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SEMI-ANALYTIC TABLEAUX FOR PROPOSITIONAL NORMAL MODAL LOGICS WITH APPLICATION TO NONMONOTONICITY

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Abstract

The propositional monotonic modal logics *K45*, *K45D*, *S4.2*, *S4R* and *S4F* elegantly capture the semantics of many current *nonmonotonic* formalisms as long as (strong) deducibility of A from a theory Γ, $\Gamma \vdash A$, allows the use of necessitation on the members of Γ. This is usually forbidden in modal logic where Γ is required to be empty, resulting in a weaker notion of deducibility.

Recently, Marek, Schwarz and Truszczyński have given algorithms to compute the stable expansions of a *finite* theory Γ in various such nonmonotonic formalisms. Their algorithms assume the existence of procedures for deciding (strong) deducibility in these monotonic modal logics and consequently such decision procedures are important for automating nonmonotonic deduction.

We first give a sound, (weakly) complete and cut-free, semi-analytic tableau calculus for monotonic *S4R*, thus extending the cut elimination results of Schwarz for monotonic *K45* and *K45D*. We then give sound and complete semi-analytic tableau calculi for monotonic *K45*, *K45D*, *S4.2* and *S4F* by adding an (analytic) cut rule. The proofs of tableau completeness yield a deterministic satisfiability test to determine theoremhood (weak deducibility), $\vdash_L A$, because all proofs are constructive. The techniques are due to Hintikka and Rautenberg. We then show that the tableau calculi extend trivially to handle (strong) deducibility, $\Gamma \vdash A$, for *finite* Γ.

Using a general theorem due to Rautenberg we also obtain the (weak) interpolation theorem for *K45*, *K45D*, *S4.2* and *S4R*.

Keywords: modal theorem proving, semi-analytic tableaux, nonmonotonic modal logic.

1. *Introduction*

Propositional modal logics have been used to model epistemic notions like knowledge and belief for quite a while now where the formula $\Box A$ is read as "A is believed" or as "A is known". Given a (monotonic) modal logic S and a set of formulae Γ, the formula A is a monotonic consequence of Γ in S if it is deducible in S from Γ, usually written as $\Gamma \vdash_S A$. The set Γ is usually called a theory and the monotonic consequences of Γ in S are all the formulae deducible from Γ in S; that is $Cn_S(\Gamma) = \{A \mid \Gamma \vdash_S A\}$. The system is "monotonic" in that if A is in $Cn_S(\Gamma)$ then it will be in $Cn_S(\Gamma')$ for any superset Γ' of Γ.

To obtain nonmonotonicity we assume $\neg\Box A$ ("A is not known") if there is no deduction of A in S from Γ and previous assumptions. More formally, the theory T is an S-expansion of theory Γ if it satisfies the equation $T = Cn_S(\Gamma \cup \{\neg\Box A \mid A \notin T\})$. Since T appears in the right hand side, the definition is circular, and consequently, a theory Γ may have zero, one, or more S-expansions. To compensate for this phenomenon, the set of *nonmonotonic* consequences of Γ in S is usually defined as the intersection of all S-expansions of Γ. The new system is "nonmonotonic" because, although A may be a nonmonotonic consequence of Γ, it may not be a nonmonotonic consequence of a superset of Γ.

Concurrently, various nonmodal formalisms have also been used to model epistemic notions giving rise to default logics and autoepistemic logics. Recently, the nonmonotonic modal logics based on the (monotonic) modal logics *K45*, *K45D*, *S4R*, *S4.2* and *S4F* have been shown to capture the minimal model semantics for some of these *nonmodal* nonmonotonic formalisms; for example, nonmonotonic *K45D* captures the semantics of autoepistemic logic while nonmonotonic *S4F* "naturally generalises default logic and autoepistemic logic"[Sch, Trua, Trub, ST]. Indeed, Marek, Schwarz and Truszczyński [MST91] give algorithms to compute the S-expansions of a *finite* theory Γ in a wide class of nonmonotonic modal logics. However their algorithms assume the existence of effective procedures for deciding deducibility, $\Gamma \vdash_S A$, in the underlying (monotonic) modal logic S. Therefore decision procedures for deducibility in these particular (monotonic) modal logics are crucial for automating nonmonotonic deduction in both *modal and nonmodal* formulations.

