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# SIMPLIFIED GENTZENIZATIONS FOR CONTRACTION-LESS LOGICS

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Gentzenizations have been achieved for the contraction-less logics, DW, TW, EW, RW and RWK in [2] and [3], and these are characterized by the use of signed formulae, TA and FA, representing the truth and falsity of the formula A, and the contraposed forms of the (T→ ||-) and ( ||-T→) rules. There is an additional contraposed form of the (T→ ||-) rule and an additional premise in the ( ||-T→) rule, in comparison to that required in the Gentzenizations of positive logics, as introduced in Dunn [1] for $R_+$ and furthered by Giambrone in [4] for $TW_+$ and $RW_+$. Further, the Gentzenization of EW contains the sentential constant t, which represents the conjunction of all theorems. For this paper, we use the simplified version for the Gentzenization of RW that appears at the end of [3].

We will simplify the form of the consecutions in these Gentzen systems by replacing them with single structures, this having the advantage of reducing the number of rules virtually by half, in that only one form is generally required for each connective and sign, instead of the two forms for each side of the turnstile ||-, as at present. In the cases of RW and RWK, this will yield a substantial reduction in the number of branches being required in a proof, due to ( ||-T→) being replaced by a single premise rule. This can also be done for DW, TW and EW, but the reduction in branches only occurs for a special case of the ( ||-T→) rule, the full reduction being possible with the addition of an extra structural rule. We also reduce the kinds of structures that can occur in the Gentzenization of RWK, bringing them into line with that for RW. Further, we eliminate the commutativity rules, the associativity rules and the weakening rules, in the appropriate systems, with a view to reducing the overall work in applying these systems, whether manually or by computer.([1])

([1]) None of these simplifications as such will provide any improvement in the prospects of Gentzenization of a wider range of logics, including the quantified versions of the logics under study. The reason is that any such improvement would require a Cut Elimination argument for the new logic, but here I will rely on the Cut Elimination arguments from [2] and [3], which are subject to the constraints given at the end of [2]. However, I have directly constructed Gentzenizations for some logics, using the kinds of structures introduced here

We first present Hilbert-style axiomatizations for the logics, DW, TW, EW, RW and RWK. We take as primitives: $\sim$, &, $\vee$, $\rightarrow$. We consider the following axioms and rules:

*Axioms.*

A1. $A \rightarrow A$.
A2. $A \& B \rightarrow A$.
A3. $A \& B \rightarrow B$.
A4. $(A \rightarrow B) \& (A \rightarrow C) \rightarrow . A \rightarrow B \& C$.
A5. $A \rightarrow A \vee B$.
A6. $B \rightarrow A \vee B$.
A7. $(A \rightarrow C) \& (B \rightarrow C) \rightarrow . A \vee B \rightarrow C$.
A8. $A \& (B \vee C) \rightarrow (A \& B) \vee (A \& C)$.
A9. $\sim\sim A \rightarrow A$.
A10. $A \rightarrow \sim B \rightarrow . B \rightarrow \sim A$.
A11. $A \rightarrow B \rightarrow . B \rightarrow C \rightarrow . A \rightarrow C$.
A12. $A \rightarrow B \rightarrow . C \rightarrow A \rightarrow . C \rightarrow B$.
A13. $A \rightarrow . A \rightarrow B \rightarrow B$.
A14. $A \rightarrow . B \rightarrow A$.

*Rules.*

R1. $A, A \rightarrow B \Rightarrow B$.
R2. $A, B \Rightarrow A \& B$.
R3. $A \rightarrow B, C \rightarrow D \Rightarrow B \rightarrow C \rightarrow . A \rightarrow D$.
R4. $A \Rightarrow A \rightarrow B \rightarrow B$.

*Systems.*

DW = A1-10, R1-3.
TW = DW + A11 + A12 - R3.
EW = TW + R4.
RW = TW + A13.
RWK = RW + A14.

and thus bypassing consecutions, but this work will appear separately.

## 1. *The Gentzenizations LDW, LTW, LEW$^t$, LRW and LRWK.*

We present the Gentzenizations LDW, LTW, LEW$^t$, LRW and LRWK from [3], for the logics DW, TW, EW, RW and RWK, respectively. '$\mathtt{t}$' is added as a sentential constant only for the Gentzenization of EW. We make some minor changes in presentation.

Let $\alpha$, $\beta$, $\gamma$, $\alpha_1$ ... range over structures. We recursively define structures as follows:

(i) A signed formula SA is a structure, where S is a sign, T or F.
(ii) If $\alpha_1, \dots, \alpha_n$ are structures of type (i), (iii) or (iv), then the sequence $\alpha_1, \dots, \alpha_n$ is a structure, where $n \geq 2$.
(iii) If $\alpha$ and $\beta$ are structures then $(\alpha{:}\beta)$ is a structure.
(iv) If $\alpha$ and $\beta$ are structures then $(\alpha;\beta)$ is a structure.

Structures of type (ii) are called *extensional sequences* or e-sequences, structures of type (iii) are called *i-intensional pairs* or i-pairs, and structures of type (iv) are called *j-intensional sequences* or j-pairs.

A *consecution* is an expression of either the form '$\alpha \Vdash$SA' or the form '$\Vdash$SA', where $\alpha$ is a structure, S is a sign (T or F), and A is a formula. We use square brackets to pick out substructures of a structure, e.g. $\gamma[\alpha]$ is a structure $\gamma$ which includes the substructure $\alpha$. Whilst structures $\alpha$ are non-empty, unless otherwise indicated, the containing structures $\gamma[\alpha]$ can be just $\alpha$.

(i) We begin by setting out the Gentzenization LDW of DW, as follows:

*Axioms.*
(Ax.) SA $\Vdash$SA

*Structural Rules.*

(Ke $\Vdash$) $\dfrac{\gamma[\alpha] \Vdash SC}{\gamma[\alpha,\beta] \Vdash SC}$

(We $\Vdash$) $\dfrac{\gamma[\alpha,\alpha] \Vdash SC}{\gamma[\alpha] \Vdash SC}$

(Ce $\Vdash$) $\dfrac{\gamma[\alpha,\beta] \Vdash SC}{\gamma[\beta,\alpha] \Vdash SC}$

(Ci $\Vdash$) $\dfrac{\gamma[(\alpha:\beta)] \Vdash SC}{\gamma[(\beta:\alpha)] \Vdash SC}$

*Logical Rules.*

(T& $\Vdash$) $\dfrac{\gamma[TA] \Vdash SC}{\gamma[T(A\&B)] \Vdash SC}$ $\qquad \dfrac{\gamma[TB] \Vdash SC}{\gamma[T(A\&B)] \Vdash SC}$

( $\Vdash$T&) $\dfrac{\alpha \Vdash TA \qquad \alpha \Vdash TB}{\alpha \Vdash T(A\&B)}$ , where $\alpha$ can be null.

(F& $\Vdash$) $\dfrac{\gamma[FA] \Vdash SC \qquad \gamma[FB] \Vdash SC}{\gamma[F(A\&B)] \Vdash SC}$

( $\Vdash$F&) $\dfrac{\alpha \Vdash FA}{\alpha \Vdash F(A\&B)}$ $\qquad \dfrac{\alpha \Vdash FB}{\alpha \Vdash F(A\&B)}$ , where $\alpha$ can be null.

