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SOME STRUCTURAL AND LOGICAL ASPECTS OF THE NOTION OF SUPERVENIENCE

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Abstract

The sophisticated philosophical literature on supervenience stands in need of supplementation by a treatment of more fundamental questions about what features this notion possesses solely in virtue of the form of the definition it is standardly given. We provide a discussion of these features without getting involved in the merits of particular supervenience claims advanced and contested in that literature. The discussion touches on the relation between supervenience and the notion of a closure operation (§§2,3); the more 'logical' part of the discussion—in the sense that we consider formal languages and valuations (truth-value assignments)—takes us into the relationship between what we call inference-determined and supervenience- determined consequence relations (§§4,5). An 'edited' reading of the paper is available for those wishing to ignore the Appendix to §1, and the various passages explicitly marked 'Digression'.

1. *Introduction*

A common way of introducing the idea of supervenience is as follows. One class P of properties is *supervenient on* another class Q just in case it is not possible for objects to be alike in respect of all the properties in Q without also being alike in respect of all the properties in P. In calling objects x and y 'alike' in respect of a property what is meant is that either x and y both have the property or else x and y both lack the property. An alternative way of putting this is to say that x and y *agree on* (or *agree with respect to*) the property concerned. Another—and this time dangerously misleading—alternative terminology which you may sometimes see (and often hear) would say that x and y *share* the property in question. The reason that this is misleading is that we ordinarily understand talk of x and y's sharing a property to mean that x and y both possess ('have') the property, whereas x and y also count as agreeing, no less equally, in respect of the properties that they both *lack*.

Now the above way of introducing the terminology of supervenience, although not suggesting that the supervenience relation is symmetric, still exhibits a certain symmetry between the two classes of properties—P, the supervenient class, and Q, the 'subvenient' class, in that it takes both as inhabiting the same set-theoretic level: both are *classes* of properties.(1) For example, when we instantiate the general definition taking P and Q to comprise, respectively, the moral properties and the non-moral properties (supposing for the sake of exposition that we have some idea of what distinction between properties is being alluded to here), then the familiar supervenience claim of ethics is the claim that it is not possible for things alike in respect of their non-moral properties to differ in respect of their moral properties: no moral difference without a non-moral difference. This makes it sound as though the supervenient class (here, the class of moral properties) and the subvenient class (the class of non-moral properties) are on an equal footing as relata of the supervenience relation. And though there is nothing wrong with our general definition, there is something wrong with this impression it may convey. A supervenience claim of the above form involves a hidden asymmetry between the supervenient and the subvenient: for we can see properties individually as supervenient on classes of properties, with the supervenience of a class P on a class Q as the supervenience of each element of P on the class Q, whereas we cannot see the supervenience of P on Q as the supervenience of P on each element of the class Q. That is to say, though there is technically nothing amiss with our opening definition, we could have proceeded by initially defining the relation of supervenience to hold between a property P and a class of properties Q, thus:

P is *supervenient*$_1$ *on* Q just in case it is not possible for objects to be alike in respect of all the properties in Q without also being alike in respect of the property P.

The subscript '1' has been used to distinguish this relation, which holds

(1) R. M. Hare —to whose usage the current popularity of the term 'supervenient' is due— is said to regard the commonly heard 'subvenient', for a putative converse, as a barbarism, preferring 'subjacent' in this role (on etymological grounds). We stick with the common though disfavoured term here, however, in order to reduce terminological novelties.

between a property and a class of properties, from that introduced in our opening definition which relates, instead, two classes of properties. The connexion between supervenience and supervenience$_1$ is obvious enough:

$$\text{P is supervenient on Q iff for each } P \in \text{P}, P \text{ is supervenient}_1 \text{ on Q} \quad (1)$$

What we have here is a kind of reduction in set-theoretic level (or 'rank') for one of the two relata—namely the first—of the original relation of supervenience. Accordingly, we will say the relation is *reducible* in the first position. (Actually, in a closer examination of these matters in the Appendix to this section, we will say, to record the 'for each P' part of (1), that the relation is '$\forall$-reducible' in the position in question.)

Warning: Various notions going under the name of reducibility in the literature on supervenience pertain to the idea of reducing—in the sense of reductionist theses—some (concepts of) properties to others, have nothing to do with the present structural notion of reducibility. (See Hellman and Thompson [1975], Tennant [1985], Bacon [1986], Petrie [1987] for the notion(s) here distinguished from current concerns.)

Not only is supervenience definable in terms of supervenience$_1$, as in (1), but, conversely, starting with supervenience as defined in our opening paragraph, we can use (2) to define supervenience$_1$ in terms of supervenience:

$$P \text{ is supervenient}_1 \text{ on Q iff } \{P\} \text{ is supervenient on Q.} \quad (2)$$

There is, by contrast, no similar reducibility in respect of the second position. This point can be established by applying the general necessary and sufficient conditions for reducibility in a position, provided in the Appendix, but for the moment we can make it visible by contrasting the two positions of the universal quantifiers over properties in the definition of supervenience. With the aid of a sloppy but convenient notation, we can write the *definiens* as (3):

$$\forall x \forall y[\forall Q \in \text{Q}(Qx \leftrightarrow Qy) \rightarrow \forall P \in \text{P}(Px \leftrightarrow Py)] \quad (3)$$

(This formulation is sloppy because the variables 'P', 'Q' occupy term positions in the quantifier prefixes in which they are followed by '$\in$', but predicate position in 'Px', etc.) Then the point is that we can always move universal quantifiers on the consequent of a conditional so that they have broader scope, as in passing to (3*a*), from which we move to (3*b*) by 'commuting like quantifiers':

$$\forall x \forall y \forall P \in \mathrm{P}\ [\forall Q \in \mathrm{Q}(Qx \leftrightarrow Qy) \rightarrow (Px \leftrightarrow Py)] \tag{3a}$$
$$\forall P \in \mathrm{P} \forall x \forall y [\forall Q \in \mathrm{Q}(Qx \leftrightarrow Qy) \rightarrow (Px \leftrightarrow Py)] \tag{3b}$$

And (3*b*) makes clear the availability of the above definition of supervenience$_1$: wiping out the initial '$\forall P \in \mathrm{P}$' gives the condition for a property P to be supervenient$_1$ on Q, so the equivalence of (3*b*) with (3) means that P is supervenient on Q iff every property in P is supervenient$_1$ on Q. But we cannot similarly move the universal quantifier '$\forall Q \in \mathrm{Q}$' from the antecedent of (3) so that it has broad scope, since in general $\forall v \varphi(v) \rightarrow \psi$ (with the variable v free only as shown) is not equivalent to $\forall v(\varphi(v) \rightarrow \psi)$. (This corresponds to the difference between 'if every... then_' and 'if any... then_'.)

A well-known anomaly of classical logic is that, the latter non-equivalence notwithstanding $\forall v \varphi(v) \rightarrow \psi$, is indeed equivalent to $\exists v(\varphi(v) \rightarrow \psi)$, which may seem to offer hope of some kind of reducibility ('existential' perhaps, rather than universal: see Appendix) in the second position. There is a minor hitch here in that we have used $\exists$-restricted quantifiers and $(\forall v \in X.\varphi(v)) \rightarrow \psi$ is only equivalent to $\exists v \in X(\varphi(v) \rightarrow \psi)$ with the additional assumption that $\exists v.v \in X$. In our case, this amounts to the assumption that Q is non-empty. Making that assumption, we arrive at (3*c*)

$$\forall x \forall y \exists Q \in \mathrm{Q}[(Qx \leftrightarrow Qy) \rightarrow \forall P \in \mathrm{P}(Px \leftrightarrow Py)] \tag{3c}$$

and here a major hitch arises, since we cannot move the '$\exists Q \in \mathrm{Q}$' out to the front (unlike quantifiers not commuting) and get the desired existential reduction, of the form: P is supervenient on Q iff for some property Q in Q... (the dots being filled by something analogous to supervenience$_1$).

Before taking up, in the following section, the general question of when a relation (between classes of, *e.g.*, properties) admits of the kind of reducibility in one position but not another which we have noted the supervenience relation to possess in its first though not its second position, we pause to address a worry that may have been raised by our recent talk of classical logic and the properties it accords to conditionals. The worry is that perhaps we should not have been involving ourselves with these material conditionals at all: wouldn't strict conditionals (in a modal predicate logic) have been more appropriate to capture the original idea? After all, according to that characterization, P supervenes on Q when it is not *possible* for objects to be alike in respect of all the properties in Q without also being alike in respect of all the properties in P. The exact role of modal notions in characterizing supervenience is a somewhat delicate topic of considerable logical interest—*cf.* especially the works by McFetridge and Bacon cited below—but on which we shall say nothing here beyond this: we may satisfy ourselves that the modal element has not been omitted by noting a particular way our official rendering of that characterization, namely (3), can be interpreted. We repeat (3) here, for convenience

$$\forall x \forall y[\forall Q \in \mathrm{Q}(Qx \leftrightarrow Qy) \rightarrow \forall P \in \mathrm{P}(Px \leftrightarrow Py)] \tag{3}$$

The way to interpret (3) so that the required modal force is present even though explicit modal operators are absent is that urged by David Lewis (in many places, but see pp.14-117 of Lewis [1986], in which such formulations of supervenience theses are explicitly discussed): think of the individual variables ('x','y') as ranging over all possible objects, and make the assumption that no object exists in more than one world. (The latter assumption allows us to think of instantiating the property variables—'P', 'Q'—to such things as 'being round', rather than to 'being round in world w'. Properties are taken as simply classes of possible objects.) This corresponds to what is called *strong* (or 'inter-world') supervenience in the literature, and is the only notion of supervenience we consider here. For distinctions and relations between the various other notions and this one, see Kim [1984, 1987, 1988, 1990], Teller [1985], McFetridge [1985], Bacon [1986], Noonan [1987].