Using the technique of semi-analytic tableaux we provide deterministic and nondeterministic decision procedures for deducibility in monotonic *K45*, *K45D*, *S4R*, *S4.2* and *S4F*. Specifically, we give sound and complete (nondeterministic) tableau systems for deciding *theoremhood* in each of these (monotonic) modal logics. Each proof of tableau completeness is constructive thereby yielding a *deterministic* test for satisfiability and hence a *deterministic* test for theoremhood. The system for *S4R* does not have the subformula property but is cut-free proving that Gentzen's cut-elimination theorem holds for *S4R*. The systems for *S4.2* and *S4F* also break the subformula property and require an analytic cut rule for completeness but remain decidable. Cut-free systems for monotonic *K45* and *K45D* have been given already by Schwarz [Shv89] in sequent form, but by adding an analytic cut rule we obtain greatly simplified completeness proofs. The techniques are due to Hintikka and Rautenberg [Rau83, Rau85].

In (monotonic) modal logic, the ability to decide *theoremhood* in a logic *L does not* automatically enable us to decide (strong) deducibility, $\Gamma \vdash_L A$, in L because theoremhood is defined only when Γ is empty. The tableau calculi can be extended to handle deducibility, in a trivial way, as long as Γ is finite.

The outline of the paper is as follows. In Section 2 we give definitions of the syntactic and semantic concepts we need. In Section 3 we review Rautenberg's tableau formulation which is slightly different than the standard one due to Smullyan and Fitting [Fit83]. In Section 4 we prove soundness and completeness. In Section 5 we discuss the nondeterministic and deterministic decision procedures obtainable from these proofs. In Section 6 we show that the structural rules in our systems are also eliminable. In 7 we show how to extend our tableau procedures to handle strong deducibility and in Section 8 we relate the tableaux systems to sequent systems. In Sections 9 and 10 we mention related work and further work.

2. *Definitions and Notational Conventions*

2.1 *Propositional Normal Modal Logics*

We consider only propositional modal logics. We use a denumerable set of primitive propositions $\mathcal{P} = \{p_1, p_2, \ldots\}$ and use $\wedge$, $\neg$ and $\Box$ as primitives. Then the other usual connectives are defined as abbreviations: $(A \vee B) = (\neg(\neg A \wedge \neg B))$; $(A \Rightarrow B) = (\neg(A \wedge \neg B))$; and $(\Diamond A) = (\neg \Box \neg A)$ where the $=$ sign is merely a meta-linguistic notation. We also use 0 to denote a constant false proposition and use $\varnothing$ to denote the empty set.

The definition of (well formed) formulae is as usual. Lower case letters like p and q denote members of $\mathcal{P}$. Upper case letters from the beginning of the alphabet like A and B together with P and Q (all possibly annotated) denote formulae. Upper case letters from the end of the alphabet like X, Y, Z (possibly annotated) denote *finite* (possibly empty) sets of formulae.

The logics we consider are all normal extensions of the minimal normal modal logic K and are axiomatised by taking the rules of necessitation and modus ponens as inference rules, by taking the axiom schemas of classical propositional logic, and by taking K and combinations of formulae from Figure 1 as further axiom schemas. For an introduction to modal logics see [HC84].

The name of a logic is usually formed by concatenating the names of its (modal) axiom schemas to K to denote that the logic is a normal extension of K. However we use the traditional names of the more famous logics; for example *S4* is *KT4* and *S4.2* is *KT42*. The logic *S4R* is also known in the literature as *S4.4* and as *SW5*, while *S4F* is also known as *S4.3.2* [Seg71].

2.2 *Deducibility and Theoremhood*

A deduction of a formula A in logic L from a finite set of formulae Γ is a finite sequence of formulae $A_1, A_2, \ldots, A_n$ such that $A_n = A$ and each A_i is: (1) a member of Γ; or (2) an instance of an axiom schema of L; or (3) equal to $\Box A_j$ for some $j < i$; or (4) obtained from some A_j and A_k via modus ponens where $k < i$ and $j < i$. We write $\Gamma \vdash_L A$ to indicate there is a deduction of A in L from Γ.