(T $\vee$ $\Vdash$) $\dfrac{\gamma[TA] \Vdash SC \qquad \gamma[TB] \Vdash SC}{\gamma[T(A \vee B)] \Vdash SC}$

( $\Vdash$T $\vee$ ) $\dfrac{\alpha \Vdash TA}{\alpha \Vdash T(A \vee B)}$ $\qquad \dfrac{\alpha \Vdash TB}{\alpha \Vdash T(A \vee B)}$ , where $\alpha$ can be null.

(F $\vee$ $\Vdash$) $\dfrac{\gamma[FA] \Vdash SC}{\gamma[F(A \vee B)] \Vdash SC}$ $\qquad \dfrac{\gamma[FB] \Vdash SC}{\gamma[F(A \vee B)] \Vdash SC}$

( $\Vdash$F $\vee$ ) $\dfrac{\alpha \Vdash FA \qquad \alpha \Vdash FB}{\alpha \Vdash F(A \vee B)}$ , where $\alpha$ can be null.

(T $\sim$ $\Vdash$) $\dfrac{\gamma[FA] \Vdash SC}{\gamma[T{\sim}A] \Vdash SC}$

( $\Vdash$T $\sim$ ) $\dfrac{\alpha \Vdash FA}{\alpha \Vdash T{\sim}A}$ , where $\alpha$ can be null.

(F $\sim$ $\Vdash$) $\dfrac{\gamma[TA] \Vdash SC}{\gamma[F{\sim}A] \Vdash SC}$

( $\Vdash$F $\sim$ ) $\dfrac{\alpha \Vdash TA}{\alpha \Vdash F{\sim}A}$ , where $\alpha$ can be null.

(T→ ‖⊢) $\dfrac{\alpha \Vdash TA \quad \gamma[TB] \Vdash SC}{\gamma[(T(A\to B);\alpha)] \Vdash SC}$ $\qquad$ $\dfrac{\alpha \Vdash FB \quad \gamma[FA] \Vdash SC}{\gamma[(T(A\to B);\alpha)] \Vdash SC}$

( ‖⊢T→) $\dfrac{(\alpha;TA) \Vdash TB \quad (\alpha;FB) \Vdash FA}{\alpha \Vdash T(A\to B)}$ , where $\alpha$ can be null, in which case the adjacent ';' 's are removed.

(F→ ‖⊢) $\dfrac{\gamma[(TA:FB)] \Vdash SC}{\gamma[F(A\to B)] \Vdash SC}$

( ‖⊢F→) $\dfrac{\alpha \Vdash TA \quad \beta \Vdash FB}{(\alpha:\beta) \Vdash F(A\to B)}$

(ii) To obtain LTW, we add a further 4 rules:

($Bj_1$ ‖⊢) $\dfrac{\gamma[(\alpha;(\beta;\delta))] \Vdash SC}{\gamma[((\alpha;\beta);\delta)] \Vdash SC}$

($Bj_2$ ‖⊢) $\dfrac{\gamma[(\alpha;(\beta;\delta))] \Vdash SC}{\gamma[((\beta;\alpha);\delta)] \Vdash SC}$

($Bij_1$ ‖⊢) $\dfrac{\gamma[((\alpha;\beta):\delta)] \Vdash SC}{\gamma[(\alpha;(\beta:\delta))] \Vdash SC}$

($Bij_2$ ‖⊢) $\dfrac{\gamma[(\alpha:(\beta;\delta))] \Vdash SC}{\gamma[(\beta:(\alpha:\delta))] \Vdash SC}$

(iii) $LEW^t$ is LTW, with no consecutions of the form ' ‖⊢SA', with $\alpha$ non-null in ( ‖⊢T&), ( ‖⊢F&), ( ‖⊢T∨), ( ‖⊢F∨), ( ‖⊢T~), ( ‖⊢F~) and ( ‖⊢T→), and with the addition of the following $\mathtt{t}$-rules:

(T$\mathtt{t}$jI ‖⊢) $\dfrac{\gamma[\alpha] \Vdash SC}{\gamma[(T\mathtt{t};\alpha)] \Vdash SC}$

(T$\mathtt{t}$iE ‖⊢) $\dfrac{\gamma[(T\mathtt{t}:\alpha)] \Vdash SC}{\gamma[\alpha] \Vdash SC}$

(T$\mathtt{t}$jE ‖⊢) $\dfrac{\gamma[(T\mathtt{t};\alpha)] \Vdash SC}{\gamma[\alpha] \Vdash SC}$ $\qquad$ $\dfrac{\gamma[(\alpha;T\mathtt{t})] \Vdash SC}{\gamma[\alpha] \Vdash SC}$

(iv) LRW is LDW without the structural connective ';', with the addition of the following structural rule (Bi $\Vdash$), with the replacement of ';' by ':' in the (T→ $\Vdash$) and ( $\Vdash$T→) rules, and where $\alpha$ can be null in the (T→ $\Vdash$) rule, with deletion of the adjacent ':', and $\alpha$ and/or $\beta$ can be null in the ( $\Vdash$F→) rule, with deletion of the adjacent ':'.

(Bi $\Vdash$) $$\frac{\gamma[((\alpha:\beta):\delta)] \Vdash SC}{\gamma[(\alpha:\delta):\beta)] \Vdash SC}$$

(v) LRWK is LRW with the addition of consecutions of the form '$\alpha \Vdash$' and '$\Vdash$', with 'SC' in each structural and logical rule replaced by '(SC)', indicating that the position after the '$\Vdash$' can be empty, and with the addition of the following rules:

(S′ $\Vdash$S) $$\frac{\alpha \Vdash SA}{\alpha{:}S'A \Vdash}$$ , where $\alpha$ can be null.

(Ki $\Vdash$) $$\frac{\gamma[\alpha] \Vdash (SC)}{\gamma[(\alpha:\beta)] \Vdash (SC)}$$

( $\Vdash$Ki) $$\frac{\alpha \Vdash}{\alpha \Vdash SC}$$ , where $\alpha$ can be null.

In LDW, LTW, LRW and LRWK, we say that a formula *A is derivable* iff the consecution $\Vdash$TA is derivable using the above axioms and rules of the appropriate system. In LEW$^t$, we say *A is derivable* iff Tt $\Vdash$TA is.

## 2. *The Replacement of Consecutions by Structures.*

We now proceed to make the following uniform replacements in each consecution, for each of the systems, except for LRWK which will be dealt with later in a similar manner. These are similar to the translations used by McRobbie in [5], pp.65-7, where he shows that a range of Gentzen systems are equivalent to their respective left-handed versions.

(I) Replace 'T' by 'F' and 'F' by 'T' in the signed formula after the turnstile '$\Vdash$'.

(II) Replace '$\Vdash$' by ':', provided there is a structure to the left of '$\Vdash$'. If no such structure exists, just delete '$\Vdash$'.