Appendix to §1: Reducibility in a Position

If R is an n-ary relation between sets we will use capital italics to range over such sets, lower case italics to range over their elements, drawn from some underlying set U (so that $\mathrm{R} \subseteq \mathscr{P}(U)^n$) and write '$<X_1,\ldots,X_n> \in \mathrm{R}$' rather than '$\mathrm{R}(X_1,\ldots,X_n)$'. Such a relation will be described as $\forall$-*reducible* in its k$^{\text{th}}$ position (where $1 \leq tk \leq n$) just in case there is some relation R'

$\mathscr{P}(U)^{k-1} \times U \times \mathscr{P}(U)^{n-k}$ such that for all $X_1,\ldots,X_n$ ($X_i \in \mathscr{P}(U)$) we have

$$<X_1,\ldots,X_n> \in \mathtt{R} \textit{ iff } \forall x \in X_k. <X_1,\ldots,X_{k-1},x,X_{k+1},\ldots,X_n> \in \mathtt{R}' \tag{4}$$

The definition of what we shall call *∃-reducibility in the k^{th}* position for R is similar except that the '∀' in (4) is to be replaced by '∈'. Our primary interest is in ∀-reducibility. What we saw in §1 was that taking R as the relation of supervenience (so $n = 2$), R was ∀-reducible in the first position *via* R′ as the relation we called supervenience$_1$. (*Warning*: for this example, the underlying set U contains properties, not the individuals which have or lack those properties.) To think about these matters in a general way, it will pay us to consider first the even simpler case in which $n = 1$. We further identify the ordered 1-tuple $<X>$ with its sole element, (the set) X. In this case, there is only one position concerning which the question of ∀-reducibility can arise, and the above definition tells us that a 1-ary relation (a collection of sets, that is) R is ∀-reducible (in that solitary position) just in case there is some subset R of U for which (5), below, holds, for all $X \subseteq U$. (We write 'R' rather than 'R′' to stay in line with the idea of using italic capitals for a set-theoretic level one lower than whatever level the courier characters—R, R′,...—range over. In the general case, given by (4), still has all but its k^{th} position at the level of subsets of U, so we use 'R′' for the 'reducing' relation there.)

$$X \in \mathtt{R} \textit{ iff } \forall x \in X. x \in R \tag{5}$$

Given R, we may enquire as to when there exists $R \subseteq U$ for which (5) holds. Two conditions come to mind immediately as necessary, namely that R should be closed under subsets and that R should be closed under unions; in other words, that (6) and (7) are satisfied:

$$X \in \mathtt{R},\ Y \subseteq X \textit{ together imply: } Y \in \mathtt{R} \tag{6}$$
$$X \in \mathtt{R},\ Y \in \mathtt{R} \textit{ together imply: } X \cup Y \in \mathtt{R} \tag{7}$$

As to (6), if $X \in \mathtt{R}$ this means, by (5), that every element of X belongs to R, so if in addition $Y \subseteq X$, every element of Y also belongs to R, and by (5) again, we should have $Y \in \mathtt{R}$. As to (7), the hypotheses mean that all elements of X, as well as all elements of Y, belong to R, in which case all elements of $X \cup Y$ belong to R, and we have the conclusion of (7). This latter justification of (7) works for arbitrary unions, so what we should really write for the 'closure under unions' condition is

$$\mathrm{R}_0 \subseteq \mathrm{R} \textit{ implies } \bigcup\{X|X \in \mathrm{R}_0\} \in \mathrm{R} \tag{7a}$$

(7) was the special case in which $\mathrm{R}_0 = \{X,Y\}$. (7*a*) is more general, in including closure under infinite unions, as well—moving in the opposite direction—as in subsuming the case of $\mathrm{R}_0 = \varnothing$; here the antecedent is automatically satisfied (*i.e.*, is satisfied for any R), and since the union of the empty set (thought of as a collection of sets) is the empty set, (7*a*) tells us that $\varnothing \in \mathrm{R}$ if R is $\forall$-reducible. Looking back to (5), we notice that its right-hand side is automatically satisfied when we take X as $\varnothing$.

We have seen that the conditions (6) and (7*a*) are necessary for the $\forall$-reducibility of R; we shall now show that they are sufficient. Suppose, then, that R satisfies (6) and (7*a*). We must find R for which (5) holds, for all X. Define $R = \{x \in U|\{x\} \in \mathrm{R}\}$. (Compare (2) in §1.) We show that (5) holds with R so defined. In other words, that we have

$$X \in \mathrm{R} \textit{ iff } \forall x \in X.\{x\} \in \mathrm{R} \tag{8}$$

But the 'only if' direction of (8) follows from the supposition that R is closed under subsets (*alias* (6)), since we are looking here at the singleton subsets, and the 'if' direction of (8) follows from the supposition that R is closed under arbitrary unions (*alias* (7*a*)), since any set is the union of its singleton subsets (or 'atoms').

In a moment, we pass to the general case of $\forall$-reducibility in the k^{th} position, but before doing so, we should look at $\exists$-reducibility for 1-ary relations. Instead of (5), what we are concerned with here is those $\mathrm{R} \subseteq \mathscr{P}(U)$ for which there exists $R \subseteq U$ such that for all $X \subseteq U$

$$\mathrm{X} \in \mathrm{R} \textit{ iff } \exists x \in X.x \in R \tag{8a}$$

In view of the duality between '$\forall$' and '$\exists$', one might expect that necessary and sufficient conditions for R to be $\exists$-reducible in this sense, are provided by: closure under supersets, and closure under (arbitrary) intersections. But this is overhasty dualization. The first expected condition is indeed necessary: if $X \in \mathrm{R}$ and this is a matter, as (8*a*) says, of X's containing some element which belongs to R, then any superset of X will contain that element (or those elements), and so likewise belong to R. But the second expectation is not satisfied. X and Y could each belong to R in virtue of different elements, neither of which survives into $X \cap Y$, belonging to R. What we need, corresponding to (7), is rather

$X \cup Y \in \mathrm{R}$ *implies: either* $X \in \mathrm{R}$ *or* $Y \in \mathrm{R}$ (9)

or, more generally, corresponding to (7*a*):

$\bigcup\{X | X \in \mathrm{R}_0\} \in \mathrm{R}$ *implies: for some* $X \in \mathrm{R}_0$, $X \in \mathrm{R}$ (9*a*)

It is clear that conditions (7) and (9*a*) are necessary for R to be ∃-reducible, and a proof that they are sufficient proceeds alongs the same lines as that given above in the case of ∀-reducibility. (In particular, given R satisfying those conditions, we can define R, as before, by putting $x \in R$ iff $\{x\} \in \mathrm{R}$.)

So much for the one-place case. The results established for that case can be applied for the general case, once we define appropriately generalized versions of the conditions of closure under subsets and arbitrary unions ((6) and (7*a*) above: we concentrate on the conditions for ∀-reducibility). Accordingly, understand by the condition of closure under subsets in the k^{th} position, for a relation $\mathrm{R} \subseteq \mathscr{P}(U)^n$, with $1 \leq k \leq n$, the following condition:

$<X_1,\ldots,X_n> \in \mathrm{R}$, $Y \subseteq X_k$ imply: $<X_1,\ldots,X_{k-1},Y,X_{k+1},\ldots,X_n> \in \mathrm{R}$ (10)

and by the condition of closure under arbitrary unions in the k^{th} position, the condition:

If, for some $\mathrm{R}_0 \subseteq \mathscr{P}(U)$, $X_k = \bigcup\{Y | Y \in \mathrm{R}_0\}$ *and for each* $Y \in \mathrm{R}_0$, $<X_1,\ldots,X_{k-1},Y,X_{k+1},\ldots,X_n> \in \mathrm{R}$, *then* $<X_1,\ldots,X_n> \in \mathrm{R}$. (11)

THEOREM 1.1 *Any relation* $\mathrm{R} \subseteq \mathscr{P}(U)^n$ *is ∀-reducible in its* k^{th} *position if and only if* R *is closed under subsets and arbitrary unions in the* k^{th} *position.*

To conclude this discussion, we apply the above Theorem taking R as the relation of supervenience; thus $n = 2$ for the case in which we are interested. We have already seen that this relation is ∀-reducible in the first position. For a similar reducibility to obtain in the second position, we should need (according to the Theorem) closure under subsets in this position, as well as closure under unions there. We state these as (12) and (13); the latter is the special case of binary unions.

$\mathbf{P}$ supervenient on $\mathbf{Q}$, $\mathbf{Q}' \cup \mathbf{Q}$, *together imply* $\mathbf{P}$ supervenient on $\mathbf{Q}'$. (12)
$\mathbf{P}$ supervenient on $\mathbf{Q}$, $\mathbf{P}$ supervenient on $\mathbf{Q}'$,
imply $\mathbf{P}$ supervenient on $\mathbf{Q} \cup \mathbf{Q}'$. (13)

Now (12) is clearly false (consider $\mathbf{Q}' = \varnothing$), so one of the two necessary conditions for $\forall$-reducibility fails. But let us look at (13) anyway. (13) is clearly true, with its consequent following from either antecedent taken separately. This is because our relation is closed under—not subsets but—*supersets* in its second position. (Understand this as given by (10) but with the inclusion in the antecedent reversed.) So perhaps we have a case of $\exists$-reducibility in this position. However, though we have not formulated the general analogue of condition (9*a*), it is not hard to see that even the special 'binary unions' case of this condition, here further specialized to $n = k = 2$, is not satisfied:

$\mathbf{P}$ supervenient on $\mathbf{Q} \cup \mathbf{Q}'$ *implies* $\mathbf{P}$ supervenient on $\mathbf{Q}$
or $\mathbf{P}$ supervenient on $\mathbf{Q}'$ (14)

For a counterexample to (14), let U consist of properties of points on the surface of the earth, with $\mathbf{Q}$ = the class of latitude properties (*i.e.*, for each latitude, the property of having that latitude) and $\mathbf{Q}'$ the class of longitude properties. For any $\mathbf{P}$, $\mathbf{P}$ is supervenient on $\mathbf{Q} \cup \mathbf{Q}'$, since if points x, y, have the same latitude and the same longitude, then x is the same point as y. But obviously we can find $\mathbf{P}$ not supervenient on either $\mathbf{Q}$ or $\mathbf{Q}'$ separately (*e.g.*, $\mathbf{P} = \mathbf{Q} \cup \mathbf{Q}'$). So the supervenience relation is neither $\forall$- nor $\exists$-reducible in its second position.