This notion of deduction is stronger than the one usually employed in modal logic where "$\Gamma \vdash_L A$" corresponds to the statement that for some finite subset $\{A_1, A_2, \ldots, A_n\}$ of Γ, we have $\vdash_L A_1 \wedge A_2 \wedge \ldots \wedge A_n \Rightarrow A$ [Gol87]. For example, $p \vdash_L \Box p$ is a perfectly legal deduction in our formulation but $\vdash_L p \Rightarrow \Box p$ is rarely a theorem in modal logics. If Γ is empty then the two notions coincide and we say that A is a *theorem* of L if $\vdash_L A$. Clearly, the deduction theorem is the key but for most of this paper we shall deal with the weaker notion called theoremhood.

Axiom Name	Defining Formula	Alternative Names
K	$\Box(A \Rightarrow B) \Rightarrow (\Box A \Rightarrow \Box B)$	
T	$\Box A \Rightarrow A$	M
4	$\Box A \Rightarrow \Box\Box A$	
5	$\Diamond A \Rightarrow \Box\Diamond A$	E
D	$\Box A \Rightarrow \Diamond A$	
R	$\Diamond\Box A \Rightarrow (A \Rightarrow \Box A)$	$W5$
F	$(\Box(\Box A \Rightarrow B)) \vee (\Diamond\Box B \Rightarrow A)$	
2	$\Diamond\Box A \Rightarrow \Box\Diamond A$	

Figure 1 : Axiom names and associated schemas.

L	L-frame
$K45$	type 1 : al lone cluster (degenerate or nondegenerate); or type 2 : a degenerate cluster followed by a nondegenerate cluster
$K45D$	type 1 : a lone nondegenerate cluster; or type 2 : a degenerate cluster followed by a nondegenerate cluster
$S4R$	type 1 : a lone nondegenerate cluster; or type 2 : a simple cluster followed by a nondegenerate cluster
$S4F$	a sequence of at most two nondegenerate clusters
$S4$	any reflexive and transitive frame
$S4.2$	any reflexive, transitive and convergent frame

Figure 2 : Definition of L-frames.

2.3 *Kripke Semantics*

We assume familiarity with the notions of Kripke frames $\langle W, R\rangle$ and Kripke models $\langle W, R, V\rangle$. A possible world w satisfies an atomic formula p if and only if $w \in V(p)$. We write this as $w \models p$ and write $w \not\models p$ to mean "not $w \models p$". The semantics of the other connectives and modal operators are as usual.

We often use annotated names like w_1 and w_2 to denote possible worlds. Unless stated explicitly, there is no reason why w_1 and w_2 cannot name the same world. We use R as a name of an axiom schema and also for the reachability relation but the context should make clear which R is meant.

In any model $\langle W, R, V\rangle$, a formula A is *true in a world* $w \in W$ if $w \models$ A. A formula A is *valid in a model* $M = \langle W, R, V\rangle$, written as $M \models A$, if it is true in all worlds in that model; that is, if $\forall w \in W, w \models A$. A formula A is *valid in a frame* $F = \langle W, R\rangle$, written as $F \models A$, if A is valid in all models based on F ; that is, if $\forall V, \langle W, R, V\rangle \models A$. Suppose C is a class of models, or of frames. A formula A is *valid in a class* C, written as C $\models A$, if it is valid in every member of C. An axiom is said to be valid in a model (valid in a frame) if all instances of that axiom have that property. If we have a set of formulae X then M $\models X$ (F $\models X$) denotes that all members of X are valid in M (F).

Let C be either a collection of models, or of frames. Then logic L is *sound with respect to C* if for every formula A we have that $\vdash_L A$ implies C $\models A$ [HC84]. Logic L is *complete with respect to C* if for every formula A we have that C $\models A$ implies $\vdash_L A$ [HC84]. A logic L is *determined* or *characterised* by a class C if it is both sound and complete with respect to C; that is, when C $\models A$ iff $\vdash_L A$.

A frame $\langle W, R\rangle$ is: *reflexive* if $\forall w \in W, w\, R\, w$; *transitive* if $\forall\, w_1, w_2, w_3 \in W$, $w_1\, R\, w_2$ and $w_2\, R\, w_3$ implies $w_1\, R\, w_3$; *serial* if $\forall\, w_1 \in W, \exists\, w_2 \in W, w_1\, R\, w_2$; *symmetric* if $\forall\, w_1, w_2 \in W$, $w_1\, R\, w_2$ implies $w_2\, R\, w_1$; and *convergent* if $\forall\, w_1, w_2 \in W, \exists\, w_3 \in W, w_1\, R^i\, w_3$ and $w_2\, R^j\, w_3$ (where i and j are positive integers).