We end up with structures, rather than consecutions. We can then tidy up the rules, simplifying the presentations and eliminating most of the right-hand versions of the signed connective rules. In the process, we add the following version of the (Ci) rule:

$$\frac{\alpha:\beta}{\beta:\alpha}$$

We represent these new structural Gentzen systems, $L_1DW$, $L_1TW$, $L_1EW^t$, $L_1RW$ and $L_1RWK$, as follows :

We define structures, as before.

A *C-structure* is a structure in which ':' is the main structural connective.

The following structural restriction applies to the given systems, with the remaining systems to be specified later:

Each structure of $L_1DW$, $L_1TW$ and $L_1RW$ is either a C-structure or a signed formula.

The structures $\alpha$ are non-empty, unless otherwise stated, and the containing structures $\gamma[\alpha]$ can be just $\alpha$, provided this is allowed by the structural conditions placed on the system.

The rules of each of these systems are applied in accordance with the structural restrictions on the system.

*$L_1DW$.*

*Axioms.*

(Ax.) TA:FA

*Structural Rules.*

(Ke) $\dfrac{\gamma[\alpha]}{\gamma[\alpha,\beta]}$

(We) $\dfrac{\gamma[\alpha,\alpha]}{\gamma[\alpha]}$

(Ce) $\dfrac{\gamma[\alpha,\beta]}{\gamma[\beta,\alpha]}$

(Ci) $\dfrac{\gamma[(\alpha:\beta)]}{\gamma[(\beta:\alpha)]}$

*Logical Rules.*

(T&) $\dfrac{\gamma[\mathrm{TA}]}{\gamma[\mathrm{T(A\&B)}]}$ $\qquad$ $\dfrac{\gamma[\mathrm{TB}]}{\gamma[\mathrm{T(A\&B)}]}$

(F&) $\dfrac{\gamma[\mathrm{FA}] \qquad \gamma[\mathrm{FB}]}{\gamma[\mathrm{F(A\&B)}]}$

(T∨) $\dfrac{\gamma[\mathrm{TA}] \qquad \gamma[\mathrm{TB}]}{\gamma[\mathrm{T(A \vee B)}]}$

(F∨) $\dfrac{\gamma[\mathrm{FA}]}{\gamma[\mathrm{F(A \vee B)}]}$ $\qquad$ $\dfrac{\gamma[\mathrm{FB}]}{\gamma[\mathrm{F(A \vee B)}]}$

(T~) $\dfrac{\gamma[\mathrm{FA}]}{\gamma[\mathrm{T}\sim\mathrm{A}]}$

(F~) $\dfrac{\gamma[\mathrm{TA}]}{\gamma[\mathrm{F}\sim\mathrm{A}]}$

(T→i) $\dfrac{\alpha:\mathrm{FA} \qquad \beta:\mathrm{TB}}{(\alpha:\beta):\mathrm{T(A\to B)}}$

(T→j) $\dfrac{\alpha:\mathrm{FA} \qquad \gamma[\mathrm{TB}]}{\gamma[(\mathrm{T(A\to B)};\alpha)]}$ $\qquad$ $\dfrac{\alpha:\mathrm{TB} \qquad \gamma[\mathrm{FA}]}{\gamma[(\mathrm{T(A\to B)};\alpha)]}$

(F→i) $\dfrac{\gamma[(\mathrm{TA}:\mathrm{FB})]}{\gamma[\mathrm{F(A\to B)}]}$

(F→j) $\dfrac{(\alpha;\mathrm{TA}):\mathrm{FB} \qquad (\alpha;\mathrm{FB}):\mathrm{TA}}{\alpha:\mathrm{F(A\to B)}}$

*$L_1TW$.*

Add to $\mathrm{L_1DW}$, the following rules:

($\mathrm{Bj_1}$) $\dfrac{\gamma[(\alpha;(\beta;\delta))]}{\gamma[((\alpha;\beta);\delta)]}$

(Bj$_2$) $\dfrac{\gamma[(\alpha;(\beta;\delta))]}{\gamma[((\beta;\alpha);\delta)]}$

(Bij$_1$) $\dfrac{\gamma[((\alpha;\beta):\delta)]}{\gamma[(\alpha;(\beta:\delta))]}$

(Bij$_2$) $\dfrac{\gamma[(\alpha:(\beta;\delta))]}{\gamma[(\beta:(\alpha:\delta))]}$

*$L_1EW^t$.*

Each structure in $L_1EW^t$ is a C-structure. Add to $L_1TW$ the following $\mathtt{t}$-rules:

(T$\mathtt{t}$jI) $\dfrac{\gamma[\alpha]}{\gamma[(\mathrm{T}\mathtt{t};\alpha)]}$

(T$\mathtt{t}$iE) $\dfrac{\gamma[(\mathrm{T}\mathtt{t}:\alpha)]}{\gamma[\alpha]}$

(T$\mathtt{t}$jE) $\dfrac{\gamma[(\mathrm{T}\mathtt{t};\alpha)]}{\gamma[\alpha]}$ $\qquad$ $\dfrac{\gamma[(\alpha;\mathrm{T}\mathtt{t})]}{\gamma[\alpha]}$

*$L_1RW$.*

Replace all occurrences of ';' in $L_1DW$ with ':', add the rule (Bi) below, and replace the 4 '→'-rules of $L_1DW$ by the following (T→) and (F→) rules:

(Bi) $\dfrac{\gamma[((\alpha:\beta):\delta)]}{\gamma[((\alpha:\delta):\beta)]}$

(T→) $\dfrac{\alpha:\mathrm{FA} \qquad \gamma[\mathrm{TB}]}{\gamma[(\mathrm{T}(\mathrm{A}\to\mathrm{B}):\alpha)]}$ $\qquad$ $\dfrac{\alpha:\mathrm{TB} \qquad \gamma[\mathrm{FA}]}{\gamma[(\mathrm{T}(\mathrm{A}\to\mathrm{B}):\alpha)]}$ , where $\alpha$ can be null, in which case the adjacent ':' 's are removed. Note that $\gamma$[TB] can be TB.

(F→) $$\frac{\gamma[(TA{:}FB)]}{\gamma[F(A\to B)]}$$

*$L_1RWK$.*

Each structure in $L_1RWK$ can be arbitrary, i.e. extensional sequences of C-structures and signed formulae are allowed as well. We add to $L_1RW$ the following rule:

(Ki) $$\frac{\gamma[\alpha]}{\gamma[(\alpha{:}\beta)]}$$

In $L_1DW$, $L_1TW$, $L_1RW$ and $L_1RWK$, we say that a formula *A is derivable* iff the structure FA is derivable using the above axioms and rules of the appropriate system. In $L_1EW^t$, we say *A is derivable* iff Tt:FA is.

*Theorem 1.* For all formulae A,
(i) If A is derivable in LDW then A is derivable in $L_1DW$.
(ii) If A is derivable in LTW then A is derivable in $L_1TW$.
(iii) If A is derivable in $LEW^t$ then A is derivable in $L_1EW^t$.
(iv) If A is derivable in LRW then A is derivable in $L_1RW$.
(v) If A is derivable in LRWK then A is derivable in $L_1RWK$.

*Proof.*
(i) Make the replacements (I) and (II), given above, in a proof of $\Vdash$TA in LDW. Checking each axiom and rule of LDW, we see that, under this replacement, a proof of FA is obtainable in $L_1DW$. [Note that for ($\Vdash$T→), when $\alpha$ is null, only one premise is used.] Each step in such proofs only contains C-structures and signed formulae.
(ii) Doing the same for a proof of $\Vdash$TA in LTW, we again see that FA is provable in $L_1TW$.
(iii) Similarly, a proof of Tt:FA is derivable in $L_1EW^t$ where each step is a C-structure.
(iv) For LRW, any proof of $\Vdash$TA can be converted, under the replacements, into a proof of FA, but for ($\Vdash$T→), use is made of (Ci), (Bi) and (F→), and for ($\Vdash$F→), use is made of (Ci), (Bi) and (T→). [Note that only one premise of ($\Vdash$T→) is used in this proof.]
(v) In LRWK, the structure $\Vdash$ is not derivable as there is no rule that can be applied to yield it. We are left with consecutions of the forms, '$\alpha \Vdash$SA',

'$\Vdash$SA' and '$\alpha \Vdash$'. So, we make the replacements (I) and (II), where (I) is applied only when there is a signed formula after the '$\Vdash$', and for (II) we replace '$\Vdash$' by ':', provided there is a structure to the left and to the right of '$\Vdash$'. If either of these structures do not exist, we just delete '$\Vdash$'. Any proof of $\Vdash$TA in LRWK can be converted into a proof of FA in $L_1$RWK, noting that for (S$'$ $\Vdash$S) no rule is needed, and (Ki) suffices for both (Ki $\Vdash$) and ($\Vdash$Ki).

The immediate advantage of the Gentzen systems, $L_1$DW through $L_1$RWK, except for $L_1EW^t$, is that they achieve some saving of branches, in comparison to their original systems, LDW, LTW, LRW and LRWK. Savings in $L_1$DW and $L_1$TW occur when $\alpha$ is null in ($\Vdash$T$\rightarrow$), in that the single premise rule (F$\rightarrow$) is used in the new system. This saving would generally occur at the start of proofs of formulae with '$\rightarrow$' as main connective. Additional savings occur for $L_1$RW and $L_1$RWK whenever ($\Vdash$T$\rightarrow$) is used, since (F$\rightarrow$), together with (Ci) and (Bi), suffices.(2)

We can save further branches in $L_1$DW and $L_1$TW, and save these branches in $L_1EW^t$ as well, by introducing the additional structural rule, (Ij), which is as follows :

(Ij) $$\frac{(\alpha;\beta):\gamma}{\alpha:(\beta:\gamma)}$$

This would enable (F$\rightarrow$j) and (T$\rightarrow$i) to be derived using (F$\rightarrow$i) and (T$\rightarrow$j), respectively. The lack of (F$\rightarrow$j) saves branching, whilst the lack of (T$\rightarrow$i) gives a tidier presentation. Under "left-handed" interpretation, (Ij) represents the $\circ$-rule, $(A \circ B)\rightarrow C \Rightarrow A\rightarrow.B\rightarrow C$, derivable in the '$\circ$'-extension of all three logics. However, the presence of (Ij) would open up more deductive possibilities which may make deductions more complicated than the presence of the extra two rules and the associated branching.

## 3. *The Reduction of the Structures in LRWK.*

We next reduce the structures that can occur in a proof in $L_1$RWK to C-structures or signed formulae, in order to bring $L_1$RWK into line with the

(2) The savings here would reduce the number of steps a computer would need to apply in a proof-search tree, but the computational complexity classification would still be exponential.

other Gentzenizations and to afford some further simplicity.

*Theorem 2.*
Every step in the derivation of a formula in $L_1$RWK can be restricted to be either a C-structure or a signed formula.
*Proof.* Generally, any step in a proof in $L_1$RWK can be taken to consist of an extensional sequence of one or more elements, where the elements are either C-structures or signed formulae. We prove, by induction on proof steps, that one of these elements is independently provable, i.e. that a C-structure or signed formula is provable for every step. We construct a transformed proof in $L_1$RWK, each step of which is a single element of the corresponding extensional sequence in the main proof. We indicate, for each step in the main proof, what rule is to be applied, if any, to the element in the premise that is proved in the transform, in order to yield a provable element in the conclusion.

(Ax). It is a C-structure, and will also be used in the transform.
(Ke). (i) *$\alpha$ is a proper substructure of the provable element.* Apply
(ii) *$\alpha$ is, contains or is separate from the provable element.* No rule needed.
(We). (i) *$\alpha$ is a proper substructure of the provable element.* Apply (We).
(ii) *$\alpha$ is, contains or is separate from the provable element.* No rule needed.
(Ce). (i) *$\alpha$ and $\beta$ are proper substructures of the provable element.* Apply (Ce).
(ii) *One of $\alpha$ or $\beta$ is, contains or is separate from the provable element.* No rule needed.
(Ci). (i) *$\alpha$:$\beta$ is or is a proper substructure of the provable element.* Apply (Ci).
(ii) *$\alpha$:$\beta$ is or is a proper substructure of one of the other elements.* No rule needed.
(Bi). (i) *($\alpha$:$\beta$):$\delta$ is or is a proper substructure of the provable element.* Apply (Bi).
(ii) *($\alpha$:$\beta$):$\delta$ is or is a proper substructure of one of the other elements.* No rule needed.
(Ki). (i) *$\alpha$ is or is a proper substructure of the provable element.* Apply (Ki).
(ii) *$\alpha$ contains the provable element.* Apply (Ke) and (Ce) to

obtain $\alpha$ and then apply (Ki).

(iii) *$\alpha$ is separate from the provable element.* No rule needed.

(T&). (i) *TA (or TB) is or is a proper substructure of the provable element.* Apply (T&).

(ii) *TA (or TB) is or is a proper substructure of one of the other elements.* No rule needed.

(F&). (i) *FA is or is a proper substructure of the provable element in one branch and FB is or is a proper substructure of the provable element in the other branch.* Apply (F&).

(ii) *FA (FB) is or is a proper substructure of the provable element in one branch and FB (FA) is or is a proper substructure of one of the other elements in the second branch.* No rule is needed, since the provable element of the second branch appears as an element in the conclusion of the main proof.

(iii) *FA is or is a proper substructure of one of the other elements in one branch and FB is or is a proper substructure of one of the other elements in the second branch.* No rule is needed, since the provable elements in both these branches appear in the conclusion. We just make a choice of one, if they are different.

(T $\vee$). Similar to (F&).

(F $\vee$), (T $\sim$), (F $\sim$). Similar to (T&).

(T$\rightarrow$). (i) *TB (or FA) in the second premise is or is a proper substructure of the provable element.* Apply (T$\rightarrow$).

(ii) *TB (or FA) is or is a proper substructure of one of the other elements.* No rule needed, the first premise not being used.

(F$\rightarrow$). (i) *TA:FB is or is a proper substructure of the provable element.* Apply (F$\rightarrow$).

(ii) *TA:FB is or is a proper substructure of one of the other elements.* No rule needed.

The last step, which is of the form FA, is also used in the transform. We have thus constructed a transformed proof in $L_1$RWK, consisting solely of C-structures and signed formulae. We will use this reduced form of $L_1$RWK in what follows.

## 4. *The Elimination of Commutativity and Associativity.*

We next proceed to eliminate the commutativity rules, (Ce) and (Ci), from all five Gentzen systems, and the associativity rule, (Bi), from $L_1RW$ and $L_1RWK$. We do this by replacing the extensional sequences by extensional multisets in all systems, by absorbing (Ci) into the axiom and the forms of appropriate rules in $L_1DW$, $L_1TW$ and $L_1EW^t$, and by replacing the i-pairs by intensional multisets in $L_1RW$ and $L_1RWK$.

We first form the Gentzen systems, $L_2DW$ through $L_2RWK$, which are the corresponding systems, $L_1DW$ through $L_1RWK$, without (Ce) and with the structural formation step (ii) replaced by :

(ii)′ If $\alpha_1, \ldots, \alpha_n$ are signed formulae or i- (or j-)intensional pairs then the multiset $\alpha_1, \ldots, \alpha_n$ is a structure, called an extensional multiset. [j-intensional pairs are applicable to $L_2DW$, $L_2TW$ and $L_2EW^t$.]

The (Ke) and (We) rules, each of which specifically contain a comma, would then be modified so that what was part of an extensional sequence in $L_1DW$-$L_1RWK$ becomes an extensional multiset or submultiset in $L_2DW$-$L_2RWK$.

*Theorem 3.*
Any formula provable in $L_1DW$, $L_1TW$, $L_1EW^t$, $L_1RW$ or $L_1RWK$ is also provable in the corresponding system, $L_2DW$, $L_2TW$, $L_2EW^t$, $L_2RW$ or $L_2RWK$.

*Proof.* We replace, in proofs in $L_1DW$, $L_1TW$, $L_1EW^t$, $L_1RW$ and $L_1RWK$, each extensional sequence, occurring as a substructure of some step in the proof, by a multiset of the same elements. Then, when (Ce) is applied in the proof in $L_1DW$-$L_1RWK$, there is no change to the transformed proof as the corresponding multisets containing the displayed extensional pairs are unchanged. When (Ke) or (We) is applied, the transformed proof uses the corresponding extensional multiset instead of the extensional sequence containing the displayed extensional pair. The remaining rules are straightforwardly transformed using instances of the same rule.

We next eliminate (Ci) from $L_2DW$, $L_2TW$ and $L_2EW^t$ by absorbing (Ci) into (Ax) and the conclusions of (T→i) and (F→j) and, for $L_2TW$ and $L_2EW^t$, the conclusions of ($Bij_1$) and ($Bij_2$). So, we form $L_3DW$, $L_3TW$ and $L_3EW^t$, which are $L_2DW$, $L_2TW$ and $L_2EW^t$, respectively, without (Ci), but with the following expanded forms of (Ax), (T→i), (F→j), and, for $L_3TW$ and

$L_3EW^t$, expanded forms of ($Bij_1$) and ($Bij_2$) :

$$(Ax)' \qquad TA{:}FA \qquad FA{:}TA$$

$$(T\to i)' \qquad \frac{\alpha{:}FA \qquad \beta{:}TB}{(\alpha{:}\beta){:}T(A\to B)} \qquad \frac{\alpha{:}FA \qquad \beta{:}TB}{T(A\to B){:}(\alpha{:}\beta)}$$

$$\frac{\alpha{:}FA \qquad \beta{:}TB}{(\beta{:}\alpha){:}T(A\to B)} \qquad \frac{\alpha{:}FA \qquad \beta{:}TB}{T(A\to B){:}(\beta{:}\alpha)}$$

$$(F\to j)' \qquad \frac{(\alpha;TA){:}FB \qquad (\alpha;FB){:}TA}{\alpha{:}F(A\to B)} \qquad \frac{(\alpha;TA){:}FB \qquad (\alpha;FB){:}TA}{F(A\to B){:}\alpha}$$

$$(Bij_1)' \qquad \frac{\gamma[((\alpha;\beta){:}\delta)]}{\gamma[(\alpha;(\beta{:}\delta))]} \qquad \frac{\gamma[((\alpha;\beta){:}\delta)]}{\gamma[(\alpha;(\delta{:}\beta))]}$$

$$(Bij_2)' \qquad \frac{\gamma[(\alpha{:}(\beta;\delta))]}{\gamma[(\beta{:}(\alpha{:}\delta))]} \qquad \frac{\gamma[(\alpha{:}(\beta;\delta))]}{\gamma[((\alpha{:}\delta){:}\beta)]}$$

$$\frac{\gamma[(\alpha{:}(\beta;\delta))]}{\gamma[(\beta{:}(\delta{:}\alpha))]} \qquad \frac{\gamma[(\alpha{:}(\beta;\delta))]}{\gamma[((\delta{:}\alpha){:}\beta)]}$$

*Theorem 4.*
Any formula provable in $L_2DW$, $L_2TW$ or $L_2EW^t$ is also provable in the corresponding system, $L_3DW$, $L_3TW$ or $L_3EW^t$.
*Proof.* We add to $L_2DW$, $L_2TW$ and $L_2EW^t$, the expanded forms $(Ax)'$, $(T\to i)'$ and $(F\to j)'$, and further add to $L_2TW$ and $L_2EW^t$, the expanded forms $(Bij_1)'$ and $(Bij_2)'$, and then eliminate the expanded form $(Ci)'$ of $(Ci)$ by induction on the number of proof steps.

$$(Ci)' \qquad \frac{\gamma[(\alpha_1{:}\beta_1)]\ \ldots[(\alpha_n{:}\beta_n)]}{\gamma[(\beta_1{:}\alpha_1)]\ \ldots[(\beta_n{:}\alpha_n)]}\ ,$$

where each of the $(\alpha_i{:}\beta_i)$ can occur in disjoint parts of $\gamma$ and are replaced respectively by $(\beta_i{:}\alpha_i)$. The disjointness entails that $\alpha_i{:}\beta_i$ does not occur inside an $\alpha_j$ or $\beta_j$, for any i and j. We include the case where $\gamma$ is just $\alpha_1{:}\beta_1$.

There are two types of interaction between $(Ax)'$ or a rule of $L_2DW$, $L_2TW$ or $L_2EW^t$ and $(Ci)'$. In the transform, the conclusion of $(Ci)'$ may be independently derivable, i.e. derivable without the use of $(Ci)'$, or derivable by applying (Ci)$'$ one step earlier, followed by a re-application of the rule concerned. We will simply call this latter one a re-application of the rule.

We tabulate the cases, accordingly.

(Ax)$'$. Independent derivation using (Ax)$'$.

(Ke). Re-apply (Ke), but, if (Ci)$'$ is totally applied within $\beta$, then the conclusion is independently derivable using (Ke).

($\mathrm{Bij}_1$)$'$. Re-apply ($\mathrm{Bij}_1$)$'$, except where (Ci)$'$ is only applied to $\beta{:}\delta$ or $\delta{:}\beta$, in which case, the conclusion is independently derivable.

($\mathrm{Bij}_2$)$'$. Re-apply ($\mathrm{Bij}_2$)$'$, except where (Ci)$'$ is only applied to $\beta{:}(\alpha{:}\delta)$, $(\alpha{:}\delta){:}\beta$, $\beta{:}(\delta{:}\alpha)$ or $(\delta{:}\alpha){:}\beta$, the conclusion is independently derivable.

(T$\rightarrow$i)$'$. If (Ci)$'$ is applied totally within $\alpha$ and/or $\beta$ then re-apply (T$\rightarrow$i)$'$. If (Ci)$'$ is applied to one of the displayed ':' 's then the conclusion is independently derivable.

(F$\rightarrow$j)$'$. If (Ci)$'$ is applied totally within $\alpha$ then re-apply (F$\rightarrow$j)$'$. If (Ci)$'$ is applied to the displayed ':' then the conclusion is independently derivable.

All the remaining rules involve a straight-forward re-application of the same rule after (Ci)$'$.

We proceed to eliminate both (Ci) and (Bi) in the systems $L_2$RW and $L_2$RWK by replacing the i-pairs by intensional multisets. We form the Gentzen systems, $L_3$RW and $L_3$RWK, which are the corresponding systems, $L_2$RW and $L_2$RWK, without (Ci) and (Bi), and with the structural formation step (iii) replaced by :

(iii)$'$ If $\alpha_1, \ldots, \alpha_n$ are signed formulae or extensional multisets then the multiset $(\alpha_1{:} \ldots {:}\alpha_n)$ is a structure, called an intensional multiset. (Ax) and the (Ki), (T$\rightarrow$) and (F$\rightarrow$) rules, each of which specifically contain a ':', would then be modified so that what was an i-pair in $L_2$RW or $L_2$RWK becomes an intensional multiset or submultiset in $L_2$RW or $L_2$RWK.

We will need the following definition for $L_2$RW and $L_2$RWK :

Where an intensional sequence is a bracketed sequence of elements formed by successive intensional pairing, a *maximal intensional sequence* is one whose elements are signed formulae or extensional multisets, and which cannot be expanded any further by intensional pairing within the confines of the structure of which it is a part.

*Theorem 5.*
Any formula provable in $L_2$RW or $L_2$RWK is also provable in the corresponding system, $L_3$RW or $L_3$RWK.
*Proof.* We replace, in proofs in $L_2$RW and $L_2$RWK, each maximal intensional sequence of signed formulae and extensional multisets, occurring as a substructure of some step in the proof, by an intensional multiset of the same elements. Then, when (Ci) or (Bi) is applied in the proof in $L_2$RW or $L_2$RWK, there is no change to the transformed proof as the multisets containing the displayed intensional pairs are unchanged. When (Ax), (Ki), (T→) or (F→) is applied, the transformed proof uses an intensional multiset or submultiset instead of the displayed i-pair. The remaining rules are straight-forwardly transformed using instances of the same rule.

## 5. *Elimination of Weakening Rules.*

We eliminate the weakening rule (Ke) in all 5 logics and eliminate (Ki) in $L_3$RWK. We proceed to eliminate (Ke) in $L_3$DW, $L_3$TW, $L_3EW^t$ and $L_3$RW, and then to eliminate both (Ke) and (Ki) in $L_3$RWK. First we eliminate (Ke) by absorbing it into (Ax) for $L_3$DW, $L_3$TW, $L_3EW^t$ and $L_3$RW, into the conclusions of (T→i), (T→j) and (F→j), for $L_3$DW, $L_3$TW and $L_3EW^t$, into the conclusion of (T→) for $L_3$RW, into the conclusion of the 4 B-rules for $L_3$TW and $L_3EW^t$, and into the conclusion of (T$t$jI) for $L_3EW^t$.

So, we form $L_4$DW, $L_4$TW, $L_4EW^t$ and $L_4$RW by dropping (Ke) from their respective systems, $L_3$DW, $L_3$TW, $L_3EW^t$ and $L_3$RW, and adding expanded forms of (Ax)′, (T→i)′, (T→j), (F→j)′, (T→), ($Bj_1$), ($Bj_2$), ($Bij_1$)′, ($Bij_2$)′ and (T$t$jI), all given below, to the appropriate logics.

(Ax)″ $\quad \ldots,TA,\ldots{:}\ldots,FA,\ldots \qquad \ldots,FA,\ldots{:}\ldots,TA,\ldots$ , where for $L_4$RW only one of these forms is necessary.

(T→i)″
$$\frac{\alpha{:}FA \qquad \beta{:}TB}{\ldots,(\alpha{:}\beta),\ldots{:}\ldots,T(A\to B),\ldots} \qquad \frac{\alpha{:}FA \qquad \beta{:}TB}{\ldots,T(A\to B),\ldots{:}\ldots,(\alpha{:}\beta),\ldots}$$

$$\frac{\alpha{:}FA \qquad \beta{:}TB}{\ldots,(\beta{:}\alpha),\ldots{:}\ldots,T(A\to B),\ldots} \qquad \frac{\alpha{:}FA \qquad \beta{:}TB}{\ldots,T(A\to B),\ldots{:}\ldots,(\beta{:}\alpha),\ldots}$$

$$(\mathrm{T}{\to}\mathrm{j})' \quad \frac{\alpha{:}\mathrm{FA} \qquad \gamma[\mathrm{TB}]}{\gamma[(\ldots,\mathrm{T}(\mathrm{A}{\to}\mathrm{B}),\ldots;\alpha)]} \qquad \frac{\alpha{:}\mathrm{TB} \qquad \gamma[\mathrm{FA}]}{\gamma[(\ldots,\mathrm{T}(\mathrm{A}{\to}\mathrm{B}),\ldots;\alpha)]}$$

$$(\mathrm{F}{\to}\mathrm{j})'' \quad \frac{(\alpha;\mathrm{TA}){:}\mathrm{FB} \qquad (\alpha;\mathrm{FB}){:}\mathrm{TA}}{\alpha{:}\ldots,\mathrm{F}(\mathrm{A}{\to}\mathrm{B}),\ldots} \qquad \frac{(\alpha;\mathrm{TA}){:}\mathrm{FB} \qquad (\alpha;\mathrm{FB}){:}\mathrm{TA}}{\ldots,\mathrm{F}(\mathrm{A}{\to}\mathrm{B}),\ldots{:}\alpha}$$

$$(\mathrm{T}{\to})' \quad \frac{\alpha{:}\mathrm{FA} \qquad \gamma[\mathrm{TB}]}{\gamma[(\ldots,\mathrm{T}(\mathrm{A}{\to}\mathrm{B}),\ldots{:}\alpha)]} \qquad \frac{\alpha{:}\mathrm{TB} \qquad \gamma[\mathrm{FA}]}{\gamma[(\ldots,\mathrm{T}(\mathrm{A}{\to}\mathrm{B}),\ldots{:}\alpha)]}\ ,$$

where $\alpha$ can be null, in which case the adjacent ':' 's and extensional multisets around T(A→B) are removed. Note that $\gamma$[TB] can be TB.

$$(\mathrm{Bj}_1)' \quad \frac{\gamma[(\alpha;(\beta;\delta))]}{\gamma[(\ldots,(\alpha;\beta),\ldots;\delta)]}$$

$$(\mathrm{Bj}_2)' \quad \frac{\gamma[(\alpha;(\beta;\delta))]}{\gamma[(\ldots,(\beta;\alpha),\ldots;\delta)]}$$

$$(\mathrm{Bij}_1)'' \quad \frac{\gamma[((\alpha;\beta){:}\delta)]}{\gamma[(\alpha;\ldots,(\beta{:}\delta),\ldots)]} \qquad \frac{\gamma[((\alpha;\beta){:}\delta)]}{\gamma[(\alpha;\ldots,(\delta{:}\beta),\ldots)]}$$

$$(\mathrm{Bij}_2)'' \quad \frac{\gamma[(\alpha{:}(\beta;\delta))]}{\gamma[(\beta{:}\ldots,(\alpha{:}\delta),\ldots)]} \qquad \frac{\gamma[(\alpha{:}(\beta;\delta))]}{\gamma[(\ldots,(\alpha{:}\delta),\ldots{:}\beta)]}$$

$$\frac{\gamma[(\alpha{:}(\beta;\delta))]}{\gamma[(\beta{:}\ldots,(\delta{:}\alpha),\ldots)]} \qquad \frac{\gamma[(\alpha{:}(\beta;\delta))]}{\gamma[(\ldots,(\delta{:}\alpha),\ldots{:}\beta)]}$$

$$(\mathrm{TtjI})' \quad \frac{\gamma[\alpha]}{\gamma[(\ldots,\mathrm{Tt},\ldots;\alpha)]}$$

*Theorem 6.*
Any formula provable in $L_3DW$, $L_3TW$, $L_3EW^t$ or $L_3RW$ is also provable in the corresponding system, $L_4DW$, $L_4TW$, $L_4EW^t$ or $L_4RW$.
*Proof.* We add to $L_3DW$, $L_3TW$, $L_3EW^t$ and $L_3RW$ the expanded forms (Ax)'', (T→i)'', (T→j)', (F→j)'', (T→)', $(\mathrm{Bj}_1)'$, $(\mathrm{Bj}_2)'$, $(\mathrm{Bij}_1)''$, $(\mathrm{Bij}_2)''$ and (TtjI)' of the rules appropriate to the logics, and then eliminate the expanded form (Ke)' of (Ke) by induction on the number of proof steps.

(Ke)′ $\dfrac{\gamma[\alpha_1] \ldots [\alpha_n]}{\gamma[\alpha_1,\beta_1] \ldots [\alpha_n,\beta_n]}$ ,

where each of the $\alpha_i$ 's occur in disjoint parts of $\gamma$. The types of interaction between (Ax)″ or a rule of $L_3$DW, $L_3$TW, $L_3EW^t$ or $L_3$RW and (Ke)′ are of a similar sort to that for the elimination of (Ci). So, we use the same terminology here.
We tabulate the cases.

(Ax)″. Independent derivation using (Ax)″.
(We). Re-apply (We).
($Bj_1$)′. Re-apply ($Bj_1$)′, unless (Ke)′ is applied solely to element(s) of the extensional multiset, ...,($\alpha$;$\beta$),..., in which case, the conclusion can be independently derived using ($Bj_1$)′.
($Bj_2$)′, ($Bij_1$)″, ($Bij_2$)″. Similar to ($Bj_1$)′.
(T→i)″. Re-apply (T→i)″, except where (Ke)′ is applied solely to the element(s) of either or both of the displayed extensional multisets, in which case, independently derive the conclusion by using (T→i)″.
(T→j)′. Re-apply (T→j)′, except where (Ke)′ is applied solely to the element(s) of the extensional multiset, ...,T(A→B),..., in which case, the conclusion can be independently derived using (T→j)′.
(F→j)″. Similar to (T→j)′, but with multiset, ...,F(A→B),...
(T→)′. Similar to (T→j)′.
(TtjI)′. Similar to (T→j)′, but with multiset, ...,Tt,....
For the remaining rules, just reapply the appropriate rule after (Ke)′.

We now eliminate both (Ke) and (Ki) from $L_3$RWK, by absorbing the two rules together into (Ax) and the conclusion of (T→). Thus, we form $L_4$RWK, by dropping (Ke) and (Ki), and adding the expanded forms of (Ax) and (T→), given below.

(Ax)‴. $\alpha$[TA]:$\beta$[FA]

(T→)″. $\dfrac{\alpha\text{:FA} \qquad \gamma[\text{TB}]}{\gamma[(\beta[\text{T(A}\to\text{B)}]:\alpha)]}$ $\qquad$ $\dfrac{\alpha\text{:TB} \qquad \gamma[\text{FA}]}{\gamma[(\beta[\text{T(A}\to\text{B)}]:\alpha)]}$ ,

where $\alpha$ can be null, in which case the adjacent ':' 's and the contexts $\beta$ around T(A→B) are removed. Note that $\gamma$[TB] can be TB.

For all the remaining rules, just re-apply the respective rule after (Ke)′ or (Ki)′.

*Theorem 7.*
Any formula provable in $L_3RWK$ is also provable in $L_4RWK$.
*Proof.* We add to $L_3RWK$ the expanded forms (Ax)‴ and (T→)″, and then simultaneously eliminate the expanded forms (Ke)′ and (Ki)′ of (Ke) and (Ki), respectively, by induction on the number of proof steps.

$$\text{(Ke)}' \qquad \frac{\gamma[\alpha_1] \;\ldots\; [\alpha_n]}{\gamma[\alpha_1,\beta_1] \;\ldots\; [\alpha_n,\beta_n]} \quad ,$$

where each of the $\alpha_i$'s occur in disjoint parts of $\gamma$.

$$\text{(Ki)}' \qquad \frac{\gamma[\alpha_1] \;\ldots\; [\alpha_n]}{\gamma[\alpha_1{:}\beta_1] \;\ldots\; [\alpha_n{:}\beta_n]} \quad ,$$

where each of the $\alpha_i$ 's occur in disjoint parts of $\gamma$. In tabulating the cases, we consider both (Ke)′ and (Ki)′ together.

(Ax)‴. Independent derivation using (Ax)‴, given that (Ax)‴ just requires TA and FA to be contained in superstructures which are elements of the main intensional multiset.

(T→)″. If (Ke) or (Ki) are applied totally within $\beta[T(A\rightarrow B)]$ then the conclusion is independently derived using (T→)″. Otherwise, re-apply (T→)″.

This completes the elimination of the weakening rules and leaves the contraction rule (We) as the only structural rule, except for the 4 B-rules of $L_4TW$ and $L_4EW^t$. However, using Giambrone's reduction method of [4], for each of the five Gentzen systems, $L_4DW$-$L_4RWK$, there is never any need for more than one repetition of an element in an extensional multiset, and thus (We) is never applied to the same elements more than once in succession. In the process of proving this, (Ke) can be assumed, as we have shown it to be admissible. Giambrone's overall decision procedure does not apply to $L_4EW^t$, because of the t-elimination rules, but, nevertheless, his use of reduced derivations does still apply, which yields the above restriction on structures.

## 6. *Containment of the Gentzen Systems in the Corresponding Hilbert Systems.*

In order to relate the Gentzen systems above with each of the five logics, we need to show that the derivable formulae of each of $L_4$DW-$L_4$RWK are contained in their corresponding Hilbert system, DW-RWK, which, together with the containment of these Hilbert systems in their respective Gentzenizations, LDW-LRWK, will complete the cycle of containments. Thus, all of the various systems representing each of the five logics will have the same set of derivable formulae.

*Theorem 8.*
Any formula derivable in $L_4$DW, $L_4$TW, $L_4$EW$^t$, $L_4$RW or $L_4$RWK is also derivable in the corresponding Hilbert system, DW, TW, EW, RW or RWK.
*Proof.* We require an interpretation into the corresponding Hilbert system, for each structure in $L_4$DW, $L_4$TW, $L_4$EW$^t$, $L_4$RW and $L_4$RWK. This is determined inductively on structures, as follows :

(i) $I(TA) = A$.
(ii) $I(FA) = \sim A$.
(iii) $I(\alpha_1, \ldots, \alpha_n) = I(\alpha_1) \,\&\, \ldots \,\&I(\alpha_n)$.
(iv) $I(\alpha{:}\beta) = I(\alpha)\oplus I(\beta))$, for $L_4$DW, $L_4$TW and $L_4$EW$^t$.
(v) $I(\alpha_1{:} \ldots {:}\alpha_n) = I(\alpha_1)\oplus \ldots \oplus I(\alpha_n)$, for $L_4$RW and $L_4$RWK, where '$\oplus$' is associative.
(vi) $I(\alpha;\beta) = I(\alpha) \circ I(\beta)$, for $L_4$DW, $L_4$TW and $L_4$EW$^t$.

For all these logics, $A\oplus B =_{df} \sim(A\rightarrow\sim B)$,
for $L_4$DW, $L_4$TW and $L_4$EW$^t$, '$\circ$' is represented by the additional two-way rule, $A \rightarrow .B\rightarrow C \Leftrightarrow A \circ B \rightarrow C$, and
for $L_4$EW$^t$, '$\mathtt{t}$' is represented by the additional rule, $A \Leftrightarrow \mathtt{t}\rightarrow A$.

Once the interpretation I is determined for a structure proved in one of $L_4$DW-$L_4$RWK, we finally prefix '$\sim$' as the structures are regarded as "left-handed". It is this negated interpretation which is used for mapping structures of $L_4$DW-$L_4$RWK to formulae of the respective Hilbert system, DW°, TW°, EW°$^t$, RW° or RWK°.

We show that, under this negated interpretation, each axiom of $L_4$DW-$L_4$RWK is a theorem of the respective DW°-RWK° and each rule of

$L_4$DW-$L_4$RWK yields a derived rule of the respective DW°-RWK°. Most steps are straight-forward and we just pick out the key theorems and derived rules of DW°-RWK°.

The following derived rules and theorems apply, for all 5 logics :

$$
\begin{aligned}
A \to B \;&\Rightarrow\; C\&A \to C\&B\\
&\Rightarrow\; A\&C \to B\&C\\
&\Rightarrow\; C \oplus A \to C \oplus B\\
&\Rightarrow\; A \oplus C \to B \oplus C\\
&\Rightarrow\; C \circ A \to C \circ B\\
&\Rightarrow\; A \circ C \to B \circ C
\end{aligned}
$$

(These yield $A \to B \Rightarrow \gamma'[A] \to \gamma'[B]$, for any context $\gamma'$, made up of '&', '$\oplus$' and '$\circ$'. This is used for all single premise rules.)

$$
\begin{aligned}
&(A \vee B)\&C \to (A\&C) \vee (B\&C)\\
&C\&(A \vee B) \to (C\&A) \vee (C\&B)\\
&(A \vee B) \oplus C \to (A \oplus C) \vee (B \oplus C)\\
&C \oplus (A \vee B) \to (C \oplus A) \vee (C \oplus B)\\
&(A \vee B) \circ C \to (A \circ C) \vee (B \circ C)\\
&C \circ (A \vee B) \to (C \circ A) \vee (C \circ B)
\end{aligned}
$$

(These yield $\gamma'[A \vee B] \to \gamma'[A] \vee \gamma'[B]$, where $\gamma'$ is a context as above, and are for (F&) and (T$\vee$).)
The following are additional key theorems of TW° and EW°$^t$.

$$
\begin{aligned}
&(A \circ B) \circ C \to A \circ (B \circ C)\\
&(A \circ B) \circ C \to B \circ (A \circ C)\\
&A \circ (B \oplus C) \to (A \circ B) \oplus C\\
&A \oplus (B \oplus C) \to B \oplus (A \circ C)
\end{aligned}
$$

The following are additional key theorems of EW°$^t$ :

$$
\begin{aligned}
&A \to A \circ t\\
&A \to t \oplus A
\end{aligned}
$$

For RW°, we add the key theorem, $A \circ (B \circ C) \leftrightarrow (A \circ B) \circ C$.

This will then suffice to show that any formula derivable in one of $L_4$DW-

$L_4$RWK will be derivable in its corresponding Hilbert system, DW°, TW°, EW°$^t$, RW° or RWK°. Due to the conservative extension results for '$\circ$' and '$t$' for the respective logics, given in [6], pp. 350-4 and 365-6, since the formulae involved do not contain '$\circ$' or '$t$', the above derivability extends to DW, TW, EW, RW or RWK. This completes the theorem.

By the results of [2] and [3], any formula which is derivable in one of the Hilbert systems, DW, TW, EW, RW or RWK, is also derivable in its respective Gentzen system, LDW, LTW, LEW$^t$, LRW or LRWK. This then completes the cycle of containments and yields the following theorem.

*Theorem 9.*
(i) DW $\subseteq$ LDW $\subseteq$ $L_1$DW $\subseteq$ $L_2$DW $\subseteq$ $L_3$DW $\subseteq$ $L_4$DW $\subseteq$ DW.
(ii) TW $\subseteq$ LTW $\subseteq$ $L_1$TW $\subseteq$ $L_2$TW $\subseteq$ $L_3$TW $\subseteq$ $L_4$TW $\subseteq$ TW.
(iii) EW $\subseteq$ LEW$^t$ $\subseteq$ $L_1$EW$^t$ $\subseteq$ $L_2$EW$^t$ $\subseteq$ $L_3$EW$^t$ $\subseteq$ $L_4$EW$^t$ $\subseteq$ EW.
(iv) RW $\subseteq$ LRW $\subseteq$ $L_1$RW $\subseteq$ $L_2$RW $\subseteq$ $L_3$RW $\subseteq$ $L_4$RW $\subseteq$ RW.
(v) RWK $\subseteq$ LRWK $\subseteq$ $L_1$RWK $\subseteq$ $L_2$RWK $\subseteq$ $L_3$RWK $\subseteq$ $L_4$RWK $\subseteq$ RWK.

*Proof.* By [2] and [3], and Theorems 1-8.

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