2. *Supervenience and Closure*

Consider the mapping *Spv* which takes a collection $\mathbf{Q}$ of properties to the class of all properties which are supervenient$_1$ on $\mathbf{Q}$. In view of the $\forall$-reducibility of the relation of supervenience, we have not only (15) but also (16):

P is supervenient$_1$ on $\mathbf{Q}$ *iff* $P \in Spv(\mathbf{Q})$ (15)
$\mathbf{P}$ is supervenient on $\mathbf{Q}$ *iff* $\mathbf{P} \subseteq Spv(\mathbf{Q})$ (16)

In the discussion following (13) in the Appendix to §1, we had occasion to observe that the relation of supervenience was closed closed under supersets in its second position. Let us write this observation in terms of *Spv*; it emerges as a monotonicity property:

$$Q \subseteq Q' \text{ implies } Spv(Q) \subseteq Spv(Q') \tag{17}$$

Now the condition (17) describes as satisfied by *Spv* is one of three conditions customarily used to circumscribe the class of 'closure' operations (or 'closure operators'). It does not take much reflection to see that the other two are also satisfied by *Spv*. The general definition is as follows. Given a non-empty set U we say a function C: $\mathscr{P}(U) \rightarrow \mathscr{P}(U)$ is a *closure* operation on U iff for all $X, Y \subseteq U$

$$X \subseteq Y \text{ implies } C(X) \subseteq C(Y);\ X \subseteq C(X);\ C(C(X)) \subseteq C(X). \tag{18}$$

In our case, U should be taken to be some set of properties (again echoing our earlier warning: *not* the set of individuals which have or lack those properties).

The following further terminology is used in connexion with closure operations. If C is such an operation on U then the subsets X of U for which $C(X) = X$ are called *closed*, and in general when $C(X) = Y$, X is called a *basis* for Y, and Y the *closure* of X. For the proof of Theorem 2.1 below, we shall need to note that U is itself a closed set since it is included in its closure by the second condition in (18) and its closure is included in it since C maps subsets of U to subsets of U. We shall also need to exploit the fact—an easy consequence of the above definitions—that the intersection of two closed sets is closed.

One might wonder whether, just in virtue of the way *Spv* was defined, it is guaranteed to satisfy further general structural principles in the style of those listed in (18), beyond such principles as follow from those listed. That is, could we usefully single out a proper subset of the closure operations as the 'supervenience closure' operations, by supplementing (18) with some further conditions? The question is vague as it stands, because we have not said what should be understood by a 'supervenience closure' operation. In making it precise, we should find a generalization of the notion of supervenience freed of the presumption that we are dealing with a relation between classes of properties: for no such presumption was present in the definition of a closure operation, and we wish to show that no special further conditions need to be laid down by showing that every closure operation can be seen as 'supervenience-like'. (A more restrictive generalization will appear in the following section, under the terminology 'strictly supervenience-like'.) To see how to abstract from the fact that supervenience itself relates classes of properties, we should first note that we can rewrite our definition (3), repeated below as (19), once we agree to abbreviate '$Qx \leftrightarrow Qy$' (etc.) as '$x \equiv_Q y$'. (Notice that the relations $\equiv_Q$ are indeed, as the notation suggests, equivalence relations.) In the terms introduced in §1, for any property Q, $\equiv_Q$ is the relation of agreement in respect of Q; we will sometimes call it Q-indiscernibility in what follows.

$$\forall x \forall y[\forall Q \in \mathrm{Q}(Qx \leftrightarrow Qy) \rightarrow \forall P \in \mathrm{P}(Px \leftrightarrow Py)] \tag{3}$$
$$\forall x \forall y[\forall Q \in \mathrm{Q}(x \equiv_Q y) \rightarrow \forall P \in \mathrm{P}(x \equiv_P y)] \tag{19}$$

Thus a given property P is, as we put it, supervenient$_1$ on Q when

$$\forall x \forall y[\forall Q \in \mathrm{Q}(x \equiv_Q y) \rightarrow (x \equiv_P y)] \tag{20}$$

In other words, as we may put it, P is supervenient$_1$ on Q when objects which are Q-indiscernible for each $Q \in \mathrm{Q}$ are always P-indiscernible. More concisely:

$$\bigcap\{\equiv_Q\}_{Q \in \mathrm{Q}} \subseteq \equiv_P \tag{21}$$

Since (21) gives what it is for P to belong to $Spv(\mathrm{Q})$, we abstract from the property-specific nature of our discussion of supervenience by allowing the equivalence relations indexed here by properties to be relations on any collection of things whatever (rather than specifically the objects which have or lack those properties): then we need no longer think of the indices as representing properties. This motivates the following definition. A closure operation C on a set U is *supervenience-like* if there is some set S, and some map assigning each $u \in U$ to an equivalence relation on S we shall denote by $\equiv_u$ (*i.e.*, $\equiv_u \subseteq S \times S$), for which (22) holds, for all $y \in U$, $X \subseteq U$:

$$\mathrm{y} \in C(X) \textit{ if and only if } \bigcap\{\equiv_x\}_{x \in X} \subseteq \equiv_y \tag{22}$$

The closure operation Spv is supervenience-like in the sense here defined, taking S as the set of individuals which may have or lack the properties in U (so that 'x', 'y' now range over those properties, not the individuals in

S), since the right-hand side of (22) is in this instance just (21). Further, it is not hard to see that if we start with some set S together with the indexed equivalence relations $\equiv_u$ on S, for $u \in U$, and treat (22) as defining the mapping C, then the map so defined is automatically a closure operation. What we are currently concerned with is the converse of this question, namely, whether or not every closure operation can be obtained in this manner. And to this question, we may return an affirmative answer:

THEOREM 2.1 *Every closure operation is supervenience-like.*

Proof. Given a closure operation C on some set U, define

$$S = \{X \subseteq U | X = C(X)\}$$

and, for $X, Y \in S$, $u \in U$, put

$$X \equiv_u Y \textit{ iff } X = Y \textit{ or } u \in X \cap Y. \tag{23}$$

It is left to the reader to verify that the relations $\equiv_u$ as defined by (23) are equivalence relations. We now claim that (22) is satisfied, and hence that C is supervenience-like. First, to show that the 'if' direction of (22) holds, suppose that $y \notin C(X)$. We need to show that we do not in that case have $\bigcap\{\equiv_x\}_{x \in X} \subseteq \equiv_y$; *i.e.*, that for some V, $W \in S$, we have $V \equiv_x W$ for all $x \in X$, but $V \not\equiv_y W$. Choose $V = U$ and $W = C(X)$. Since $U = C(U)$, as noted in the paragraph following (18) above, and $C(X) = C(C(X))$, by the second and third conditions listed in (18) itself, we have $V \in S$ and $W \in S$, as desired. Note that since $y \in U$ while $y \notin C(X)$, for this choice of V, W, we have $V \not\subseteq W$. By (23) then, for all $u \in U$, $V \equiv_u W$ iff $u \in V \cap W$. Since $V = U$, this holds iff $u \in W (= C(X))$. And since $x \in C(X)$ for all $x \in X$, by the second of the defining conditions on closure operations listed in (18), $V \equiv_x W$ for all $x \in X$, while, since $y \notin C(X)$, we have $V \not\equiv_y W$. Next, to show the 'only if' direction of (22), suppose $y \in C(X)$ but, for a contradiction, that also, for some V, $W \in S$: $V \equiv_x W$ for all $x \in X$, while $V \not\equiv_y W$. Since $V \not\equiv_y W$, $V \not\subseteq W$ (by (23)); so, as $V \equiv_x W$ for all $x \in X$, we have $x \in V \cap W$ for each $x \in X$ (consulting (23) again), and, similarly, $y \notin V \cap W$. By the first condition on closure operations listed in (18), as $X \subseteq V \cap W$, we have $C(X) \subseteq C(V \cap W)$, and so, since $y \in C(X)$, $y \in C(V \cap W)$. Now we recall the fact that S comprises the C-closed subsets of U, and the fact that the intersection of two closed sets is closed, to infer that $C(V \cap W) = V \cap W$. This gives the desired contradiction, since we have already concluded that $y \in C(V \cap W)$ while $y \notin V \cap W$.

REMARK 2.2. The rather unusual equivalence relations defined in (23) figured in a similar proof in Humberstone [1991], Theorem 4.2. One might have expected instead the more straightforward definition $X \equiv_u Y$ iff $u \in X \leftrightarrow u \in Y$; but the 'only if' part of the argument does not go through using such relations: one needs to bring in u's membership in the intersection $X \cap Y$. (And we can't simply put $X \equiv_u Y$ *iff* $u \in X \cap Y$, since this is in general not reflexive—hence not an equivalence relation. This is the point of disjoining '$X = Y$' on the right-hand side.)

Digression. What is 'unusual' (to quote the above Remark) about the equivalence relations figuring in the proof above? Ore [1942], p.583, defines a partition to be *singular* if at most one of its blocks contains more than one element; let us transfer this terminology to the associated equivalence relation. Thus an equivalence relation is singular if at most one of its equivalence classes contains more than one element. Now, as a piece of temporary terminology, let us define an equivalence relation $\equiv$ on a set S to be *special* if there is some subset S_0 of S such that for all $s, t \in S$, we have

$$s \equiv t \textit{ iff } s = t \textit{ or } s \in S_0 \text{ and } t \in S_0. \tag{23a}$$

(23*a*) looks a bit different from what was written above as (23):

$$X \equiv_u Y \textit{ iff } X = Y \textit{ or } u \in X \cap Y. \tag{23}$$

but in fact the relations $\equiv_u$ are special in the sense just defined since we may take S_0 as the set of subsets of S which contain u as an element, in which case (23*a*) says exactly what (23) says (given a change in the style of variables used). And we have:

PROPOSITION 2.3 *An equivalence relation is special iff it is singular.*

Proof.
'*If*'. Suppose $\equiv$ is a singular equivalence relation on a set S and let $P_{\equiv}$ be the associated partition. Thus at most one block of $P_{\equiv}$ has cardinality > 1.

If there is no such block, take $S_0 = \varnothing$; (23*a*) is then easily seen to held. If there is one such block, take S_0 to be that block; again one can check that (23*a*) holds.

'Only if'. Suppose $\equiv$ is a special equivalence relation on S, but $\equiv$ is not singular. Then there are distinct blocks B_1, B_2, of $P_{\equiv}$ each containing more than one element of S; say, $s_1, t_1 \in B_1$ while $s_2, t_2 \in B_2$, with $s_1 \not\subseteq t_1$, $s_2 \not\subseteq t_2$. Where S_0 is a subset of S in virtue of which $\equiv$ counts as special, in satisfying (23*a*), since $s_1 \equiv t_1$ and $s_1 \not\subseteq t_1$, (23*a*) tells us we must have s_1, $t_1 \in S_0$. Similarly, we must have $s_2, t_2 \in S_0$. By (23*a*) again, since s_1, $s_2 \in S_0$, we have $s_1 \equiv s_2$, contradicting our assumption that the blocks, B_1 and B_2, from which these elements were drawn, are distinct.