If $\langle W, R\rangle$ is a frame where R is transitive, then a *cluster* C is a maximal subset of W such that for all *distinct* worlds w and w' in C we have wRw' and $w'Rw$. A cluster is *degenerate* if it is a single irreflexive world, otherwise it is *nondegenerate*. A nondegenerate cluster is *proper* if it consists of two or more worlds. A nondegenerate cluster is *simple* if it consists of a single reflexive world. Note that in a nondegenerate cluster, R is reflexive, transitive *and symmetric*. If the frame is transitive and convergent then there is guaranteed to be a *last* cluster which is reachable from every other cluster. In the case where the clusters form a tree (or a linear sequence), the leaf nodes are known as *final clusters*. For an introduction to Kripke frames, Kripke models and the notion of clusters see Hughes and Cresswell [HC84].

A frame is an L-frame if it meets the conditions for L as shown in Figure 2. It is known that logic L is characterised by the class of all L-frames [Seg71, pages 77-78 and 160], [Sch]. Then, a model $\mathsf{M} = \langle W, R, V\rangle$ is an *L-model* if $\langle W, R\rangle$ is an L-frame. Also, M is an *L-model for* (a finite set of formulae) X if there exists $w \in W$ such that $w \models X$. Recall that $w \models X$ means that $w \models A$ for all $A \in X$. A formula A is *L-valid* iff A is valid in all L-models, and hence in all L-frames. A finite set X is *L-satisfiable* iff there exists an L-model for X. So, X is *L-unsatisfiable* iff there are no L-models for X.

3. *Semantic Tableaux and Tableau Rules*

Since our tableau systems work with *finite* sets of formulae, we use the following notational conventions: (1) P and Q stand for formulae; (2) U, X, Y, Z stand for finite (possibly empty) sets of formulae; (3) "$X; Y$" stands for $X \cup Y$; (4) "$X; P$" stands for $X \cup \{P\}$; (5) $\Box X$ stands for $\{\Box P \mid P \in X\}$ and (6) $\neg\Box X$ stands for $\{\neg\Box P \mid P \in X\}$. To minimise the number of rules, we work with primitive notation in terms of $\neg$, $\Box$ and $\wedge$. Each of our tableau rules has a dual rule which can be easily obtained by using the definition of $\Diamond$ as $\neg\Box\neg$. The tableau systems and the completeness proofs are based on those of Rautenberg [Rau83].

3.1 *Syntax of Tableau Systems*

Tableau systems consist of a collection of tableau (inference) rules. A tableau rule consists of a *numerator* above the line and a list of *denominators* (below the line). The denominators are separated by vertical bars. The numerator is a finite set of formulae and so is each denominator. We use the terms numerator and denominator rather than premiss and conclusion to avoid confusion with the sequent terminology.

Figure 3 shows the tableau rules we require. Each tableau rule is read downwards as "if the numerator is L-satisfiable, then so is one of the denominators". A tableau calculus is a finite collection of tableau rules identified with the set of its rule names. Thus the tableau calculus for propositional classical logic PC is $CPC = \{(0), (\neg), (\theta), (\wedge), (\vee)\}$. The other tableau calculi we consider are shown in Figure 4. The $(S4.2)$ rule is the only potentially dangerous rule since its denominator contains a formula to which the rule can be applied in an endless fashion. To forbid this the new formula is marked with a star and the $(S4.2)$ rule is restricted to apply only to non-starred formulae. All other rules must treat starred formula as if they were non-starred.

Following Rautenberg [Rau83, Rau85], a *CL-tableau* for X is a finite tree T with root X whose nodes carry finite formula sets stepwise constructed by the rules of CL according to: if a rule with n denominators is applied to a node then that node has n successors with the proviso that if a node E carries a set Y and Y has already appeared on the branch from the root to E then E is an end node of T. A tableau is *closed* if all its end nodes carry $\{0\}$. A set X is CL-consistent if no closed CL-tableau for X exists.

When formulated using sets rather than multisets, tableau systems include an implicit rule of contraction since the set $X; P; P$ is the same as the set $X; P$. In order to eliminate contraction, we explicitly build contraction into the rules. For example, the (T) rule contains a form of contraction on $\Box$P since $\Box$P is carried from the numerator into the denominator. We return to this point later in Section 6.

The subformula property for tableau systems in primitive notation is slightly different than that for sequent systems. In a sequent $\Gamma \longrightarrow \Delta$, the left side and right side of the sequent arrow respectively act as signs representing "true" and "false". In fact, Fitting makes these signs explicit in his signed tableau [Fit83]. In our tableau systems, the formulae from the right side of the sequent arrow appear with an extra negation sign in the tableau node carrying $\Gamma \cup \neg\Delta$. Hence the "subformulae" we need to consider in our tableaux must contain the negated versions of the sequent subformulae. The following definitions cater for this change.

$$(\wedge)\ \frac{X;P \wedge Q}{X;P;Q} \qquad (0)\ \frac{P;\neg P}{0} \qquad (\vee)\ \frac{X;\neg(P \wedge Q)}{X;\neg P \mid X;\neg Q}$$

$$(\neg)\ \frac{X;\neg\neg P}{X;P} \qquad (\theta)\ \frac{X;Y}{X} \qquad (T)\ \frac{X;\Box P}{X;\Box P;P}$$

$$(45)\ \frac{\Box X; \neg\Box Y; \neg\Box P}{X; \Box X; \neg\Box Y; \neg\Box P; \neg P} \qquad (S4)\ \frac{\Box X; \neg\Box P}{\Box X; \neg P}$$

$$(45D)\ \frac{\Box X; \neg\Box Y; \neg\Box P}{X; \Box X; \neg\Box Y; \neg\Box P; \neg P} \quad \text{where } Y;P \text{ may be empty}$$

$$(R)\ \frac{X; \neg\Box P}{X; \neg\Box P; \neg P \mid X; \neg\Box P; \Box\neg\Box P; P}$$

$$(S4F)\ \frac{U; \Box X; \neg\Box P; \neg\Box Y}{U; \Box X; \neg\Box P; \Box\neg\Box P \mid \Box X; \neg\Box P; \neg\Box Y; \neg P}$$

$$(S4.2)\ \frac{X; \neg\Box P}{X; \neg\Box P; \Box\neg\Box P \mid X; \neg\Box P; \Box(\neg\Box\neg\Box P)^*} \text{where } \neg P \text{ is not starred}$$

$$\frac{X; \neg(P \land Q)}{X; \neg P; \neg Q \mid X; \neg P; Q \mid X; P; \neg Q}$$

$$(sfc)\ \frac{X; \neg\Box P}{X; \neg\Box P; P \mid X; \neg\Box P; \neg P}$$

$$\frac{X; \Box P}{X; \Box P; P \mid X; \Box P; \neg P}$$

$$(sfcT)\ \frac{X; \neg(P \land Q)}{X; \neg P; \neg Q \mid X; \neg P; Q \mid X; P; \neg Q} \qquad \frac{X; \neg\Box P}{X; \neg\Box P; P \mid X; \neg\Box P; \neg P}$$

Figure 3 : Tableau rules

$\underline{CL}$	Static Rules	Transitional Rules	Structural Rules
CPC	$(0),(\neg),(\wedge),(\vee)$	-	(θ)
$CS4$	$(0),(\neg),(\wedge),(\vee),(T)$	$(S4)$	(θ)
$CK45$	$(0),(\neg),(\wedge),(\vee)$	(45)	(θ)
$CK45D$	$(0),(\neg),(\wedge),(\vee)$	$(45D)$	(θ)
$CS4R$	$(0),(\neg),(\wedge),(\vee),(T),(R)$	$(S4)$	(θ)
$CK45\dagger$	$(0),(\neg),(\wedge),(sfc)$	(45)	(θ)
$CK45D\dagger$	$(0),(\neg),(\wedge),(sfc)$	$(45D)$	(θ)
$CS4.2\dagger$	$(0),(\neg),(\wedge),(sfcT),(T),(S4.2)$	$(S4)$	(θ)
$CS4F\dagger$	$(0),(\neg),(\wedge),(sfcT),(T),(S4.2)$	$(S4F),(S4)$	(θ)

For a formula A, the *finite* set of all subformulae $Sf(A)$ is defined inductively as usual where $A \in Sf(A)$. For any finite set X :

- let $Sf(X)$ denote the set of all subformulae of all formulae in X;
- let $\neg Sf(X)$ denote $\{\neg P \mid P \in Sf(X)\}$;
- let $\tilde{X}$ denote the set $Sf(X) \cup \neg Sf(X) \cup \{0\}$;
- let $X^*_{CK45} = X^*_{CK45\dagger} = X^*_{CK45D} = X^*_{CK45D\dagger} = \tilde{X}$;
- let $X^*_{CS4R} = Sf(\Box\tilde{X})$.
- let $X^*_{CS4.2\dagger} = X^*_{CS4F\dagger} = Sf(\Box\neg\Box\tilde{X})$.

Thus, the set carried by *any* node in *any* CL-tableau for X must be a subset of X^*_{CL}. A tableau system CL has the subformula property if $X^*_{CL} = \tilde{X}$, and when this holds, CL is said to be *analytic* [Fit83]. If $\tilde{X} \subset X^*_{CL}$ then CL breaks the subformula property and is said to be *semi-analytic* since CL may not be a decision procedure. However, as long as X^*_{CL} is *finite* then any CL-tableau for X is guaranteed to terminate (for finite X) and there are only a finite number of such tableaux, so that even a semi-analytic CL *is* a decision procedure. Some of our systems are semi-analytic since they break the subformula property, and some even contain an (analytic) cut rule, but in each case, X^*_{CL} is finite.

Intuitively, we can associate the numerator and the denominator of a tableau rule with possible worlds. We classify a rule as a *static rule* if the numerator and the denominator correspond to the same world and classify a rule as a *transitional rule* if at least one of the denominators corresponds to a different world. Note that there is only one explicit structural rule (θ).

A set X is *closed with respect to a tableau rule* if, whenever (an instantiation of) the numerator of the rule is in X, so is (a corresponding instantiation of) at least one of the denominators of the rule. If C is a finite collection of tableau rules then a set X is *closed with respect to* C if it is closed with respect to each rule in C. For each L, a set X is *CL-saturated* if it is CL-consistent and closed with respect to the static rules of CL.

Lemma 1 For each CL-consistent X there is an effective procedure to construct some finite CL-saturated X^* *with* $X \subseteq X^* \subseteq X^*_{CL}$.

Proof: The proof is based on the facts that: (0) is not applicable at any stage since no CL-tableau for X closes by supposition; that each sequence of rule applications is guaranteed to terminate (possibly with a cycle); and that each rule application is guaranteed to give at least one CL-consistent denominator. That is, X^* is obtained as the union of stepwise tableau rule applications. Since each tableau rule maps a subset of X^*_{CL} to another subset of X^*_{CL}, we must have $X^* \subseteq X^*_{CL}$. ■

Such CL-saturated sets are important because they provide a direct connection between the syntactic and semantic aspects of tableau systems. This is the subject of the next section.

3.2 *Soundness and Completeness of Modal Tableau Systems*

For soundness we have to show that for each CL-tableau rule: if the numerator is L-satisfiable then at least one of the denominators is L-satisfiable. For completeness we have to show that if there is no closed CL-tableau for X then X has an L-model (i.e. there is an L-model which is an L-model for X). The basic idea is due to Hintikka.

The following definition from Rautenberg [Rau83] is central for the model construction mentioned above. A *model graph* for some finite fixed set of formulae X is a finite L-frame $\langle W, R\rangle$ such that all $w \in W$ are CL-saturated sets with $w \subseteq X^*_{CL}$ and

(i) $X \subseteq w_0$ for some $w_0 \in W$;

(ii) if $\neg\Box P \in w$ then there exists some $w' \in W$ with wRw' and $\neg P \in w'$;

(iii) if wRw' and $\Box P \in w$ then $P \in w'$.

Lemma 2 If $\langle W, R\rangle$ is a model graph for X then there exists an L-model for X at node w_0 [Rau83]

Proof: Suppose $\langle W, R\rangle$ is a model graph for X. Then w_0 is CL-saturated. Take the valuation V from atomic propositions to subsets of W, where V : $p \mapsto \{w \in W \mid p \in w\}$. Using simultaneous induction on the degree of $P \in w$ it is easy to show that: (a) $P \in w$ implies $w \models P$ and (b) $\neg P \in w$ implies $w \not\models P$. By (a), $w_0 \models X$, hence $\langle W, R, V\rangle$ is an L-model for X. ■

This model graph construction is similar in spirit to the subordinate frames construction of Hughes and Cresswell [HC84] except that they use maximal consistent sets and do not consider cycles, giving infinite models rather than finite models.

4 *Soundness and Completeness*

4.1 *Soundness*

Theorem 1 (soundness) : If L is one of K45, K45D, S4, S4R, S4.2 and S4F then the tableau calculi CL and CL† are sound with respect to L-frames.

Proof : It is easy to show that the rules (0), ($\neg$), ($\wedge$), (θ), and ($\vee$) are sound with respect to all our L-frames. The (*sfcT*) rule is merely a reflexive counterpart of the (*sfc*) rule.

The soundness proofs of the modal rules are very similar so we give the intuitions behind the more arcane rules.

Proof for (S4F) for S4F-frames : Suppose $\langle W, R, V\rangle$ is an $S4F$-model containing a world w_0 such that $w_0 \models U; \Box X; \neg\Box P; \neg\Box Y$. Thus there exists a world w_1 with w_0Rw_1 and $w_1 \models \Box X; \neg P$ by transitivity. There are two cases depending on whether w_1 appears in a final cluster or in a nonfinal cluster.

Case 1: If w_1 occurs in a nonfinal cluster C_{nf} then w_0 must also occur in C_{nf}. Also, since R is universal over nondegenerate clusters we must have w_1Rw_0. Hence by transitivity $w_1 \models \Box X; \neg P; \neg\Box Y; \neg\Box P$ and we are done.

Case 2: If w_1 occurs in a final cluster C_f then $w_1 \models \neg\Box P$ by reflexivity of R. But then, regardless of where w_0 occurs, we must have $w_0 \models \Box\neg\Box P$ and hence $w_0 \models U; \Box X; \neg\Box P; \neg\Box Y; \Box\neg\Box P$. ■

Proof for (45) for K45-frames: Let $\mathsf{M} = \langle W, R, V\rangle$ be a $K45$-model and suppose that $w_0 \in W$ and $w_0 \models \Box X; \neg\Box Y; \neg\Box P$. We have to show that there exists a $w' \in W$ such that $w' \models X; \Box X; \neg\Box Y; \neg\Box P; \neg P$.

Clearly, the $X; \Box X$ part will follow from the transitivity of R so we need only prove that there exists a $w' \in W$ such that $w' \models \neg\Box Y; \neg\Box P; \neg P$.

If $\langle W, R\rangle$ is of type 1 then it cannot be a single degenerate cluster since $w_0 \models \neg\Box P$. But it may be a single, proper or simple, (nondegenerate) cluster. So if $\langle W, R\rangle$ is of type 1 then there must be some $w' \in W$ such that w_0Rw' and $w' \models \neg P$. Also, w' must be reflexive since $\langle W, R\rangle$ is a nondegenerate cluster, hence $w' \models \neg\Box P; \neg P$. In a nondegenerate (reflexive, transitive) cluster, R must be symmetric as well, so $w'Rw_0$. But in a reflexive, transitive and symmetric cluster, $w_0 \models \neg\Box Y$ implies $w_0 \models \Box\neg\Box Y$, hence $w' \models \neg\Box P; \neg P; \neg\Box Y$ and we are done.

If $\langle W, R\rangle$ is of type 2 then, regardless of whether w_0 is in the first or last cluster, there must be some w' in the last (nondegenerate) cluster such that $w' \models \neg P$ since $w_0 \models \neg\Box P$. Similarly, if $Y = \{Q_1, Q_2, \ldots, Q_m\}$, then there must exist (not necessarily distinct) worlds $w_1, w_2, \ldots, w_m$ in the last nondegenerate cluster such that $w_i \models \neg Q_i$ for each $Q_i \in Y$. Since R is reflexive, transitive and symmetric over a nondegenerate cluster, this means that $w' \models \neg P; \neg\Box P; \neg\Box Y$. ■

Proof for (45D) for K45D-frames: Let $\mathsf{M} = \langle W, R, V\rangle$ be a $K45D$-model and suppose that $w_0 \in W$ and $w_0 \models \Box X; \neg\Box Y; \neg\Box P$. We have to show that there exists a $w' \in W$ such that $w' \models X; \Box X; \neg\Box Y; \neg\Box P; \neg P$ allowing for the case where the $\neg\Box Y; \neg\Box P$ part is missing. Every $K45D$-frame is a $K45$-frame hence the proof above applies when the $\neg\Box Y; \neg\Box P$ part is present. If there are no eventualities in w_0 then the seriality and transitivity of R guarantees that there is some world w' with wRw' such that $w' \models X; \Box X$ and we are done. ■

Proof for (T) and (S4) for S4, S4.2, S4R, S4F-frames: Straightforward since R is reflexive and transitive. ■

Proof for (R) for S4R-frames: Suppose $X; \neg\Box P$ is $S4R$-satisfiable, we have to show that $X; \neg P; \neg\Box P$ or $X; \neg\Box P; \neg P; \Box\neg\Box P$ is also $S4R$-satisfiable.

Suppose M $= \langle W, R, V \rangle$ is an $S4R$-model and $w_0 \in W$ and $w_0 \models X; \neg\Box P$. If M is a single nondegenerate cluster then there exists some $w_1 \in W$ with $w_0 R w_1$ and $w_1 \models \neg P$. Since R is reflexive, transitive and symmetric, $w_1 \models \neg P \Rightarrow \Box\neg\Box P$. Hence $w_0 \models X; \Box\neg\Box P$. Otherwise, if M is of type 2 then either w_0 *is* the simple (reflexive) cluster C_1 or it is in the last (reflexive, transitive and symmetric) nondegenerate cluster C_2.

If w_0 is the simple (reflexive) cluster C_1 and $w_0 \models \neg P$ then $w_0 \models X; \neg P; \neg\Box P$ by reflexivity of R and we are done.

If w_0 is the simple (reflexive) cluster C_1 and $w_0 \models P$, or if w_0 is in C_2, then there exists some $w_2 \in C_2$ with $w_0 R w_2$ and $w_2 \models \neg P$. Now if $w_0 \not\models \Box\neg\Box P$ then $w_0 \models \Diamond\Box P$ and hence all members of C_2 would satisfy P since R is reflexive, transitive and symmetric over C_2. In particular, $w_2 \models P$ giving a contradiction. Hence $w_0 \models X; \Box\neg\Box P$. ■

Proof for (S4.2) for S4.2-frames: Suppose $\langle W, R, V \rangle$ is an $S4.2$ model with some $w_0 \in W$ such that $w_0 \models U; \neg\Box P$. If $w_0 \models \Box\neg\Box P$ then we are done, otherwise $w_0 \not\models \Box\neg\Box P$, which is the same as $w_0 \models \Diamond\Box A$. Every $S4.2$-model is convergent and transitive so there must be a last cluster, and furthermore, every world w'' of this last cluster must make $\Box P$ true. Since w'' is in the last cluster we must have $w_i R w''$ for every world w_i with $w_0 R w_i$. That is, every such w_i satisfies $\Diamond\Box P$ and hence $w_0 \models \Box\Diamond\Box P; \Diamond\Box P$.■

Proof for (S4.2) for S4F-frames : Every $S4F$-frame is an $S4.2$-frame. ■

Note that, in general, the denominators of the transitional rules do not inherit all the formulae of the numerator. Thus, the transitional rules involve a loss of semantic information about the contents of the parent node.

4.2 *Completeness of the cut-free systems*

Theorem 2 (completeness) If X is a finite set of formulae and X is CL-consistent then there is an L-model for X on a finite L-frame where $L \in \{S4, K45, K45D, S4R\}$.

A formula $\neg\Box P$ is called an eventuality since it entails that eventually $\neg P$ must hold. A set w is said to fulfill an eventuality $\neg\Box P$ when $\neg P \in w$. A sequence $w_0 \prec w_1 \prec \ldots \prec w_m$ of sets is said to fulfill an eventuality $\neg\Box P$

when $\neg P \in w_i$ for some w_i in the sequence.

Proof for CS4 : The construction of the model graph is due to Rautenberg [Rau83] where $\prec$ denotes the immediate successor relation. By Lemma 1 (page 83) we can construct some *CS4*-saturated $X^* = w_0$ with $X \subseteq w_0 \subseteq X^*_{CS4}$. If no $\neg\Box P$ occurs in w_0 then $\langle\{w_0\}, \{(w_0, w_0)\}\rangle$ is the desired model graph since it is an *S4*-frame and (i)-(iii) are satisfied. Otherwise, let $Q_1, