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STRICT FINITISM AS A VIABLE ALTERNATIVE IN THE FOUNDATIONS OF MATHEMATICS*

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1. *Introduction*

Whether or not one believes that Gödel's theorems shattered Hilbert's dream of proving mathematics absolutely consistent, the term "finitism" in his philosophy applies to the metalevel only. Many philosophers - see Ernst Welti [1987] for an overview - have been intrigued by the possibility to extend Hilbert's finitism to the object level. It seemed and it still seems a neat way to escape Gödel altogether. Although Gödel himself had something quite different in mind when he wrote about "an extension of finitary mathematics that has not yet been used", the strict finitist point of view too can be seen as an extension of finitary mathematics.

Perhaps one will remark that to consider strict finitism as an extension is stretching the meaning of extension a bit, if not a bit too much. For it is a very well-known remark in the literature of philosophy of mathematics that strict finitism is most suitably characterized by the label "strict". The resulting mathematics is so poor and weak, that terms such as trivial, uninteresting and the like come to mind. It is therefore both a philosophically and mathematically interesting question to ask whether this is necessarily so. The main thesis of this paper is

* This paper is essentially an improved version of a paper that was presented at and published after the *First International Symposium on Gödel's Theorems* in Paris under a slightly different title: "On an extension of finitary mathematics which has most certainly not yet been used" (see my [1993a]). The meaning of this title is explained in the paper. More or less the same material has been presented at a lecture at the Technical University of Athens, Greece on the invitation of Aristides Baltas. My thanks to the audience there, Kostas Gavroglu in particular, for comments and criticisms. The same material but in a very early stage of its development, has been presented at the *Ecole Normale Supérieure* of Paris in the *Séminaire de Philosophie et Mathématiques* organised by Maurice Loi. My thanks to Maurice Loi and to Yehuda Rav for the invitation. Thanks also to Graham Priest for providing me with the material I needed at precisely the right time and the opportunity to discuss the material at several places in Australia. Finally, thanks to the "Leuven"-group on philosophy of mathematics, Leon Horsten in particular. A "popular" version of this material was published in my [1992] and a discussion of the relevance of this approach to Hilbert's program is to be found in my [1993b]. Applications to physics have been treated in my [1993c] and [1994].

that the answer is (most emphatically) no. Actually, a stronger claim will be made. It is possible to formulate a strict finitist version of a classical infinite mathematical theory, such that the former is a proper extension of the latter (without meaning stretching).

Probably one is tempted to think that this is an impossibility. It would be, if the additional claim were that strict finitist mathematics must have the same logical basis as classical mathematics. Here, paraconsistency enters into the discussion. Separating the notion of triviality from the notion of consistency, classical logic can be replaced by a more generous logic that provides the additional power to turn strict finitism into an extension of classical mathematics. Anyone familiar with Gödel's (neo-)Platonist philosophical ideas will realize that such an approach was quite simply impossible for him, nothing less than a philosophical heresy. Hence I might as well talk about "an extension of finitary mathematics that Gödel would most definitely not have used".

Instead of presenting the general theory right from the start, I will go through a detailed discussion of elementary number theory on the semantical level. In paragraph 2 the standard classical model is presented. In paragraph 3 the paraconsistent view is introduced and in 4 the rich finitist model is formulated with the following properties:

(i) If $\models cl\ A$ then $\models_{r^*} A$ If A is a valid statement in classical elementary number theory, then A is valid in rich finitist elementary number theory. As said before, the claim is indeed that rich finitist number theory is an extension of classical number theory.

(ii) There exists at least one non-empty subset *Fin* of the set of all formulas F of the language of elementary number theory, such that for A in Fin:

$$\models cl\ A \text{ iff } \models_{r^*} A.$$

Furthermore this subset is easy to characterize.

In a single statement this means that this part of classical mathematics can be rewritten in a strict finitist fashion without the classical mathematician's (quite understandable) worry that all kinds of nice results will disappear in the transition. What the platonist does during the day, the finitist can rewrite in the evening. In paragraph 5, I present the general method due to Graham Priest (although designed for different purposes). Finally, in 6 a philosophical discussion concludes the paper.

It might help the reader to better understand the approach outlined here, if some history is told. In previous work, I tried to formulate a type

of "strict" finitism[1] incompatible with classical mathematics. Classical theorems turned out false ("There is no largest prime number"), finitist theorems were classically speaking false ("There is number equal to its successor"). Not altogether satisfied with this result, I tried (though not very successfully) to find stronger models. Familiar with the work on relevant and paraconsistent logic, I did not realize the connection there could be between these logics and strict finitism. Reading the papers of Chris Mortensen, I noticed that paraconsistent and relevant mathematical theories typically have finite models. It seemed natural to turn the question around: is it possible to finitize theories by going paraconsistent? Unfortunately, the models Mortensen presented in his papers, were not what I needed as a strict finitist and it was not clear to me at the time how to generate other models. Thanks to a result of Graham Priest (presented in paragraph 5), such a method is available. It is actually extremely powerful. Thus, this paper can also be seen as a continuation of the work of Mortensen and Priest, not from the relevant-paraconsistent point of view but from the strict finitist perspective. As must be obvious, this does not imply that they share the latter perspective.

2. *The classical case*

Let PA be the theory of (Peano) arithmetic. The language of PA consists of the language of first-order predicate logic in its standard form with predicates restricted to "=" (equality) and functions restricted to "S" (successor), "+" (addition) and "." (multiplication). "O" is the only constant of the language.

A model M of PA is a triple $M = \langle N, I, v_I \rangle$ where N is the (standard) domain (of the natural numbers)[2], I is an interpretation function and v_I is a valuation function based on I, satisfying the following conditions:

($I1$) $I(O) = 0$ (where 0 is the number zero in the domain)

($I2$) $I(Sx) = I(x) \oplus 1$ (where $\oplus$ means addition in the model)

($I3$) $I(x + y) = I(x) \oplus I(y)$

($I4$) $I(x.y) = I(x) \otimes I(y)$ (where $\otimes$ means multiplication in the

[1] See my [1987].

[2] Non-standard models are not considered, although they do not complicate matters in any serious way. The only reason for the restriction to standard models is for clarity's sake.

model)

(*I*5) $I(=) = \{<n, n> \mid n \in N\}$

(*V*1) $v_I(x=y)=1$ iff $<I(x), I(y)> \in I(=)$
(or, equivalently, $I(x)=I(y)$)

(*V*2) $v_I(\sim A)=1$ iff $v_I(A)=0$

(*V*3) $v_I(A \vee B)=1$ iff $v_I(A)=1$ or $v_I(B)=1$

(*V*4) $v_I(\exists x A(x))=1$ iff there is an I' that differs from I at most in the value of $I(x)$ such that $v_{I'}(A(x))=1$.

A formula A is valid ($\models cl\ A$) iff for all models M, $v_I(A)=1$.

3. *Going paraconsistent*

In order to prepare the ground for the strict finitist case, one intermediate step is necessary. In the scheme of paragraph 2, v_I is a function from the formulas of PA to $\{0,1\}$. Suppose now that v_I is a function from the formulas of PA to the following set: $\{\{0\}, \{0, 1\}, \{1\}\}$[3]. Suppose further that the conditions are replaced by:

(*I*1)-(*I*4) as before

(*I*5') $I+(=) = \{<n, n> \mid n \in N\}$
$I-(=) = (N \times N) \setminus I+(=)$

Instead of talking of the extension of a predicate, we now talk of its positive and its negative extension. Although not relevant at this stage, it is a crucial step in the next paragraph.

(*V*1') $1 \in v_I(x=y)$ iff $<I(x), I(y)> \in I+(=)$
$0 \in v_I(x=y)$ iff $<I(x), I(y)> \in I-(=)$

(*V*2') $1 \in v_I(\sim A)$ iff $0 \in v_I(A)$
$0 \in v_I(\sim A)$ iff $1 \in v_I(A)$

(*V*3') $1 \in v_I(A \vee B)$ iff $1 \in v_I(A)$ or $1 \in v_I(B)$
$0 \in v_I(A \vee B)$ iff $0 \in v_I(A)$ and $0 \in v_I(B)$

[3] This type of valuation function has been proposed by Priest in a number of papers and in his [1987] book (see references there).

(*V*4') $1 \in v_I(\exists x A(x))$ iff there is an I' that differs from I at most in the value of $I(x)$ such that $1 \in v_{I'}(A(x))$.
$0 \in v_I(\exists x A(x))$ iff for all I' that differ from I at most in the value of $I(x)$, $0 \in v_{I'}(A(x))$.

A formula A is valid ($\models pc\ A$) iff $1 \in v_I(A)$ for all models M.

It is not that hard to see that the following theorem must hold:

Theorem 1. $\models pc\ B$ iff $\models cl\ B$.

Proof. It is sufficient to show that the truth value of a formula of the form $x = y$ can only be $\{0\}$ or $\{1\}$. If so, it is obvious that (*V*2'), (*V*3') and (*V*4') reduce immediately to (*V*2), (*V*3) and (*V*4). Suppose then that $v_I(x = y) = \{0, 1\}$. This is the case if $<I(x), I(y)> \in I+(=)$ and $<I(x),I(y)> \in I-(=)$, which is impossible. QED

Thus, although this formulation looks more complicated than the preceeding one, it is entirely equivalent to it. In the next step the value $\{0,1\}$ will show its importance.

4. *Rich finitism*

Starting with a classical model $M = <N, I, v>$ as defined in 3, a new model $M^* = < N^*, I^*, v^*_{I^*} >$, called *the model derived from M*, will be constructed that has the following properties:

(i) N^* is finite
(ii) if $\models cl\ B$ then $\models_{r^*} B$.