Now that we have had some practice with the notion of a singular equivalence relation, we can extract a stronger conclusion from the proof of the Theorem above, than the Theorem itself provides. A supervenience-like closure operation on U was to be defined to be one for which there could be found a set S together with a family $\{\equiv_u\}_{u \in U}$ of equivalence relations on S, such that (22) was satisfied:

$$y \in C(X) \textit{ if and only if } \bigcap\{\equiv_x\}_{x \in X} \subseteq \equiv_y \tag{22}$$

Let us now define C to be *singularly* supervenience-like if there is a set S together with a family $\{\equiv_u\}_{u \in U}$ of singular equivalence relations on S, for which (22) is satisfied. It is clear from the form of the definition that every singularly supervenience-like closure operation is a supervenience-like closure operation, but the proof of the Theorem above gives a converse to this. For, as we have noted, the equivalence relations $\equiv_u$ provided by the proof are 'special', and so—by the above Proposition—singular, so every closure operation is singularly supervenience-like. Thus the three descriptions: *closure* operation, supervenience-like closure operation, singularly supervenience-like closure operation, in fact all pick out the same class of maps. (However, we have abstracted further than is called for from the original motivating example, and the following section will call attention to a more restricted class—suggested by that example—comprising what we call *strictly* supervenience-like closure operations.)

As a coda to this Digression, let me define a binary relation on a set U to be transitive$^+$ when the following holds for all $x, y, w, z \in U$:

$$(Rxy \wedge Rwz) \rightarrow (x = y \vee w = z \vee Rxz) \tag{+}$$

One can easily check that any transitive$^+$ relation is transitive, though not conversely. Bearing in mind the usual definition of an equivalence relation as a relation which is reflexive, symmetric, and transitive, let us define a relation to be an equivalence$^+$ relation if it is reflexive, symmetric, and transitive$^+$, and note the following (easily proved):

PROPOSITION 2.4 *A relation is an equivalence$^+$* relation iff it is a singular equivalence relation.

We remark that if an addition is disjunct '$x = z$' is added to the consequent of (+) above, we obtain the defining condition given in Goodman [1951], p.83, for what it is for a relation to be 'self-complete'. For reflexive relations, the properties of self-completeness and transitivity$^+$ coincide. (On the other hand, if the '=' disjuncts of (+) are omitted from the consequent altogether, we obtain the condition on binary relations called [&] on p.369 of Humberstone [1984].) *End of Digression.*

The point was made above that if we regard (22), repeated here, as defining C, then C is guaranteed to be a closure operation:

$$y \in C(X) \textit{ if and only if } \bigcap\{\equiv_x\}_{x\in X} \subseteq \equiv_y \tag{22}$$

It is worth emphasizing that *this* fact has nothing to do with the requirement—part of our definition of 'supervenience-like'—that the relations $\equiv_u$ ($u \in U$) be equivalence relations. For any binary relations whatever, the analogue of (22) defines a closure operation. Indeed, we can generalize further and note that given any condition expressed by an open formula Φ_x $(s_1,\ldots,s_n)$ in $n+1$ free variables (written here in a manner suggestive of the special case (22)), if we put:

$$\begin{aligned} y \in C(X) \textit{ if and only if for all } s_1,\ldots,s_n \in S, \textit{ if } \Phi_x\,(s_1,\ldots,s_n) \\ \textit{for each } x \in X, \textit{ then } \Phi_y\,(s_1,\ldots,s_n) \end{aligned} \tag{24}$$

then the mapping C so defined is a closure operation. And of course, it follows from the above Theorem that every closure operation can be so represented; but what we wanted to know specifically was that such a representation could be obtained for the special case of (22) in which equivalence relations were involved.

Digresssion. It is an interesting question whether some of the extra generality here observed is of any use in the theory of properties, in the way that supervenience itself has been. (We revert here to the earlier use of variables.) Our equivalence relations $\equiv_Q$ could be replaced by partial orderings $\leq_Q$, for example, so that, just as we defined $x \equiv_Q y$ to hold when $Qx \leftrightarrow Qy$, and rewrote $(3)^-$ (= (3) without the universal quantifier on 'P') as (20):

$$\forall x \forall y[\forall Q \in \mathrm{Q}(Qx \leftrightarrow Qy) \rightarrow (Px \leftrightarrow Py)] \tag{3$^-$}$$
$$\forall x \forall y[\forall Q \in \mathrm{Q}(x \equiv_Q y) \rightarrow (x \equiv_P y)] \tag{20}$$

so we could define $x \leq_Q y$ to hold when $Qx \rightarrow Qy$, and rewrite (25) as (26):

$$\forall x \forall y[\forall Q \in \mathrm{Q}(Qx \rightarrow Qy) \rightarrow (Px \rightarrow Py)] \tag{25}$$
$$\forall x \forall y[\forall Q \in \mathrm{Q}(x \leq_Q y) \rightarrow (x \leq_P y)] \tag{26}$$

But whether any interesting relation between Q and P is here isolated remains to be seen. (Note that if the consequent of (3^-) has '$\leftrightarrow$' replaced by '$\rightarrow$', then the result is just equivalent to (3^-) itself, and also that if Q is closed under complementation, so that whenever $Q \in \mathrm{Q}$, we also have $\overline{Q} \in \mathrm{Q}$, $\overline{Q}$ being the property possessed by all and only the individuals that do not possess Q, then changing '$\leftrightarrow$' to '$\rightarrow$' in the antecedent of (3^-) also makes no difference.) *End of Digression.*

3. *Supervenience and Consequence*

An interesting special case of (24) arises when we take $n = 1$, taking U to be the set of formulas (or statements) of some language, and the auxiliary set S to be some class of truth-value assignments (*alias* valuations), with the condition $\Phi_x(s)$ which arises for this case of (24), namely

$$y \in C(X) \textit{ if and only if for all } s \in S, \quad \textit{if } \Phi_x(s) \textit{ for each } x \in X, \textit{ then } \Phi_y(s) \tag{27}$$

being the condition that (the formula) x is true according to the assignment s. Then what (27) defines as C is the *consequence operation* (determined by S); for example, if U and S comprises the respectively the usual language of sentential logic and the familiar 'boolean' valuations—truth-value assignments respecting the standard truth-table stipulations, that is—then C maps a set X of formulas to the set of all formulas which are tautological consequences of X. (These ideas will be presented in a more general setting in the following section.)

In the case of consequence operations, an alternative conceptualization is often found more convenient in practice, namely in terms of consequence *relations*. Such a relation holds between a set of statements and an individual statement, and is often denoted by some such symbol as '$\vdash$' or '$\Vdash$'. The defining conditions for a relation to be a consequence relation are so chosen that such relations are in a one-to-one correspondence with consequence operations *via* the equivalence $X \vdash y$ iff $y \in C(X)$, where C is the associated consequence operation. (Of course, a more suggestive notation would usually be employed, such as '$\Gamma \vdash A$' rather than '$X \vdash y$': we shall be employing the more suggestive notation in the following section.) We could help ourselves to an analogous approach in the case of any closure operation, such as the operation *Spv* with which we started. Say we agree for the moment to write '$Q \Vdash P$' in place of '$P \in Spv(Q)$'. This is of course just a different way of expressing what we earlier put by saying: P is supervenient$_1$ on Q. But it usefully directs our attention to the possibility of certain parallels between supervenience and logical consequence, to which we now turn.

Let us consider the operation of conjunction, symbolized by $\wedge$, regarded as a binary operation on statements. We can say what is needed for an operation to deserve this name in any of several equivalent ways. One would be say that the consequence relation, $\vdash$, we are interested in, satisfies, for all statements A, B:

$$A, B \vdash A \wedge B \qquad A \wedge B \vdash A \qquad A \wedge B \vdash B \tag{28}$$

where braces ('{', '}') have been suppressed for the sake of legibility and familiarity on the left-hand sides. Another way of doing the same job—which is to say selecting the same pairs $< \vdash, \wedge >$ as standing in a relation we might put by saying that the consequence relation $\vdash$ treats the operation $\wedge$ as conjunction—would be to say that $A \wedge B$ satisfies the first condition laid down in (28), and is such that that for any statement C satisfying that condition (*i.e.*, $A, B \vdash C$) we have $A \wedge B \vdash C$. While (28) gives essentially the 'introduction- and elimination-rules' characterization of conjunction, this last formulation can be paraphrased—familiarly enough—by saying that the conjunction of two statements is the *strongest* statement (to within equivalence) which is consequence of the two of them taken together. Yet another characterization expresses the idea that the conjunction of two statements is the *weakest* statement from which both those statements follow. And there are numerous further (equivalent) characterizations which could be given.

Now let us ask after a parallel in the case of supervenience. Instead of an operation on statements, we will want an operation on properties. To steer clear of some distracting associations, let us not write this as '$\wedge$', but as '$\circ$'. But the idea is to relate properties P and Q to a property $P \circ Q$ using the relation $\Vdash$ (the converse of the relation *supervenience*$_1$) in the same way that statements A and B are related to the statement A $\wedge$ B using the relation(s) $\vdash$ in (28). Accordingly we first transcribe (28) into this new setting and then ask what the result might mean:

$$P, Q \Vdash P \circ Q \qquad P \circ Q \Vdash P \qquad P \circ Q \Vdash Q \tag{29}$$

It is perhaps already clear why we don't want to write '$\wedge$' for $\circ$, the formal similarities notwithstanding. For there is already a perfectly good notion of the conjunction of two properties, P, Q, as the property possessed by precisely those individuals which possess P and also possess Q. That is a deserving thing for '$P \wedge Q$' to mean, and is near enough what $P \circ Q$ would have meant if we had used '$\vdash$' in place of '$\Vdash$' in (29), and adapted the idea of consequence (in the obvious way) so as to apply to properties rather than statements.

Given that $\circ$, satisfying (29) for all properties P and Q, is not property-conjunction, what is it? What, for a given pair P, Q, of properties, is the property $P \circ Q$? Again, we should think of the corresponding logical question: what, for a given pair A, B, of statements is the statement A $\wedge$ B? In a moment, we shall refine the analogy a little, but for now, we can notice what is needed by way of a supplementation to the answer: it is the statement satisfying, for the given A, B, the conditions listed in (28). The needed supplementation is a justification of the use of the definite description 'the statement satisfying...' in this answer. There are two aspects of such a use of 'the': an existence claim and a uniqueness claim. In the case of sentence connectives (operations on statements—perhaps some restricted subset thereof), these questions were raised in Belnap [1962]. The uniqueness claim needs modifying if 'statement' is intended (as I have been intending it) linguistically: we should say that (28) uniquely characterizes $\wedge$ *to within equivalence*, meaning by this that for any statements C and C′ satisfying, for a given A, B, the conditions imposed by (28) on A $\wedge$ B, we have C $\dashv\vdash$ C′. (It is an easy exercise to demonstrate this uniqueness property—all but explicit in the 'inferential strength' characterizations given *à propos* of (28) above—for the present case.) For the existence question, Belnap suggested that we think of matters from the point of view of a consequence relation to be extended to one satisfying such conditions as (28) imposes, and accept that the compounds (here A $\wedge$ B) involved indeed exist as long as the extension was conservative. But it seems there may be additional grounds for denying the existence of a connective with given inferential powers other than the non-conservative extension its postulation would yield of some antecedently favoured consequence relation, and we do not wish to go into these questions here. Rather, we proceed to what was described above as the task of refining the analogy here in play.

There are two respects in which the discussion of operations like $\wedge$ on statements and operations like $\circ$ on properties are disanalogous which should be cleared away so that the present point of interest emerges more clearly. First of all, we are not thinking of properties as linguistic expressions (predicates, for instance), but rather as something (to use a deliberately vague word:) *signified* by such expressions. In fact, at the end of §1 it was suggested that we think of properties David Lewis's way: as classes of *possibilia*. By contrast, we have been thinking of statements as linguistic expressions. To iron out this difference, we should have instead, non-linguistic potential *significata* for such expressions; the term 'proposition' is traditional in this role, and for definiteness, we again follow Lewis (and others) and conceive of propositions as sets of possible worlds. But there is a second way in which our presentation of (28) and (29) has involved a gratuitous disanalogy. This is best brought out by imagining the preceding disanalogy ironed out in the opposite direction: by sticking with statements, and, to have a linguistic analogue in the property case, moving to talk of predicates for the discussion of supervenience. Then although we have linguistic expressions in both cases, the remaining disanalogy is that we are treating predicates as having a fixed interpretation, while subjecting statements to varying interpretations. Such variability was introduced in the discussion following (27), where a restriction amongst arbitrary truth-value assignments to the boolean valuations amounted to treating all but the sentential connectives as uninterpreted vocabulary. This uninterpretedness can be represented at the non-linguistic level of propositions by considering the assignment of a proposition to a statement—or, better, a formula (emphasizing the uninterpretedness)—as relative to a Kripke model. To eliminate this feature, we should think (or at least pretend to think) of an intended such model, in which the assignment to a statement is of the proposition (= set of worlds) at which that statement, considered as fully interpreted (*e.g.*, as some disambiguated indexical-free declarative sentence of English or some other natural language), is true. (We return to the 'uninterpreted' perspective in §4, where A, B,... will be taken to represent formulas in some formal language.) If we now think of the variables in (28) as ranging over such propositions, rather than statements, then (28) in effect identifies $A \wedge B$ with the proposition $A \cap B$, and in general, rather than writing (30), we can write (31):

$$A_1,\ldots,A_n \vdash B \tag{30}$$
$$A_1 \cap \ldots \cap A_n \subseteq B \tag{31}$$

What becomes of the existence and uniqueness questions now? They become special cases of the corresponding questions concerning intersections of sets. The existence question is not really settled by appeal to this broader setting, however, since although one may be inclined to say that it is obvious that any family of sets has an intersection, any further explanation of the point to someone sceptical on this score will simply make use of conjunction (as when one says that $A_1 \cap A_2$ has as elements precisely those things which are elements of A_1 *and* elements of A_2). The uniqueness question is answered more satisfactorily, however, since any sets including only the common elements of A_1 and A_2 will be included in each other and hence be the *same* set. (Thus we do not have to say that *propositional* conjunction is uniquely characterized 'up to equivalence' any more, as we did when considering conjunction as an operation on statements.)

We return to the case of the envisaged operation $\circ$ on properties, governed by (29) above, repeated here:

$$P, Q \Vdash P \circ Q \qquad P \circ Q \Vdash P \qquad P \circ Q \Vdash Q \tag{29}$$

The existence and uniqueness questions for our operation are quite interesting, and we shall consider the latter first. The formal similarity between (28) and (29) may lead us to expect the uniqueness question to be resolved affirmatively. Writing '$P \dashVdash Q$' (etc.) for '$P \Vdash Q$ and $Q \Vdash P$', we easily obtain the result that if $\circ$ and $\circ'$ are two operations each satisfying all three conditions in (29), then we have (for all P, Q):

$$P \circ Q \mathrel{+\!\!\!\!\!\Vert} P \circ' Q \tag{32}$$

and so it might seem that a unique property has been fixed as the result of applying the operation $\circ$ to any given pair of properties. But this would be quite wrong. All that a claim of the form $R \mathrel{+\!\!\!\!\!\Vert} S$ (for properties R, S) says is that each property is supervenient$_1$ on the (unit set of the) other. When such a claim is correct we may call the properties concerned *equivenient*. And although equivenience, so understood, is an equivalence relation, a pair of equivenient properties can be about as far from 'equivalent' in any everyday sense of the term, as it is possible to imagine: for example, a property and its complementary property are equivenient. Thus (32) can be true while the properties (if there are such properties—we have postponed our discussion of existence, to deal with uniqueness first) mentioned on the left and the right are not identical, so we must agree that the uniqueness question has not successfully been resolved in the affirmative by pointing to (32).

Digression. Before proceeding, we pause to correct the misleading impression conveyed by saying, in the last sentence but one, '*for example*, a property and its complementary property are equivenient', as though this is an example plucked from a welter of alternatives. In fact, we can easily show that if properties P and Q are equivenient, then either $Q = P$ or else $Q = \overline{P}$. For if $Q \not\subset P$ then there is some individual a such that either Qa and $\overline{P}a$ or else $\overline{Q}a$ and Pa, and if $Q \not\subset \overline{P}$, then there is some individual b such that either Qb and Pb or else $\overline{Q}b$ and $\overline{P}b$. So if Q is identical neither with P nor with P, then one of the four composite situations obtains:

(i) $Qa, \overline{P}a, Qb, Pb$
(ii) $Qa, \overline{P}a, \overline{Q}b, \overline{P}b$
(iii) $\overline{Q}a, Pa, Qb, Pb$
(iv) $\overline{Q}a, Pa, \overline{Q}b, \overline{P}b$

But none of these can arise if P and Q are equivenient: (i) and (iv) contradict the supervenience$_1$ of P on Q (*i.e.*, on $\{Q\}$), since in these cases we have $a \equiv_Q b$ without $a \equiv_P b$, while (ii) and (iii) similarly contradict the supervenience$_1$ of Q on P. *End of Digression.*

What the equivenience of R and S means is that the equivalence relations $\equiv_R$ and $\equiv_S$ are identical, not that the properties R and S are. (Recall that for any property Q, we defined $\equiv_Q$ as the relation—"Q-indiscernibility"—of agreeing in respect of Q.) In general, as we rewrote (30) to (31), we can re-express (33) as (34)

$$P_1,\ldots,P_n \Vdash Q \tag{33}$$
$$\equiv_{P_1} \cap \ldots \cap \equiv_{P_n} \subseteq \equiv_Q \tag{34}$$

It was keeping these equivalence relations out of the picture that was responsible for any impression there may have been that (32) showed the properties constructed by the $\circ$ operation to be fixed uniquely. What is uniquely fixed here is rather the associated equivalence relations: so what one should say is that the properties concerned are fixed uniquely *to within equivenience*. This bears directly on the question of existence, to which we now turn. At the level of equivalence relations, we can certainly make sense of what (29) is trying to do, writing '$\equiv$' (unsubscripted):

$$\equiv \; = \; \equiv_P \cap \equiv_Q \tag{35}$$

We can regard (35) as a definition of the relation symbolized on the left, and there is nothing misleading about the notation since the intersection of two equivalence relations is indeed an equivalence relation.(2) But what (29) itself requires is that, given properties P, Q, there is a property $P \circ Q$ such that

$$\equiv_{P\circ Q} = \; \equiv_P \cap \equiv_Q \tag{35a}$$

and this claim, as we shall see in a moment, is false. So not only the uniqueness question, but also the existence question, associated with the putative operation $\circ$ must receive a negative answer.

The falsity of the claim that for every P, Q there is a property $P \circ Q$ satisfying (35) is easily seen. Call an equivalence relation *bipartite* if it has

(2) Incidentally, it was to avoid the complications that would arise here if we had considered a supervenience analogue of disjunction —since the preceding claim would be false with 'union' replacing 'intersection'— that we chose to consider the parallel with conjunction.

at most two equivalence classes (the associated partition has at most two blocks, that is). In other words, $\equiv$ is bipartite when for all a, b, c:

$$Either\ a \equiv b\ or\ a \equiv c\ or\ b \equiv c$$

It is clear that the equivalence relations $\equiv_R$ for a property R are all bipartite. For a property possessed by some but not all individuals, the associated partition has two blocks, whereas a property possessed by all individuals or by none induces a one-block partition (*the* one-block partition of the set of individuals, indeed). Conversely, though this is not something we need to exploit for present purposes, every bipartite equivalence relation is of the form $\equiv_R$ for some property R. Now, using the fact that all equivalence relations of the form in question are bipartite, we can see that there is in general no such property $P \circ Q$ as (29)/(35) requires. For the intersection of two equivalence relations is not in general bipartite, even if the relations intersected are bipartite. By way of example, consider the intersection, call it $\equiv$, of the two relations $\equiv_M$ and $\equiv_U$, associated with the properties M, of being male, and U, of being under thirty years of age (now, say). Then $\equiv$ has four equivalence classes, collecting together (i) individuals which are both male and both under thirty, (ii) those that are neither of them male and both of them under thirty, (iii) those that are both male with neither of them being under thirty, and (iv) those neither of which is male and neither of which is under thirty. Thus $\equiv$ is not bipartite, and so not of the form $\equiv_R$ for any property R; in particular, then, there is no property $M \circ U$ such that $\equiv$ is $\equiv_{M \circ U}$.

The above negative results suggest that such operations as our imaginary $\circ$ be specified using the conditions in (29) re-written so that the property variables P, Q, are replaced by variables standing for *partitions* (or equivalence relations). The conditions then define $\circ$ as the meet operation in the lattice of partitions of the set of individuals (having or lacking the properties concerned). We shall not pursue this theme further here, however, though aspects of the behaviour of a sentential connective analogous to '$\circ$' will occupy us again in §5, at the end of which we shall also revisit the conclusions of the preceding paragraph.

The introduction of the notion of a bipartite equivalence relation suggests a strengthening of the concept of a supervenience-like closure operation. Let us recall the definition given earlier. We described a closure operation C on a set U to be *supervenience-like* if there was some set S, and some map assigning to each $u \in U$ an equivalence relation $\equiv_u$ on S, such that for all $y \in U$, $X \subseteq U$:

$$y \in C(X) \textit{ if and only if } \bigcap\{\equiv_x\}_{x \in X} \subseteq \equiv_y \tag{22}$$

And we pointed out that *Spv* is a supervenience-like closure operation in this sense on the set U of properties, since we may take S as the set of individuals which may have or lack the properties in U. But we now observe that the original case of *Spv* satisfies a stronger condition than has explicitly been built into the definition of 'supervenience-like': namely, the equivalence relations in this case are all bipartite. This suggests a stricter notion of supervenience-likeness, capturing this feature of the prototype *Spv*. Accordingly, let us say that a closure operation C on a set U is *strictly supervenience-like* when there exist a set S and a map assigning each $u \in U$ a bipartite equivalence relation $\equiv_u$ on S, such that for all $y \in$ U, $X \subseteq U$, (22) above holds. The question then naturally arises as to whether every closure operation is strictly supervenience-like. Inspection of the proof of Theorem 2.1 (to the effect that every closure operation is supervenience-like) certainly does not enable us to extract an affirmative answer to this question, since the equivalence relations defined in that proof are not bipartite. (Recall that in the proof of Theorem 2.1, we put, for C-closed subsets X, Y of U, and elements u of U: $X \equiv_u Y$ *iff* $X = Y$ *or* $u \in$ X $\cap$ Y, and noted in the Digression following Remark 2.2 that this made $\equiv_u$ 'singular', in the sense of having at most one equivalence-class with more than one element. There is no limit built in here to the number of one-element equivalence classes, so in general the relations will not be bipartite.) Further, it is easily seen that *not* every closure operation is strictly supervenience-like, since all strictly supervenience-like operations satisfy various conditions whose satisfaction is by no means guaranteed by the definition of the notion of a closure operation. Here are two examples taken from Humberstone [1993]. We use the relational notation ('$X \Vdash y$' for '$y \in C(X)$', further abbreviating '$\{x,u\}$' to 'x,u', etc.):

$$x \Vdash y \textit{ implies either } y \Vdash x \textit{ or } \Vdash y \textit{ (i.e., } \varnothing \Vdash y) \tag{36}$$
$$x, u \Vdash y \textit{ and } x, y \Vdash u \textit{ imply either } u, y \Vdash x \textit{ or } u \Vdash y \tag{37}$$

What it would be good to have would be some conditions which are not only, as (36) and (37) are, *necessary* for C to be strictly supervenience-like, but also *sufficient* for this. This problem remains unsolved as of the time of writing, however.

Note. (36) and (37) are special cases of the more general condition:

$$U, x \Vdash y \text{ and } x, y \Vdash u \text{ for each } u \in U \text{ imply} \\ \text{either } U, y \Vdash x \text{ or } U \Vdash y \tag{38}$$

(36) arises by taking $U = \varnothing$ and (37) by taking $U = \{u\}$. (38) is the strongest condition the author has been able to find which is satisfied by all strictly supervenience-like $\Vdash$.

4. *The Logic of the Strictly Supervenience-like*

The question unanswered at the end of the preceding section can be thought of as a straightforward question about consequence relations in a formal language L. Recall the usual notion of such a relation, $\vdash$ being *determined by* a class $\mathcal{V}$ of valuations for L (maps from L to {T,F}; we often suppress the 'for L' when this can be appropriately recovered from the context): for all $\Gamma \subseteq L$, $B \in L$, we have $\Gamma \vdash B$ if and only if for each $v \in \mathcal{V}$ such that $v(A) = T$ for all $A \in \Gamma$, $v(B) = T$. Thus when $\vdash$ is presented by means of some proof-system, if $\vdash$ is determined by $\mathcal{V}$, we may think of $\mathcal{V}$ as affording a semantics w.r.t. which the proof-system is 'sound' (the 'only if' part of the above condition) and 'complete' (the 'if' part). In the interests of greater clarity, we relabel this relationship between $\vdash$ and $\mathcal{V}$: $\vdash$ is *inference-determined* by $\mathcal{V}$. The intended contrast (as in Humberstone [1993]) is with the following notion. Given again a class of valuations $\mathcal{V}$ we say that $\vdash$ is *supervenience-determined* by $\mathcal{V}$ when for all $\Gamma \subseteq L$, $B \in L$: $\Gamma \vdash B$ if and only if for all $u, v \in \mathcal{V}$ such that $u(A) = v(A)$ for each $A \in \Gamma$, we have $u(B) = v(B)$. The problem at the end of the preceding section then transmutes into: find some conditions on consequence relations which are such that a consequence relation is supervenience-determined by some class of valuations precisely when it satisfies the conditions. Note that we make no assumptions about the presence of any particular connectives in L, or about how they might behave according to $\vdash$ should they be present. (In the case of inference-determination, it is well known that every consequence relation is inference-determined by some class of valuations, since one can always take the class of all valuations 'consistent with' $\vdash$ in the sense of not verifying all of Γ while falsifying B, for any Γ, B such that $\Gamma \vdash B$. This terminology is adapted from Scott [1974], *q.v.* for related considerations. We give a version of this argument for a simplification of the notion of a consequence relation in the proof of Theorem 4.2 below). As already indicated, conditions (36)-(38) must be satisfied, at the very least. In the present context, (36) would more naturally be written as

$$A \vdash B \textit{ implies } B \vdash A \textit{ or } \vdash B \tag{39}$$

understood as universally quantified: for all A, B $\in L$ (where $\vdash \subseteq \mathscr{P}(L) \times L$.) Note that we write '$\vdash$' indifferently, whether we are interested in inference- or in supervenience-determination (as opposed to using '$\Vdash$' for the latter, as in §3), since we wish to emphasize that in both cases we have a consequence relation.

Now, rather than discuss further the case of consequence relations, we move to a simpler setting in which the question of supervenience-determination can be answered by a few elementary considerations. What we have in mind is a restriction to the case of B's being a consequence of Γ, for Γ a unit set: $\{A\}$, say. In this case, we could regard the relation under investigation as simply a binary relation on L, think of the relation as obtaining between A and B when B is a consequence of $\{A\}$. First, we rehearse the details of this with inference-determination in mind; only then will we pass to the case of supervenience-determination.

Because we are considering a restriction of the idea of consequence relations, we shall here continue to use the notation '$\vdash$' for what is now a binary relation on (any given) L. But what kind of relation? The appropriate general conditions are simply reflexivity and transitivity—(R), (T) below—which is to say that $\vdash \subseteq L \times L$ is to be a *pre-order* on L:

(R) $\quad A \vdash A \qquad\qquad$ (T) $\quad A \vdash B$ *and* $B \vdash C$ *imply* $A \vdash C$

for all A, B, C $\in L$. It is clear that, for any class $\mathscr{V}$ of valuations, the binary relation on L defined to hold between A and B iff for all $v \in \mathscr{V}$, $v(A) = T$ implies $v(B) = T$, does indeed satisfy (R) and (T). We call this relation the pre-order *inference-determined* by $\mathscr{V}$, for continuity with the previous discussion. Are any additional conditions, not following from (R) and (T), guaranteed to be satisfied by the relations inference-determined by classes of valuations? No: every pre-order is the pre-order inference-determined by some class of valuations. For consider the class of valuations *consistent* with a pre-order $\vdash$, in the sense (specializing that given above for consequence relations) of not verifying and falsifying, respectively, any A and B for which $A \vdash B$. We use the notation $\mathscr{V}al(\vdash)$ for the class of all valuations consistent with $\vdash$, and define v_A (for A $\in L$, the language of $\vdash$) thus: v_A (C) = T iff A $\vdash$ C (for all C $\in L$). The crucial relationship between these ideas is given by:

LEMMA 4.1 *For any language L and any pre-order $\vdash$ on L,* $\{v_A \mid A \in L\} \subseteq \mathcal{Val}(\vdash)$.

Proof.
If $v_A \notin \mathcal{Val}(\vdash)$, this means that we have some B, C $\in L$ such that B $\vdash$ C, with v_A (B) = T, v_A (C) = F. That is A $\vdash$ B while A $\nvdash$C; since B $\vdash$ C, this is impossible (by (T)): so we must have $v_A \in \mathcal{Val}(\vdash)$.

We can now give the promised 'abstract completeness theorem' for the conditions (R) and (T):

THEOREM 4.2 *Any pre-order $\vdash$ is inference-determined by* $\mathcal{Val}(\vdash)$.

Proof.
We must show that for all A, B: A $\vdash$ B iff for all $v \in \mathcal{Val}(\vdash)$, v(A) = T implies v(B) = T. The 'only if' direction is an immediate consequence of the definition of the consistency of a valuation (with a pre-order). For the 'if' direction, suppose A $\nvdash$B, with a view to finding $v \in \mathcal{Val}(\vdash)$ with v(A) = T, v(B) = F. Choose the valuation v_A for this purpose. By Lemma 4.1, $v_A \in \mathcal{Val}(\vdash)$, and *ex hypothesi*, since A $\nvdash$B, v_A (B) = F; finally, by (R), v_A (A) = T.

Thus every pre-order is inference-determined by some class of valuations. Before passing to a consideration of when a pre-order is supervenience-determined by some such class, we pause to make an observation that will be of assistance in that enquiry, arising out of Lemma 4.1: can the '$\subseteq$' of that Lemma be strengthened to '='? What we need to observe is that the answer to this question is *No*. For let v_F be the unique (for a given *L*) valuation which assigns the value F to every formula (of *L*). Clearly v_F is consistent with every pre-order (on *L*), since otherwise we have A $\vdash$ B with v_F (A) = T while v_F (B) = T: but we can never have v_F (A) = T. And v_F is not v_A for any formula A, since v_A (A) = T. Dually, there is the valuation v_T which assigns T to every formula and is likewise consistent with every pre-order: in this case the same reasoning does not apply to show that v_T is not v_A for any formula A, since there may well be A $\in L$

such that A $\vdash$ C for all C, in which case v_T is v_A. (Of course, there may be other such 'all implying' formulas A′: but then $v_{A'} = v_A$.) *But there may not* be! And v_T still belongs to $\mathcal{V}al(\vdash)$.

Digression. A puzzle is set up by beginning, as above, a sentence with 'dually', and continuing with 'the same reasoning does not apply': given the symmetry between the left and the right hand sides of the '$\vdash$' in our pre-order framework, an explanation for this disanalogy is called for. The explanation lies in the definition of the valuations v_A as those assiging T to precisely those C for which A $\vdash$ C. Suppose instead we had defined valuations $\tilde{v}_B$ (for B $\in$ L) by: $\tilde{v}_B$ (C) = T iff C k B. Then the analogue of Lemma 4.1 would have held that all such $\tilde{v}_B$ belong to $\mathcal{V}al(\vdash)$, by a similar appeal to (T), since otherwise there are A, C with A $\vdash$ C, $\tilde{v}_B$ (A) = T [*i.e.*, A $\nvdash$B] and $\tilde{v}_B$ (C) = F [*i.e.*, C $\vdash$ B], and the proof of Theorem 4.2 would have ended with a similar appeal to (R): given A $\nvdash$B, we have $\tilde{v}_B$ (A) = T and $\tilde{v}_B$ (B) = F. So the asymmetry noted lies, not in the material, but in an arbitrary choice of one ('v') rather than another ('$\tilde{v}$') way of presenting that material. As for v_T and v_F, had the alternative here described been employed to begin with, we would have been observing that v_T is never of the form v_A, since $\tilde{v}_A$ (A) = F, and $\tilde{v}_F$ may or may not be of the form in question, depending on the existence of a formula which is, relative to the given pre-order (not 'all-implying') but 'all-implied'. (In terms of the partial ordering $\leq$ of valuations mentioned in the following paragraph, we can give the following characterization of v_A and $\tilde{v}_A$: v_A is the *least* A-verifying valuation consistent with $\vdash$, while $\tilde{v}_A$ is the *greatest* A-falsifying valuation consistent with $\vdash$.) *End of Digression.*

As the above Digression reveals (incidentally), there are many more valuations consistent with a pre-order $\vdash$ that those of the form v_A together with the special cases of v_T and v_F. There are even more such valuations than the above comments reveal. Consider the partial ordering $\leq$ on the class of all valuations (for a given language) defined by: $u \leq v$ iff for all A, u(A) = T only if v(A) = T. Then the class of valuations consistent with $\vdash$ is closed under least upper bounds ('disjunctive combinations of valuations') and greatest lower bounds ('conjunctive combinations') w.r.t. the ordering; in fact the status of v_F and v_T emerges as the result of taking the special case in which we take the empty collection of valuations (as a subset of $\mathcal{V}al(\vdash)$) and form, respectively, disjunctive and conjunctive combinations. (By contrast, the class of valuations consistent with a consequence relation is only guaranteed to be closed under conjunctive combinations, and need not contain v_F.) Only v_T and v_F require explicit attention from amonst the valuations consistent with a pre-order in what follows, amongst consistent valuations not of the form v_A. And we shall be not be treating these two in the same way, for our proof of the analogue of Theorem 4.2 for supervenience-determination (Theorem 4.6 below). First, we extract the following, by inspection of the proof of Theorem 4.2:

PROPOSITION 4.3 *For any given pre-order* $\vdash$ *on a language and any class* $\mathcal{V}$ *of valuations for L, if* $\{v_A \mid A \in L\} \subseteq \mathcal{V} \subseteq \mathrm{Val}(\vdash)$, *then* $\vdash$ *is inference-determined by* $\mathcal{V}$.

COROLLARY 4.4 *Where* $\mathcal{V}al^-(\vdash) = \mathcal{V}al(\vdash) - v_F$, $\vdash$ *is inference-determined by* $\mathcal{V}al^-(\vdash)$

Here we have our difference in treatment between v_T and v_F: we shall be concentrating on $\mathcal{V}al^-(\vdash)$, from which v_F has been excluded, but in which v_T remains. (In fact, any other class of valuations consistent with $\vdash$, containing v_T and all the v_A, but not v_F, would do equally well for the proof below.)

To address the topic of supervenience-determination by a class of valuations, we need to recall (39) above:

A $\vdash$ B *implies* B $\vdash$ A *or* $\vdash$ B (39)

This condition no longer makes sense for pre-orders because of the appearance of $\varnothing$ on the left of the second disjunct: we require a formula to occupy this position. The next best thing is to say that the condition holds whatever formula we put into that position:

A $\vdash$ B *implies: either* B $\vdash$ A *or, for every* C $\in L$, C $\vdash$ B (40)

Revising the notion of supervenience-determination given above for consequence relation to the present more restrictive framework, we say that a pre-order $\vdash$ is *supervenience-determined* by $\mathcal{V}$ when for all A, B $\in L$: A $\vdash$ B if and only if for all $u, v \in \mathcal{V}$ such that $u(A) = v(A)$, we have $u(B) = v(B)$. The analogue of Theorem 4.2 for inference-determination (or more accurately, of the conclusion we drew from it: that every pre-order is inference-determined by some class of valuations) is then given by the 'if' direction of Theorem 4.6, for the proof of which, we enter the following:

LEMMA 4.5 *Suppose that for some formula* B *(in the language* of a pre-order $\vdash$*),* C $\vdash$ B *for every formula* C. *Then for all* $v \in \mathcal{V}al(\vdash)$, *if* $v \neq v_F$, *then* $v(B) = T$.

Proof.
Suppose C $\vdash$ B for every formula C. If $v \neq v_F$ then there is some C such that $v(C) = T$. So if $v \in \mathcal{V}al(\vdash)$, then, since C $\vdash$ B, $v(B) = T$.

THEOREM 4.6 *A pre-order* $\vdash$ *is supervenience-determined by a class of valuations if and only if* $\vdash$ *satisfies the condition (40).*

Proof.
'*Only if*': left to the reader (or: see the proof of Prop.3.5 in Humberstone [1993].)
'*If*': Suppose that $\vdash$ satisfies (40). We show that $\vdash$ is supervenience-determined by the class $\mathcal{V}al^-(\vdash)$ defined in Coro. 4.4.
We must show that for all A, B, we have:
A $\vdash$ B iff for all $u, v \in \mathcal{V}al^-(\vdash)$, $u(A) = v(A)$ implies $u(B) = v(B)$.
'If': Suppose A $\nvdash$ B, then $v_A(A) = T$ and $v_A(B) = F$, while $v_T(A) = v_T(B) = T$. So v_A and v_T agree on A but not on B, and both valuations belong to $\mathcal{V}al^-(\vdash)$
'Only if': Suppose, for a contradiction, that A $\vdash$ B and that we have $u, v \in \mathcal{V}al^-(\vdash)$ with $u(A) = v(A)$ but $u(B) \neq v(B)$. Without loss of generality, we can take it that $u(B) = T$ and $v(B) = F$. Since $v \in \mathcal{V}al^-(\vdash)$ and $\mathcal{V}al^-(\vdash) \subseteq \mathcal{V}al(\vdash)$, $v(A) = F$ (as A $\vdash$ B). Thus we have:

	A	B
u:	F	T
v:	F	F

Now, since A $\vdash$ B, we have, by (40), either B $\vdash$ A, or C $\vdash$ B for all C. But since $u \in \mathcal{V}al^-(\vdash) \subseteq \mathcal{V}al(\vdash)$, B $\nvdash$ A. So C $\vdash$ B for all C. Therefore, as $v(B) = F$ and $v \in \mathcal{V}al^-(\vdash) \subseteq \mathcal{V}al(\vdash)$, by Lemma 4.5, it must be that $v = v_F$: but this is impossible, since $v_F \notin \mathcal{V}al^-(\vdash)$.

This concludes our discussion of the supervenience-determination of pre-orders by classes of valuations. As indicated at the end of the preceding section, the problem remains open as to how to characterize the class of consequence relations—for which more than one formula may appear to the left of the '$\vdash$'—supervenience-determined by classes of valuations. Two loose ends remain to be tidied up: the title of the present section, and the connexion between its contents and that of the earlier sections, which mainly concentrated on the supervenience of one class of properties on another.

5. *Closing Comments*

The first of the two issues just raised—over the 'strictly' in the title of §4—is easily dealt with. Consider the relations $\equiv_A$ for $A \in L$ between valuations u, v for L, defined by: $u \equiv_A v$ iff $u(A) = v(A)$. Since there are only two truth-values, each such relation is a bipartite equivalence relation, so the closure operation mapping a set of formulas Γ to the set of all $\{B \mid \Gamma \vdash B\}$, where $\vdash$ is the consequence relation supervenience-determined by some class of valuations is a strictly supervenience-like closure operation. (Our definition for what made $\vdash$ supervenience-determined by $\mathcal{V}$ was that $\Gamma \vdash B$ should hold if and only if for all $u, v \in \mathcal{V}$ such that $u(A) = v(A)$ for each $A \in \Gamma$, we have $u(B) = v(B)$: but of course this is just another way of saying that for all $u, v \in \mathcal{V}$ such that $u \equiv_A v$ for each $A \in \Gamma$, we have $u \equiv_B v$.)

As to the matter of how these investigations are related to supervenience among properties, we may think of L not as a language in the usual sense but as a set of properties; valuations are then just characteristic functions for subsets of this set. Specifically, where I is the set of (possible) individuals, interpret, for $i \in I$, the valuation v_i as mapping to T precisely those properties possessed by the individual i. Reverting to our earlier notation, of $\Vdash$ (rather than $\vdash$) for the consequence relation on L supervenience-determined (as we are now saying) by $\{v_i \mid i \in I\}$, and P, Q (rather than A, B) for elements of L, we have, for example:

$$P_1, P_2 \Vdash Q \textit{ iff for all } i, j \in I,\ v_i(P_1) = v_j(P_1) \textit{ and } v_i(P_2) = v_j(P_2) \textit{ only if } v_i(Q) = v_j(Q)$$

That is: Q is supervenient (or 'supervenient$_1$' in the more explicit terminology of §1 above) on $\{P_1, P_2\}$ when agreement in respect of each of P_1 and P_2 implies agreement in respect of Q. Admittedly we have deviated from our 'logical' treatment, with an assimilation in the direction of such formulations as (3) from §1, in saying 'for all $i, j \in I$' rather than 'for all v_i, v_j (with $i, j \in I$)': but on any conception of 'property' sufficiently liberal to allow an unencumbered discussion of supervenience (such as that proposed by Lewis and mentioned in the discussion following (3)), $v_i = v_j$ if and only if $i = j$. So this deviation is without significance.

More significant is the fact that some of the kinds of valuations we have had occasion to consider, especially in the above working-through of the case of supervenience-determined pre-orders, are without analogy amongst the v_i. First to come to mind under this heading are perhaps v_T and v_F: corresponding to these would have be individuals (elements of I) which respectively had and lacked all properties. But of course there are no such possible individuals. Even in the case of the valuations v_A which verify precisely the consequences of A, there is are no corresponding v_i. With the present shift of notation, 'v_A' becomes 'v_P', and we should want $v_P(Q) = T$ iff possession of Q followed from possession of P (or more accurately, on Lewis' conception of properties: iff $P \subseteq Q$). So for this to be v_i for some $i \in I$, i would have to possess precisely such properties as follow from possession of P. In general, no such i can be found since there will be properties Q for which neither Q nor $\overline{Q}$ (the complementary property) 'follows from' P, yet one of which must be possessed by i. As this example illustrates, the problem arises from the boolean structure of the collection of properties. A suitable reaction, then, would be to reduce the level of abstractness of the above logical discussion, and consider the issue of supervenience-determination by classes of boolean valuations for various L (closed under conjunction, negation, etc.). The same reaction is similarly called for by another analogy with the philosophical literature on supervenience: what Kim [1984, 1987] calls 'global supervenience'. A global supervenience thesis maintains that worlds cannot differ in the (claimed) supervenient respect unless they differ in the respects on which supervenience is claimed. The respects in question may be taken to be propositions (classes of worlds), a domain which again carries its own boolean structure.

We can accommodate these considerations within the 'logical' approach of the preceding section by asking about consequence relations (or, less ambitiously, pre-orders) supervenience-determined by classes of *boolean* valuations. For the sake of continuity with the discussion in §3, let us consider a language whose sole connective is $\wedge$ (conjunction). The boolean valuations for this language are then just those v such that for all formulas A, B, satisfy: $v(A \wedge B) = T$ iff $v(A) = v(B) = T$, and the consequence relation inference-determined by the class of all such valuations is the least $\vdash$ satisfying (for all A, B) the conditions listed as (28) in that section. A similarly 'syntactic' specification of the consequence relation supervenience-determined by this class of valuations may be found in Humberstone [1993], where corresponding questions for the case of various other connectives are also answered. To avoid confusion, let us revert to the notation of §3, and denote the latter consequence relation by $\Vdash$. The details (simple though they are) of the relationship between $\Vdash$ and the inference-determined consequence relation need not concern us here, beyond the observation that these relations are quite distinct. For example, we certainly do *not* have: $A \wedge B \Vdash A$ in general, since boolean valuations u, v, may agree on $A \wedge B$ without agreeing on A (by having $u(A) = T$, $v(A) = F$, $u(B) = v(B) = F$). The situation is precisely as in §3, where we noted that a putative operation $\circ$ on properties satisfying the analogues of the 'conjunction conditions' (28) but with $\Vdash$ rather than $\vdash$, would not itself *be* property conjunction. We went on to observe that there was in fact no such operation, and we can usefully see what becomes of that negative verdict when the discussion is transposed to the present setting of an uninterpreted formal language. Leaving the topic of supervenience-determination by classes of boolean valutions behind, then, our final remarks address that question.

The new conditions for the case of properties were listed as (29) in §3; trading in our property variables for schematic formula letters, we obtain

$$A, B \Vdash A \circ B \qquad A \circ B \Vdash A \qquad A \circ B \Vdash B \tag{41}$$

We will in fact restrict attention to the second and third of these two conditions: their repercussions are already quite revealing, and this allows us to treat $\Vdash$ as a pre-order, rather than a consequence relation, thereby keeping the discussion within the range of the preceding section's results. The assumption whose repercussions we are to explore, then, is that we have some L with binary connective $\circ$, on which $\Vdash$ is a pre-order supervenience-determined by some class $\mathcal{V}$ of valuations, with $\Vdash$ satisfying the second and third parts of (41). To anticipate: we shall find that this assumption has extremely restrictive consequences for what $\mathcal{V}$ can be like (Prop. 5.2 below). To deliver those consequences, we first make a rather general—and in itself quite surprising—observation about binary relations.

(The present author, at any rate, was surprised to find this, as Exercise X2120 on p.71 of Andrews [1986].)

Given a binary relation R on a non-empty set U, we call an element of U a *universal source* if it bears R to every element of U, and a *universal* target if every element of U bears R to it. Then the needed observation is:

PROPOSITION 5.1 *With respect to* $U \not\subseteq \varnothing$, $R \subseteq U \times U$, *if every element of* U *is either a universal source or a universal target, then some element of* U *is a universal source, and some element of* U *is a universal target.*

Proof.
Assume every element is either a universal source or else a universal target. We show that U contains some universal source. (The argument for the case of universal targets is analogous.)
Take $a \in U$. If a is a universal source, we have the desired conclusion. So suppose a is not a universal source, *i.e.*, there is $b \in U$ such that not aRb. In that case b is not a universal target, so by our assumption, b must be a universal source.

We return to our pre-order $\Vdash$ on L supervenience-determined by some class $\mathcal{V}$ of valuations, and assumed to satisfy the second and third conditions listed under (41). The second condition tells us that $A \circ B \Vdash A$, for all $A, B \in L$, so by Theorem 4.6 we have, for all such A, B: either $A \Vdash A \circ B$ or else $C \Vdash A$, for every $C \in L$. By the third condition under (41), the former alternative implies $A \Vdash B$. So our assumption implies that for every $A, B, C \in L$: $A \Vdash B$ or $C \Vdash A$. This is a stronger conclusion that the already anomalous special case: for all $A, B \in L$, either $A \Vdash B$ or $B \Vdash A$ ("$\Vdash$ is a total pre-order"), and we can extract from this stronger conclusion the information we seek about $\mathcal{V}$. Reformulating that conclusion, we get:

$$\textit{For all } A \in L, \textit{ either } A \Vdash B \textit{ for every } B \in L, \quad \textit{or } C \Vdash A \textit{ for every } C \in L \tag{42}$$

Now (42) says that, with respect to the binary relation $\Vdash$ on L, every element of L is either a universal source or a universal target. So we may appeal to Proposition 5.1 to conclude, that there is a universal source in L (and that there is a universal target —but it is the former conclusion we are interested in here):

For some $A \in L$: $A \Vdash B$ *for every* $B \in L$. (43)

We shall appeal to (43) in our proof of Proposition 5.2, for a more concise formulation of which we introduce the following terminology. We call a pre-order $\Vdash$ on L *downward directed* when for every A, B $\in L$ there exists some C $\in L$ such that C $\Vdash$ A and C $\Vdash$ B. The conclusions we have been drawing ((42), (43)) do not mention '$\circ$', so the assumption that we have drawn them from—that our supervenience-determined pre-order satisfies the second and third parts of (41)—is equivalent to the assumption that $\Vdash$ is downward directed. The above reasoning can be thought of as proceeding thus: since for every A, B there is a C, possibly depending on A and B, such that C $\Vdash$ A and C $\Vdash$ B, let us pick one such C and call it A $\circ$ B (to record that dependence).

PROPOSITION 5.2 *If the pre-order supervenience-determined by some class* $\mathcal{V}$ *of valuations is downward directed, then* $|\mathcal{V}| \leq 2$.

Proof.
We have already seen that if the pre-order $\Vdash$ on a language L supervenience-determined by some $\mathcal{V}$ is downward directed, then $\Vdash$ satisfies (43). Accordingly let A $\in L$ be some universal $\Vdash$-source. Suppose, for a contradiction, that there are more than two elements in $\mathcal{V}$, so that we can choose $v_1, v_2, v_3 \in \mathcal{V}$, all distinct. Since there are only two truth-values, at least two of these three must assign the same truth-value to A. Without loss of generality, we may take it that v_1 (A) = v_2 (A). Now A $\Vdash$ B for all B $\in L$, and $\Vdash$ is supervenience-determined by $\mathcal{V}$, so v_1 (B) = v_2 (B) for all B $\in L$. Thus $v_1 = v_2$, contrary to our selection of v_1, v_2, v_3 as three distinct elements of $\mathcal{V}$.

Proposition 5.2 is illustrative of the limited prospects for mirroring the inferential behaviour of the boolean connectives by 'supervenience' analogues: unless there is a severe restriction on the classes of valuations allowed into consideration, even just the '$\wedge$-elimination' properties of conjunction cannot be mirrored by any such analogue. There is another moral, too, made available by the '$\{v_i \mid i \in I\}$' manoevre of the second paragraph of this section, and this concerns the would-be operation on properties likewise denoted by '$\circ$' in §3. Recall that, with example of the properties of being male and being under thirty, we showed that not every pair of properties had a '$\circ$-composition' *à la* (29), repeated here:

$$P, Q \Vdash P \circ Q \qquad P \circ Q \Vdash P \qquad P \circ Q \Vdash Q \qquad (29)$$

The argument for this conclusion was that such a $P \circ Q$ would have to be such that it would require $\equiv_{P \circ Q} = \equiv_P \cap \equiv_Q$, with $\equiv_{P \circ Q}$ being bipartite: but the intersection of bipartite equivalence relations is not in general bipartite. Our discussion of the sentential version of this issue, in the present section, set as it was in the framework of pre-orders rather than consequence relations, put the analogue of the first of the conditions in (29) to one side, and still concluded that a supervenience-determining $\mathscr{V}$ could not have more than two elements. This means that, returning to the case of properties, even the one-way inclusion $\equiv_{P \circ Q} \subseteq \equiv_P \cap \equiv_Q$ (corresponding to the joint imposition of the second and third conditions under (29)), with supposedly bipartite $\equiv_{P \circ Q}$, should give trouble in the presence of more than two individuals. And one can easily see the trouble it gives, by taking a, b, c as three distinct elements of I (the set of individuals), with $P = \{a\}$, $Q = \{b\}$. Then the equivalence classes of $\equiv_P$ are $\{a\}$ and I-$\{a\}$, while those of $\equiv_Q$ are $\{b\}$ and $I - \{b\}$. Thus any equivalence relation more refined than (*i.e.*, included in) $\equiv_P \cap \equiv_Q$ must have $\{a\}$, $\{b\}$ and at least one other (containing c) as equivalence classes: so it cannot be bipartite. So much the worse, then, for $\equiv_{P \circ Q}$.

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