k-splits. A feasible (i-admissible) k-split of a play structure $\wp$ is a play structure that is identical to $\wp$ up until the k-point. At the k-point a play of the feasible (i-admissible) game form belonging to that point is played.

Definition 4.2 Let $\wp$ be a play structure of Γ, $i \in N$ and k a positive integer. Let p_k denote the k-play of $\wp$. A play structure $\wp' = [(q'_1, p'_1), (q'_2, p'_2), \dots]$ of Γ is called

(1) a *feasible k-split* of $\wp$ if
  (a) for all $t < k$: $(q'_t, p'_t) = (q_t, p_t)$;
  (b) p'_k is a play of the feasible game form $\Sigma(q'_k)$;
(2) an *i-admissible k-split* of $\wp$ if
  (a) for all $t < k$: $(q'_t, p'_t) = (q_t, p_t)$;
  (b) p'_k is a play of the admissible game form that forms the i-component of $\Xi(q'_k)$.[4]

We are now ready to define the notion of truth within a model structure.

Definition 4.3 Let $M = < \Gamma, \wp, \Im >$ be a model structure of DLA. For any wff φ of DLA, any temporal symbols t_i, t_j, any individual symbols i, j, and all action statements μ of type i/t_j, we say that φ *is true in (holds in) M* if in case

(1)	$\varphi = (t_j, x)$:	the $\Im(t_j)$-point of $\wp$ is an element of $\Im(x)$;
(2)	$\varphi = \mathrm{Do}_i(t_j, \psi)$:	ψ is true in $< \Gamma, \wp', \Im >$ for every $[N\text{-}\Im(i),\Im(t_j)]$-variant $\wp'$ of $\wp$;
(3)	$\varphi = \mathrm{D\acute{o}}_i(t_j, \psi)$:	(a) $\mathrm{Do}_i(t_j, \psi)$ is true in M, and (b) there is at least one feasible $\Im(t_j)$-split $\wp'$ of $\wp$ such that $\mathrm{Do}_i(t_j, \psi)$ is not true in $< \Gamma, \wp', \Im >$;
(4)	$\varphi = \mathrm{Can}\mu$:	μ is true in $< \Gamma, \wp', \Im >$ for some feasible $\Im(t_j)$-split $\wp'$ of $\wp$;
(5)	$\varphi = \mathrm{May}\mu$:	μ is true in $< \Gamma, \wp', \Im >$ for some $\Im(i)$-admissible $\Im(t_j)$-split $\wp'$ of $\wp$;
(6)	$\varphi = \sim\psi$:	ψ is not true in M;
(7)	$\varphi = (\psi \,\&\, \omega)$:	both ψ and ω are true in M;
(8)	$\varphi = (t_i = t_j)$:	$\Im(t_i) = \Im(t_j)$;
(9)	$\varphi = (t_i < t_j)$:	$\Im(t_i) < \Im(t_j)$;
(10)	$\varphi = (i = j)$:	$\Im(i) = \Im(j)$.

Finally, we define the notions of satisfiability and validity.

[4]Note that a play structure $\wp$ can be a feasible k-split or i-admissible k-split of itself.

Definition 4.4 Let φ be a wff of DLA and $M = < \Gamma, \wp, \Im >$ a model structure of DLA. We say that φ is *satisfied by M* if (1) $\wp$ is a feasible play structure of Γ of DLA, and (2) φ is true in M. A formula is *satisfiable* when there is at least one model structure by which it is satisfied. It is *valid* if its negation is not satisfiable.

The basic idea of the semantics can be explained as follows. First of all, given the interpretation $\Im$, each point of the tree (each element of X) represents a possible world. The world is described by the basic propositions which have been assigned to it by the interpretation. A play structure, which describes a path in the tree, is a possible history of the world. It describes which possible worlds are realized at which points in time and hence which basic propositions are true at the various points in time. To find out what an individual is able to do in a certain world we look at the feasible game form. It describes the feasible actions of the individuals and is assumed to be identical for all. A person can see to a certain state of affairs at a certain point in time if he or she has, in the feasible game form assigned to the world through which the play structure passes as that time, a strategy which always leads to an outcome in which that state of affairs is realized. The person cannot avoid seeing to it if he or she has only such strategies in the feasible game form. Because the feasible game form belonging to a certain point is identical for all individuals, the actions of one individual have consequences for the set of feasible actions of other individuals. For instance, if I can make absolutely sure that a state of affairs x arises, then it is impossible that you can see to it that x will not arise: hence the formula '$(i \neq j)$ & $\text{CanDo}_i(t_k, \varphi) \rightarrow {\sim}\text{CanDo}_j(t_k, {\sim}\varphi)$' is valid.

The admissible game form assigned to the individual describes his admissible actions, i.e. the things an individual may or shall do. An individual *may* do something at a certain point of time if he or she has, in the admissible game form assigned to her in the world through which the play structure passes at that time, a strategy which always leads to an outcome in which that state of affairs is realized. Since each individual has been assigned his or her own admissible game form, which may be different from the feasible game form, we do not preclude the possibility that an individual has —in his or her own admissible game form— a strategy leading to a particular state of affairs, which the individual does not have in the feasible game form. The reason is obvious: in real life one often has the permission to realize a state of affairs even though one cannot actually bring it about ('I may see to it that this candidate wins the next elections').

Similarly, since the admissible game forms of individuals need not be the same, we cannot make inferences about what a person is allowed to do from a description of the permissions of the other individuals. To find out whether I may see to it that x is the case, I look at a decision situation —the admissible actions described by *my* admissible game form— which need not be identical to the decision situation that describes the actions you are allowed to take. Accordingly, the fact that I may see to it that, for instance, x arises does not entail anything about the things other human beings are allowed to do. Hence they may well have the permission to see to it that x will not arise, i.e. the formula '$(i \neq j)$ & $\text{MayDo}_i(t_k, \varphi) \rightarrow {\sim}\text{MayDo}_j(t_k, {\sim}\varphi)$' is not valid. This makes perfect sense: not every permission is accompanied by an obligation on the others not to interfere. One might, however, want to study model structures in which permissions always do entail an obligation on the others not to interfere. Such a principle holds, for instance, in all model structures in which the admissible game forms assigned to the individuals are identical at each point.

It is not very difficult to use the game-theoretic logic to derive statements that are always true. A list of such valid formulas of DLA is presented in the appendix.

5. *Concluding remarks*

As stated in the introduction, our aim was to describe a model that integrates Kripkean semantics with the game-theoretic analysis of rights. Essentially, our model structure can be seen as a structure of possible worlds. Each point represents a possible world situated at a point in time. Kripkean models of possible worlds are well known in deontic logic. In such models a binary relation representing the notion of deontic perfectness is defined over a set of possible worlds. A formula of the form Mayφ is said to be true in a particular world if there is another world that is deontically perfect and in which φ is true. Similarly, in Kripkean models of action logic a relation of accessibility is defined over a set of possible worlds. A formula of the form Canφ is then said to be true if there is another world that is accessible and in which φ is true. In our model the relations of deontic perfectness and accessibility are represented by the plays of the admissible game forms and the feasible game forms respectively. A point (or possible world) r forms a deontic perfect alternative to another point q if it is the outcome of one of the admissible game forms belonging to q. It is accessible from q if it can be reached by a play of the feasible game form belonging to q.

The use of two types of game forms is based on the importance of distinguishing feasible from admissible human actions. Although we have defined the types of game forms independently of each other, it is perfectly possible to study model structures in which logical connections are defined between the various game forms (see Van Hees 1995). For instance, as was noted in the previous section, one might want to study model structures in which the various admissible game forms belonging to a point are always identical. Another possibility is the study of model structures in which the admissible game forms are related to the feasible game forms in such a way that a principle like 'shall implies can' always applies.

Furthermore, we remark that the model permits the definition of different types of right in a game-theoretic context. Several logicians have defined classifications of individual rights in terms of the permissions and obligations of individuals (Kanger and Kanger 1966; Lindahl 1977). Translating these definitions in terms of a model structure then yields a game-theoretic 'categorization' of rights (Gärdenfors 1981; Van Hees 1995).

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Appendix

Each of the following formulas is valid:

(1) $\mathrm{Do}_i(t_j, \varphi) \rightarrow \varphi$
(2) $\mathrm{Do}_i(t_j, \varphi) \rightarrow {\sim}\mathrm{Do}_k(t_j, {\sim}\varphi)$
(3) $\mathrm{Do}_i(t_j, \varphi)\ \&\ \mathrm{Do}_i(t_j, \psi) \leftrightarrow \mathrm{Do}_i(t_j, \varphi\ \&\ \psi)$
(4) $\mathrm{Do}_i(t_j, \varphi) \rightarrow \mathrm{Do}_i(t_j, \varphi \vee \psi)$
(5) $\mathrm{Do}_i(t_j, \varphi) \rightarrow \mathrm{Do}_i(t_j, \psi \rightarrow \varphi)$
(6) $\mathrm{Do}_i(t_j, \varphi) \rightarrow \mathrm{Do}_i(t_j, {\sim}\varphi \rightarrow \psi)$
(7) $\mathrm{Do}_i(t_j, \varphi \rightarrow \psi) \rightarrow (\mathrm{Do}_i(t_j, \varphi) \rightarrow \mathrm{Do}_i(t_j, \psi))$
(8) $\mathrm{Do}_i(t_i, \mathrm{Do}_j(t_j, \varphi)) \rightarrow \mathrm{Do}_i(t_i, \varphi)$
(9) ${\sim}\mathrm{Do}_i(t_k, \varphi) \rightarrow {\sim}\mathrm{Do}_j(t_k, \varphi)$
(10) $\mathrm{Do}_i(t_k, {\sim}\varphi) \rightarrow {\sim}\mathrm{Do}_i(t_k, \varphi)$
(11) $\mathrm{Do}_i(t_k, \varphi) \rightarrow \mathrm{CanDo}_i(t_k, \varphi)$
(12) $\acute{\mathrm{Do}}_i(t_k, \varphi) \leftrightarrow \mathrm{Do}_i(t_k, \varphi)\ \&\ \mathrm{Can}{\sim}\mathrm{Do}_i(t_k, \varphi)$
(13) $\acute{\mathrm{Do}}_i(t_j, \varphi) \rightarrow \varphi$
(14) $\acute{\mathrm{Do}}_i(t_j, \varphi) \rightarrow {\sim}\acute{\mathrm{Do}}_k(t_j, {\sim}\varphi)$
(15) $\acute{\mathrm{Do}}_i(t_j, \varphi)\ \&\ \acute{\mathrm{Do}}_i(t_j, \psi) \rightarrow \acute{\mathrm{Do}}_i(t_j, \varphi\ \&\ \psi)$
(16) $\acute{\mathrm{Do}}_i(t_j, \varphi \rightarrow \psi) \rightarrow (\acute{\mathrm{Do}}_i(t_j, \varphi) \rightarrow (\acute{\mathrm{Do}}_i(t_j, \psi))$
(17) $\acute{\mathrm{Do}}_i(t_k, \mathrm{Do}_j(t_j, \varphi)) \rightarrow \acute{\mathrm{Do}}_i(t_k, \varphi)$
(18) ${\sim}\acute{\mathrm{Do}}_i(t_k, \varphi) \rightarrow {\sim}\acute{\mathrm{Do}}_j(t_k, \varphi)$
(19) $\acute{\mathrm{Do}}_i(t_k, \varphi) \rightarrow \mathrm{Can}\acute{\mathrm{Do}}_i(t_k, \varphi)$
(20) $(i \neq j)\ \&\ \mathrm{CanDo}_i(t_k, \varphi) \rightarrow {\sim}\mathrm{CanDo}_j(t_k, {\sim}\varphi)$
(21) $\mathrm{CanDo}_i(t_k, \varphi\ \&\ \psi) \rightarrow \mathrm{CanDo}_i(t_k, \varphi)\ \&\ \mathrm{CanDo}_i(t_k, \psi)$
(22) $\mathrm{CanDo}_i(t_k, \varphi) \rightarrow \mathrm{CanDo}_i(t_k, \varphi \vee \psi)$
(23) $\mathrm{CanDo}_i(t_k, \varphi) \rightarrow \mathrm{CanDo}_i(t_k, \psi \rightarrow \varphi)$
(24) $\mathrm{CanDo}_i(t_k, \varphi) \rightarrow \mathrm{CanDo}_i(t_k, {\sim}\varphi \rightarrow \psi)$
(25) $\mathrm{CanDo}_i(t_k, \mathrm{Do}_j(t_j, \varphi)) \rightarrow \mathrm{CanDo}_i(t_k, \varphi)$
(26) $\mathrm{Can}{\sim}\mathrm{Do}_i(t_k, \varphi) \rightarrow \mathrm{Can}{\sim}\mathrm{Do}_j(t_k, \varphi)$
(27) $\mathrm{CanDo}_i(t_k, {\sim}\varphi) \rightarrow \mathrm{Can}{\sim}\mathrm{Do}_i(t_k, \varphi)$
(28) $\mathrm{UnavDo}_i(t_k, \varphi) \rightarrow \mathrm{UnavDo}_j(t_k, \varphi)$
(29) $\mathrm{UnavDo}_i(t_k, \varphi) \rightarrow \mathrm{CanDo}_i(t_k, \varphi)$
(30) $\mathrm{UnavDo}_i(t_k, \varphi) \rightarrow \mathrm{Do}_i(t_k, \varphi)$
(31) $\mathrm{UnavDo}_i(t_k, \varphi) \rightarrow \mathrm{UnavDo}_i(t_k, \varphi \vee \psi)$

(32) $\text{UnavDo}_i(t_k, \varphi \,\&\, \psi) \leftrightarrow \text{UnavDo}_i(t_k, \varphi) \,\&\, \text{UnavDo}_i(t_k, \psi)$
(33) $\text{UnavDo}_i(t_k, \varphi \rightarrow \psi) \rightarrow (\text{UnavDo}_i(t_k, \varphi) \rightarrow \text{UnavDo}_i(t_k, \psi))$
(34) $\text{UnavDo}_i(t_k, \varphi) \rightarrow \text{UnavDo}_i(t_k, \psi \rightarrow \varphi)$
(35) $\text{UnavDo}_i(t_k, \varphi) \rightarrow \text{UnavDo}_i(t_k, \sim\varphi \rightarrow \psi)$
(36) $\text{UnavDo}_i(t_k, \varphi \rightarrow \psi) \,\&\, \text{CanDo}_j(t_k, \varphi) \rightarrow \text{CanDo}_j(t_k, \psi)$
(37) $\sim\text{UnavDó}_i(t_k, \varphi)$
(38) $\text{CanDó}_i(t_k, \varphi) \leftrightarrow \text{CanDo}_i(t_k, \varphi) \,\&\, \text{Can}\sim\text{Do}_i(t_k, \varphi)$
(39) $\text{Can}\sim\text{Dó}_i(t_k, \varphi)$
(40) $\text{CanDó}_i(t_k, \varphi \,\&\, \psi) \rightarrow \text{CanDó}_i(t_k, \varphi) \vee \text{CanDó}_i(t_k, \psi)$
(41) $(i \neq j) \,\&\, \text{CanDó}_i(t_k, \varphi) \rightarrow \sim\text{CanDó}_j(t_k, \sim\varphi)$
(42) $\text{CanDó}_i(t_k, \text{Do}_j(t_j, \varphi)) \rightarrow \text{CanDó}_i(t_k, \varphi)$
(43) $\text{MayDo}_i(t_k, \sim\varphi) \rightarrow \text{May}\sim\text{Do}_i(t_k, \varphi)$
(44) $\text{MayDo}_i(t_k, \varphi \,\&\, \psi) \rightarrow \text{MayDo}_i(t_k, \varphi) \,\&\, \text{MayDo}_i(t_k, \psi)$
(45) $\text{MayDo}_i(t_k, \varphi) \rightarrow \text{MayDo}_i(t_k, \varphi \vee \psi)$
(46) $\text{MayDo}_i(t_k, \varphi) \rightarrow \text{MayDo}_i(t_k, \psi \rightarrow \varphi)$
(47) $\text{MayDo}_i(t_k, \varphi) \rightarrow \text{MayDo}_i(t_k, \sim\varphi \rightarrow \psi)$
(48) $\text{MayDo}_i(t_k, \text{Do}_j(t_j, \varphi)) \rightarrow \text{MayDo}_i(t_k, \varphi)$
(49) $\text{MayDó}_i(t_k, \varphi) \leftrightarrow \text{MayDo}_i(t_k, \varphi) \,\&\, \text{Can}\sim\text{Do}_i(t_k, \varphi)$
(50) $\text{MayDó}_i(t_k, \varphi \,\&\, \psi) \rightarrow \text{MayDó}_i(t_k, \varphi) \vee \text{MayDó}_i(t_k, \psi)$
(51) $\text{MayDó}_i(t_k, \text{Do}_j(t_j, \varphi)) \rightarrow \text{MayDó}_i(t_k, \varphi)$
(52) $\text{ShallDo}_i(t_k, \varphi) \rightarrow \text{MayDo}_i(t_k, \varphi)$
(53) $\text{ShallDo}_i(t_k, \varphi) \rightarrow \text{ShallDo}_i(t_k, \varphi \vee \psi)$
(54) $\text{ShallDo}_i(t_k, \varphi \,\&\, \psi) \leftrightarrow \text{ShallDo}_i(t_k, \varphi) \,\&\, \text{ShallDo}_i(t_k, \psi)$
(55) $\text{ShallDo}_i(t_k, \varphi \rightarrow \psi) \rightarrow (\text{ShallDo}_i(t_k, \varphi) \rightarrow \text{ShallDo}_i(t_k, \psi))$
(56) $\text{ShallDo}_i(t_k, \varphi) \rightarrow \text{ShallDo}_i(t_k, \psi \rightarrow \varphi)$
(57) $\text{ShallDo}_i(t_k, \varphi) \rightarrow \text{ShallDo}_i(t_k, \sim\varphi \rightarrow \psi)$
(58) $\text{ShallDo}_i(t_k, \varphi \rightarrow \psi) \,\&\, \text{MayDo}_j(t_k, \varphi) \rightarrow \text{MayDo}_j(t_k, \psi)$
(59) $\text{ShallDó}_i(t_k, \varphi) \leftrightarrow \text{ShallDo}_i(t_k, \varphi) \,\&\, \text{Can}\sim\text{Do}_i(t_k, \varphi)$
(60) $\text{ShallDó}_i(t_k, \varphi \,\&\, \psi) \rightarrow \text{ShallDó}_i(t_k, \varphi) \vee \text{ShallDó}_i(t_k, \psi)$
(61) $\text{ShallDó}_i(t_k, \text{Do}_j(t_j, \varphi)) \rightarrow \text{ShallDó}_i(t_k, \varphi)$

It is not difficult to show the validity of each of these formulas. We prove (27) and (56) to indicate the nature of the proofs.

(27) Assume that $\text{CanDo}_i(t_k, \sim\varphi)$ is true in a model structure $M = <\Gamma, \wp, \Im>$. We have to show that $\text{Can}\sim\text{Do}_i(t_k, \varphi)$ is also true in M. The truth of $\text{CanDo}_i(t_k, \sim\varphi)$ and clauses (2), (4) and (6) of definition 4.3 imply that φ is not true in $<\Gamma, \wp', \Im>$ for every $[N\text{-}\Im(i),\Im(t_k)]$-variant $\wp'$ of at least one feasible $\Im(t_k)$-split of $\wp$. This implies that φ is not true in $<\Gamma, \wp', \Im>$ for at least one $[N\text{-}\Im(i),\Im(t_k)]$-variant $\wp'$ of at least one feasible $\Im(t_k)$-split of $\wp$. Hence, $\text{Can}\sim\text{Do}_i(t_k, \varphi)$ is true in M. ∎

(56) If $\mathrm{ShallDo}_i(t_k, \varphi)$ is true in an arbitrary $M = < \Gamma, \wp, \Im >$, then φ is true in $< \Gamma, \wp', \Im >$ for every $[N\text{-}\Im(i),\Im(t_k)]$-variant $\wp'$ of at least one $\Im(i)$-admissible $\Im(t_k)$-split of $\wp$. But then it is also the case that $(\psi \rightarrow \varphi)$ is true in $< \Gamma, \wp', \Im >$ for every $[N\text{-}\Im(i),\Im(t_k)]$-variant $\wp'$ of at least one $\Im(i)$-admissible $\Im(t_k)$-split of $\wp$. Hence, $\mathrm{ShallDo}_i(t_k, \psi \rightarrow \varphi)$ is true. ■