M^* is precisely the model that satisfies the first of the two requirements of a rich finitist model mentioned in the introduction of this paper.

N^* is the following set: $\{[0], [1], [2], ..., [L, L \oplus 1, ...]\}$. Unless otherwise indicated, L is considered to be a fixed number. The square bracket notation is meant to clarify how the elements of N^* are related to N. The easiest way is to read $[n]$ as an equivalence class under a (non-stipulated) equivalence relation, or as a partition of N in a finite set of parts.

The interpretation function I^*, derived from I, is defined as follows:

($I1^*$) $I^*(O)=[0]$

($I2^*$) $I^*(Sx)=[I(Sx)]$

($I3^*$) $I^*(x+y)=[I(x)\oplus I(y)]$

($I4^*$) $I^*(x.y)=[I(x)\otimes I(y)]$

($I5^*$) $<I^*(x),\ I^*(y)>\in I^*+(=)$ iff there is a $n\in=[I(x)]$, and there is a $m\in[I(y)]$, such that $<n,\ m>\in I+(=)$
$<I^*(x),\ I^*(y)>\in I^*-(=)$ iff there is a $n\in[I(x)]$, and there is a $m\in[I(y)]$, such that $<n,\ m>\ \in I-(=)$.

The valuation function $v^*_{I^*}$ satisfies the following conditions:

($V1^*$) $1\in v^*_{I^*}(x=y)$ iff $<I^*(x),\ I^*(y)>\in I^*+(=)$
$0\in v^*_{I^*}(x=y)$ iff $<I^*(x),\ I^*(y)>\in I^*-(=)$

($V2^*$) $1\in v^*_{I^*}(\sim A)$ iff $0\in v^*_{I^*}(A)$
$0\in v^*_{I^*}(\sim A)$ iff $1\in v^*_{I^*}(A)$

($V3^*$) $1\in v^*_{I^*}(A\vee B)$ iff $1\in v^*_{I^*}(A)$ or $1\in v^*_{I^*}(B)$
$0\in v^*_{I^*}(A\vee B)$ iff $0\in v^*_{I^*}(A)$ and $1\in v^*_{I^*}(B)$

($V4^*$) $1\in v^*_{I^*}(\exists xA(x))$ iff there is an $I^{*\prime}$ that differs from I^* at most in the value of $I^*(x)$ such that $1\in v^*_{I^{*\prime}}(A(x))$.
$0\in v^*_{I^*}(\exists xA(x))$ iff for all $I^{*\prime}$ that differ from I^* at most in the value of $I^*(x)$, $0\in v^*_{I^{*\prime}}(A(x))$.

A formula A is r^*-valid ($\models r^* A$) iff $1\in v^*_{I^*}(A)$ for all models M^*.

Lemma. For any formula A, for any classical model M and its derived model M^*, we have:
(a) If $1\in v_I(A)$, then $1\in v^*_{I^*}(A)$,,
(b) If $0\in v_I(A)$, then $0\in v^*_{I^*}(A)$.

Proof (by induction on the length of the formulas):

(i) *Basis*: note first that if t is a term, then it is easy to show $I^*(t)=[I(t)]$.
(a) Suppose now that A is $x = y$ and that $1\in v_I(x=y)$. By ($V1'$), $1\in v_I(x=y)$ iff $<I(x),\ I(y)>\ \in I+(=)$. Thus, by ($I5^*$), there is a $n(=I(x))\in[I(x)]$ and a $m(=I(y))\in[I(y)]$, such that $<n,m>\ \in I+(=)$. Hence, $<I^*(x),\ I^*(y)>\ \in I^*+(=)$, i.e. $1\in$

$v^*_{I^*}(x = y)$

(b) proceeds along precisely the same lines.

(ii) *Induction step*: three cases have to be distinguished: negation, disjunction and the existential quantifier. The three proofs are identical, I therefore restrict myself to negation. Let A be of the form $\sim C$, then:

(a) $1 \in v_I(\sim C)$ iff $0 \in v_I(C)$
then $0 \in v^*_{I^*}(C)$ by induction
iff $1 \in v^*_{I^*}(\sim C)$,

hence if $1 \in v_I(\sim C)$ then $1 \in v^*_{I^*}(\sim C)$.

(b) proceeds along precisely the same lines. QED

The central theorem is, of course, the fact that:

Theorem 2. if $\models cl\, B$ then $\models r^*\, B$.

Proof. Because of theorem 1, it is sufficient to show that if $\models pc\, B$ then $\models r^*\, B$. Now, if $\models pc\, B$ then for all v_I, $1 \in v_I(B)$ and, by the lemma, for all derived models M^*, for all $v^*_{I^*}$, $1 \in v^*_{I^*}(B)$, hence $\models r^*\, B$. QED

As said in the introduction, there is a rather easy way to strengthen theorem 2 to an equivalence. Let *Fin* be the subset of all formulas of the language of *PA* such that:

$A \in Fin$ iff there is no term t occurring in A such that
$I^*(t) = [L,\ L \oplus 1,\ \ldots]$

Theorem 3. If $B \in Fin$ then $\models cl\, B$ iff $\models rf\, B$.

Proof. It is easy to see that for i, with the exception of $[L,\ L \oplus 1,\ \ldots]$, $[i] = i$. It then follows that I^* reduces to I and hence $v^*_{I^*}$ to v_I. Thus, if $\models r^*\, B$ then $\models cl\, B$. Together with theorem 2, this shows the equivalence. QED

Remark 1. *Fin* is not necessarily the only set that satisfies the equivalence. For it is very well possible that a statement A does turn out uniquely true, even though there is a term t occurring in it, such that $I^*(t) = [L,\ L \oplus 1,\ \ldots]$. Example: if $S^{(L)}O$ stands for $SS \ldots SO$, with L occurrences of S, then the statement $O < S^{(L)}O$ (with the standard interpretation for $<$), is uniquely true. $I^*(O) = [0] = 0$ and $I^*(S^{(L)}O) = [L,\ L \oplus 1,\ \ldots]$. Therefore, given the classical interpretation function I, no matter what element k we pick from $[L,\ L \oplus 1,\ \ldots]$, we will have that 0

$< k$. Hence $v^*_{I^*}(O < S^{(L)}O) = \{1\}$.

Remark 2. It seems quite natural to ask the question whether the above result extends to the notion of semantical consequence. In other words, is it possible to extend theorem 2 to:

If $A_1, A_2, \ldots, A_n \models cl\, B$ then $A_1, A_2, \ldots, A_n \models r^* B$?

The answer is no, as the following simple counterexample shows:

$$S^{(L+1)}O = S^{(L)}O \models r^* S^{(L)}O = S^{(L-1)}O.$$

In the classical model this holds, because both premise and conclusion are false. Yet, in a finite model with domain $N^* = \{[0], [1], \ldots, [L \ominus 1], [L, L \oplus 1, \ldots]\}$, we have $v^*_{I^*}(S^{(L)}O = S^{(L-1)}O) = \{0\}$, as $I^*(S^{(L)}O) = [L, L \oplus 1, \ldots]$ and $I^*(S^{(L-1)}O) = [L \ominus 1] = L \ominus 1$, but $v^*_{I^*}(S^{(L+1)}O = S^{(L)}O) = \{0, 1\}$, as $I^*(S^{(L+1)}O) = I^*(S^{(L)}O) = [L, L \oplus 1, \ldots]$. If we take $L \in I^*(S^{(L+1)}O)$ both for n and m, then $<n, m> = <L, L> \in I+(=)$, hence $1 \in v^*_{I^*}(S^{(L+1)}O = S^{(L)}O)$. But if we take $L \in I^*(S^{(L+1)}O)$ for n and $L \oplus 1 \in I^*(S^{(L)}O)$ for m then $<n, m> = <L, L \oplus 1> \in I-(=)$, hence $0 \in v^*_{I^*}(S^{(L+1)}O = S^{(L)}O)$.

Note, however, that the following does hold:

$$\models r^* (S^{(L+1)}O = S^{(L)}O) \supset (S^{(L)}O = S^{(L-1)}O)$$

where $A \supset B$ stands for $\sim A \vee B$.. Furthermore, if all A_i and B belong to Fin, then the extension of theorem 3 holds:

If $A_1, A_2, \ldots, A_n \models cl\ B$ then $A_1, A_2, \ldots, A_n \models r^* B$.

5. *Beyond elementary number theory?*

The natural question to ask is whether this method will work for other mathematical theories? There are two answers to this question. The first is quite simply to observe that it has been done. In fact, Mortensen in his papers [1988] and [1990] has done precisely that. I repeat that in this sense, from the technical point of view, not much new is offered in this paper. Why then did I not simply state Mortensen's results? Because, and this is the second reply, I have followed a different route

leading to the same results. As a matter of fact, what I have done in this paper is to apply a method of Priest that is far more general. With just one restriction, it is possible to finitize almost any theory. Stepswise, without presenting all the details, it goes like this:

(i) Take any first-order theory T satisfying the only restriction that the number of predicates of T is finite. Let M be a model of T.
(ii) Reformulate T in a paraconsistent fashion, extending the truth values to $\{\{0\}, \{0,1\}, \{1\}\}$ instead of $\{0,1\}$.
(iii) If the models of M are infinite, define an equivalence relation R over the domain D of M, such that D/R is finite. Or, equivalently, define a partition in a finite set of parts of the domain D of M. Let the resulting model be M/R or M^*.
(iv) The model M/R or M^* is a finite paraconsistent model of the given first-order theory T such that validity is extended. Thus M/R is a strict and rich finitist extension of M.

The most important thing to note is that any equivalence relation or partition will do. This leaves room for an almost unlimited number of possibilities. Indeed, Mortensen instead of using N^*, preferred to work with $N/mod\ n$. However, for the strict finitist, the non-trivial problem that remains, is to find an R such that the resulting model deserves to be called a "natural" model. That a model is "natural" is indicated, e.g., by the existence of theorems such as theorem 3. More generally, borrowing some terminology of the structuralist school in philosophy of science, if there is a set of intended applications of a mathematical theory, say in physics, then both the original infinite theory and its rich finitist extension should agree on this set. In more mundane language and for the case of PA, this means that if we count with small, accessible (not in the set-theoretic sense of the term, of course) or feasible numbers, then both theories should agree. If, e.g. the truth-value of $1 = 8$ were $\{0,1\}$, then I would not consider this a "natural" model. As shown in this paper, for PA such models exist.

The extension to the integers is a rather dull exercise. Apart from existing proposals, it is obvious that the most "natural" partition of Z, is: $Z^* = \{[..., -(L+1), -L], ..., [-2], [-1], [0], [1], [2], ..., [L, L \oplus 1, ...]\}$. However, entering the domain of the rationals, things become different. One possible finite partition of Q that leads to a "natural" model is this (here the square brackets refer to open or closed intervals):

(a) $Q_L = [x \mid x \in Q \& x \geq L],\ Q_{-L} = [x \mid x \in Q \& x \leq -L]$
(b) $Q_d = \{[p/q] \mid p/q \in Q \& -L < p,\ q < L \& q \neq 0\},$

(c) $$Q_{ind} = \{]x, y[\mid x,\ y \in Q \& (x, y \in Q_d \text{ or } x = -L \text{ or } y = L) \\ \& \sim (\exists z)(x < z < y \& z \in Q_d)\}.$$

A simple example may illustrate this rather complicated description. Take $L = 3$. Then Q is split up in:

(a) all fractions larger than or equal to 3 (Q_3) and all fractions smaller than or equal to -3 (Q_{-3}),

(b) the set of determined fractions:
$Q_d = \{-2,\ -1,\ -1/2,\ 0,\ 1/2,\ 1,\ 2\}$,

(c) all in-between intervals, i.e. Q_{ind}= {]-3,-2[,]-2,-1[,] -1,-1/2[,]-1/2,0[,]0,1/2[,]1/2,1[,]1,2[,]2,3[}.

There are two senses in which this partition can be considered natural. First, note that, if all terms are interpreted in Q_d, we do have that $v^*_{I^*}((\exists y)(x.y = 1)) = \{1\}$. For, if $I^*(x) = [p/q] \in Q_d$, then $I^*(y) = [q/p]$, but that is an element of Q_d as well, thus the term $(p/q).(q/p)$ is uniquely determined, namely 1. Thus if, say, $L > 10$, then $(1/7).7 = 1$ holds in the strict finitist theory. Note that this approach avoids all mention of rounding-off criteria, approximate results, and the like. The second sense is related to N and Z. If A and B are two partitions then B is an extension of A (A ext B) iff for every element $a \in A$, there is an element $b \in B$ such that $a \subseteq b$. Clearly, we have N^* ext Z^*. Somewhat less clearly (but immediately clear from the example): Z^* ext Q^*. However, it is certainly (and unfortunately) not the case that this Q^* is the only partition that satisfies Z^* ext Q^*. Additional criteria are required. A good reason to reject a partition proposal is if it turns out that too many statements are true-false. Thus, what might appear to be the most "natural" partition of Q, namely

$$Q^* = \{[x,\ y[\mid x = k.\delta \& y = (k+1).\delta \& -L \le k < L-1\} \cup \\ \{]-\infty,\ -L.\delta[\} \cup \{[L.\delta,\ +\infty[\}, \text{ where } \delta \text{ is a number close to 0,}$$

actually is not. Using this model, for every x, $v^*_{I^*}(x.x^{-1} = 1) = \{0,\ 1\}$. Likewise for every choice of x, $v^*_{I^*}(x + O = x) = \{0,\ 1\}$ including the integer values of x. In Z^* (and in N^*) these statements were exclusively true. Hence, the first model presented is a better candidate than the second one.

Although the full details will be presented in papers to follow, including strict and rich finitist geometry - see my [1987a] for an intuitive ap-

proach - let me just mention that introducing (some) irrationals is quite easily achieved.

Suppose one would like to have $\sqrt{2}$. Replace the interval $]x, y[\in Q_{ind}$ that has $\sqrt{2}$ as a member by $]x, \sqrt{2}[$ and $]\sqrt{2}, y[$ and add $[\sqrt{2}]$ as a member to Q_d. It is easy to show that $v^*_{I^*}\left((\exists x)\left(x = \sqrt{2}\right)\right) = \{1\}$. To capture in a strict finitist theory the notion of an irrational is not excluded, strange though it may appear at first sight (from the finitist viewpoint, that is). The next step is the development of analysis. Mortensen in his [1990] has already developed a paraconsistent differential calculus thus showing, once again, that for the strict finitist, the problem reduces to find "natural" models.

6. *At what price rich finitism?*

6.1. From the classical point of view, the most obvious and "heaviest" price to pay is to give up consistency. Perhaps one is impressed by the idea that non-trivial strict finitist mathematics exists, but one is not willing to give up consistency.

A general argument to the contrary, is this: the whole idea of consistency proofs started with Hilbert's problem how to control the introduction of ideal elements in a mathematical theory. Typically, these ideal elements had to do with infinity. As long as everything was finite, there was no problem. Hence, as all models are all finite to start with, consistency is not of prime importance any more. Rather triviality is the key issue. We do not want trivial models, say models with all statements true-false. Thus, it seems obvious that consistency and triviality should be considered separate concepts. Thinking within a paraconsistent framework does precisely that. By dropping the *ex falso*, a theory does not become trivial then inconsistent. But classical logic is to a large extent identified by the *ex falso*. It therefore seems unavoidable that strict finitism should go hand in hand with paraconsistent logic.

A more specific argument has to do with the paradoxical nature of the very idea of a largest number (or numeral) L. The paradox is quite familiar, of course: if L is the largest number, what prevents me from writing down the next one, namely $L \oplus 1$? Apparently nothing, as I have just done precisely that. The strict finitist's answer is: there is no real paradox as $L \oplus 1 = L$. In other words, although the question can be asked: "What is the *result* if 1 is added to L?", the answer is quite simply: "The number L". Note that in the question itself, only numerals smaller or equal to L, are mentioned. Pleasing though this solution may seem, it probably will not satisfy the anti-finitist. He or she will argue

that, perhaps the set of numbers or numerals is finite, but, apparently, the number of operations, say additions, I must be able to talk about, must be larger. Therefore, the cardinality of the set of all additions is larger than L. Hence, L is not the limit, and the paradox is restored.

In my [1987], I opted for a conventionalist solution: as it does not make sense to talk about the largest number, I accepted instead a provisional largest number in full knowledge of the fact that larger finite numerals are imaginable. In the approach outlined here, another solution is possible: the largest number is an inherently paradoxical idea[4]. It is easy to show that if $I^*(x) = I^*(y) = [L,\ L \oplus 1, \ldots]$, then both $1 \in v^*(x = y)$ and $0 \in I^*(x = y)$, hence the truth-value of $x = y$ is $\{0, 1\}$. Thus, if $[L, L \oplus 1, \ldots]$ is identified as the largest element of N^* (a quite natural suggestion), then the rich finitist model tells us that it is both true and false that the largest natural number is equal to itself.

Another objection might be raised. If one claims that $L = L \oplus 1$ is true-false, why then does it not follow that $O = S\ O$ (or $0 = 1$) is true-false? What is wrong with the following argument? Finitistically speaking, as the strict finitist theory extends the classical theory, the well-known PA-axiom must hold:

$$(\forall x)(\forall y)((Sx = Sy) \supset (x = y)) \qquad (*)$$

Instantiate x by $S^{(L)}O$ and y by $S^{(L-1)}O$, and (*) becomes:

$$(SS^{(L)}O = SS^{(L-1)}O) \supset (S^{(L)}O = S^{(L-1)}O) \qquad (**)$$

(or, if a slight abuse of language is allowed for,
$(L+1 = L) \supset (L = L-1)$.)

Repeat the argument, and $O = SO$) is derivable. The answer is: because, as we already indicated, the strict finitist theory does not extend the notion of semantical consequence. It is sufficient to explicate the phrase "repeat the argument" in the reasoning above. To arrive at the conclusion $O = SO$, one needs modus ponens. Thus, the first step will be:

$$SS^{(L)}O = SS^{(L-1)}O,\ (SS^{(L)}O = SS^{(L-1)}O) \supset (S^{(L)}O = S^{(L-1)}O)$$
$$\models_{r^*} S^{(L)}O = S^{(L-1)}O$$

The first premise is true-false as has been shown. Likewise, the second premise is true-false. It is sufficient to note that $v^*_{I^*}(SS^{(L)}O =$

[4] In Priest's terminology, the largest finite number would qualify as a dialetheia (see Priest [1987], pp. 3-9).

$SS^{(L-1)}O) = \{0, 1\}$ and $v^*_{l^*}(S^{(L)}O = S^{(L-1)}O) = \{0\}$. However, the conclusion $S^{(L)}O = S^{(L-1)}O$ is exclusively false. Hence, there is a case such that the premises are true and the conclusion false. Thus the conclusion is not a semantical consequence of the premises, and the repetition breaks down.

6.2. One might object that the use of the truth-value $\{0, 1\}$ is a trick needed to prove the richness of the finitist theory. Suppose that instead of the set of truth-values $\{\{0\}, \{0, 1\}, \{1\}\}$ we had used the set $\{0, 1/2, 1\}$, thus obtaining a familiar three-valued logic. Statements with truth-value $\{0, 1\}$, now have truth-value 1/2. Instead of true-false, they are now undecided. Does not this reflect more closely the finitist's attitude? Actually, as has been shown by Priest, the three-valued logic is equivalent to the paraconsistent logic, if validity is extended to 1/2. A formula A is valid iff for all models M, $v(A) = 1$ or 1/2. Thus, the problem really comes down to this: are there any arguments for the finitist to accept a statement that is undecided, in some cases as valid? Formulated thus, the answer is yes. The undecided cases are precisely those cases that are not accessible to the strict finitist (semantically speaking), thus they should not play any part in the determination whether a sentence is valid or not. In a way, the choice of $\{0,1\}$ is much clearer than 1/2. "Undecided" does not preclude that at a later stage, it either becomes "true" or "false". The value $\{0, 1\}$, however, is a determined value.

6.3. Let me try yet another argument to defend the use of paraconsistent logic as the underlying logic of strict finitism. There is something paradoxical about the fact that, on the one hand, semantically, everything is strictly finite, while, on the other hand, syntactically, the strict finitist and the classical mathematician are speaking the same language. Thus, both are able to assert the statement $S^{(L)}O = S^{(L+1)}O$. They will, however, speak about different models, as for the infinitist, the statement is false, whereas for the strict finitist it is true-false.

These considerations - as Priest remarks in his [1991] paper - must remind one of the Löwenheim-Skolem theorems[5]. Actually, the correspondence is quite strong. As an example, think about the notion of a largest prime number. Although on the syntactical level, there will be a proof of the statement that there is no largest prime number, in the model there definitely is. But this semantical fact is not expressible in the theory itself. It is reflected however in the fact that the theorem

[5] David Isles has developed a similar idea (though not in a paraconsistent framework) in his [1994]. His work is closer to the original work of Yessenin-Volpin, one of the founding fathers of modern strict finitism. See Welti [1987] for details.

"There is no largest prime number" will turn out to be true-false in any particular model M. This can be seen as follows. Consider the two statements

$$(\forall x)(S^{(L)}O + x + SO = S^{(L)}O + x) \text{ and}$$
$$(\forall x)(S^{(L)}O + x + SO \neq S^{(L)}O + x).$$

Both are true-false. In the first case, all numerals above L collapse, and thus all prime numbers among them collapse as well, hence there is a largest prime number. The theorem is false. But in the second case, everything looks quite classical - all numbers larger than or equal to L are distinct -thus the theorem is true. Given Priest's general method outlined in paragraph 5 of this paper, the finitist analogue of Löwenheim-Skolem theorem is: Given a paraconsistent first-order theory T with a finite number of predicates, if the theory has models (countable or uncountable), it has finite models.

7. *Strict finitism without classical mathematics in the background.*

The strongest criticism imaginable is no doubt this: granted that a form of strict finitism is possible, it is still the case that to be able to formulate it, you need classical standard mathematics in the background. To be more precise, the domain N^* of the strict finite model M^* is a partition of a classical domain N of a classical model M. What if these are not available? Or, if you like, should strict finitism not have its own proper foundations? The answer is yes.

Suppose that a strict finitist starts with a limited domain Nf of natural numbers (or numerals), $Nf = \{0, 1, 2, ..., L\}$. Let us, for the moment, ignore the problem of there being a largest number or numeral L and how the strict finitist is supposed to find it[6]. Suppose further that the finitist wants to do mathematics over Nf. He or she defines a successor function, addition and multiplication over Nf, (possibly, though not necessarily) in the following way:

(D1) Succ: $Nf \rightarrow Nf$, such that: if $n < L$, then $\text{Succ}(n) = m$ (whatever m is) and if $n = L$, then $\text{Succ}(n) = L$.

(D2) $\oplus$: $Nf \times Nf \rightarrow Nf$, such that: $n \oplus 0 = n$

[6] My [1987] focuses mainly on this type of problem, especially the first chapters where an artificial mathematician, the sheet-mathematician, is introduced (or shemath for short).

$n \oplus \mathrm{Succ}(m) = \mathrm{Succ}(n \oplus m)$

(D3) $\otimes$: $Nf \times Nf \rightarrow Nf$, such that: $n \otimes 0 = 0$

$n \otimes \mathrm{Succ}(m) = (n \otimes m) \oplus n$

Note that in (D2) and (D3), it is unnecessary to add the limit L. This is taken care of by (D1). Example: suppose that $Nf = \{0, 1, 2\}$, i.e. $L = 2$, and that the result of $2 \oplus 2$ is asked for. Then $2 \oplus 2 = 2 \oplus \mathrm{Succ}(1) = \mathrm{Succ}(2 \oplus 1) = \mathrm{Succ}(2 \oplus \mathrm{Succ}(0)) = \mathrm{Succ}(\mathrm{Succ}(2 \oplus 0)) = \mathrm{Succ}(\mathrm{Succ}(2)) = \mathrm{Succ}(2) = 2$.

Given such a set Nf with the appropriate functions, the language of classical mathematics can be interpreted as follows:

$Mf = \langle Nf, If, vf_{If} \rangle$ is a strict finitist model, such that:

(*I1f*) $If(O) = 0$

(*I2f*) $If(Sx) = \mathrm{Succ}(If(x))$ if $If(x)$ is defined

$= L$, in all other cases

(*I3f*) $If(x+y) = If(x) \oplus f(y)$, if $If(x)$ and $If(y)$ are defined

$= L$, in all other cases

(*I4f*) $If(x.y) = If(x) \otimes \mathrm{If}(y)$, if $\mathrm{If}(x)$ and $If(y)$ are defined

$= L$, in all other cases

(*I5f*) $If+(=) = \{\langle If(x), If(y)\rangle \mid If(x) = If(y)\}$

$If-(=) = ((Nf \times Nf) \setminus If+(=)) \cup \{\langle L, L\rangle\}$

The valuation function vf_{If} (with range $\{\{0\},\{0,1\},\{1\}\}$) satisfies the following conditions:

(*V1f*) $1 \in vf_{If}(x = y)$ iff $\langle If(x), If(y)\rangle \in If+(=)$

$0 \in vf_{If}(x = y)$ iff $\langle If(x), If(y)\rangle \in If-(=)$

(*V2f*) $1 \in vf_{If}(\sim A)$ iff $\sim 0 \in vf_{If}(A)$

$0 \in vf_{If}(\sim A)$ iff $\sim 1 \in vf_{If}(A)$

(*V3f*) $1 \in vf_{If}(A \vee B)$ iff $1 \in vf_{If}(A)$ or $1 \in vf_{If}(B)$

$0 \in vf_{If}(A \vee B)$ iff $0 \in vf_{If}(A)$ and $0 \in vf_{If}(B)$

(*V4f*) $1 \in vf_{If}(\exists x A(x))$ iff there is an If' that differs from If at most in the value of $If(x)$ such that $1 \in vf_{If'}(A(x))$.

$0 \in vf_{If}(\exists x A(x))$ iff for all If' that differ from If at most in

the value of $If(x)$, $0 \in vf_{If'}(A(x))$.

A formula A is rf-valid $(\models rf\ A)$ iff $1 \in vf_{If}(A)$ for all models Mf.

Note that in the definition above, there is no need to refer to classical mathematics at all. Hence, it is perfectly possible to formulate strict finitist mathematics on its own (at least for elementary arithmetic). But, it is not hard to see that the following theorem can be proved:

Theorem 4. $\models rf\ A$ iff $\models r^*\ A$

Proof. Let Nf and N^* be the domains of resp. Mf and M^*. Define the function J: $Nf \rightarrow N^*$, such that $J(i) = [i]$, for $i < L$, and $J(L) = [L, L \oplus 1, ...]$. Note that J is a bijection. It is now a routine matter to show that (Iif) corresponds exactly to (Ii^*) and vice versa. The conditions on the valuation function are identical in both cases, so we need not bother about this. As an example, I will take $(I3^*)$:

$I^*(x+y) = [I(x) \oplus I(y)]$ (where I is the classical interpretation function wherefrom I^* is derived)

Now, either $If(x)$ and $If(y)$ are defined or not. In the first case, it follows that both $I(x) = If(x)$ and $I(y) = If(y)$ (or, if necessary, a permutation can be found). It then follows that:

$$I^*(x+y) = [I(x) \oplus I(y)] = J(I(x) \oplus I(y)) =$$
$$J(If(x) \oplus If(y)) = J(If(x+y)).$$

In the second case, one or both of $I(x)$ and $I(y)$ are not defined. It then follows that $I(x) > L$ and $I(y) > L$ and, certainly, $I(x+y) > L$. Thus:

$$I^*(x+y) = [L, L \oplus 1, ...] = J(L) = J(If(x+y)).$$

Summarizing, we find that for all x and y:

$$JIf(x+y) = I^*(x+y)$$

Finally, note the necessity in $(I5f)$ to include $<L, L>$ in the negative extension of $=$. This is necessary to obtain the translation. QED

Corollary: $\models cl\ A$ then $\models rf\ A$

One might still object that, on the syntactical level, the strict finitist is

still using the complete vocabulary of classical mathematics. This is indeed the case, but solely for the purpose of setting up a translation. If the strict finitist is, in any reasonable sense of the word, strict, then he or she will allow only a finite number of names for numbers. If we accept $L+1$ or $L.L$ as names, then, obviously, one cannot avoid a (potentially) infinite number of names. Thus, an additional restriction must be made, to the effect that, e.g., only terms of a limited complexity are allowed, where the limitations refer to the number of operations (additions and multiplications) mentioned in the term. It therefore follows that, syntactically, the language of the theory over *Nf* is less expressive than the language of *PA*, hence it cannot be an extension. What remains, however, is that it is still the case that, if *A* is a statement that is finitistically expressible, then if *A* holds classically, it holds finitistically. Thus, we remain as close as possible to classical mathematics.

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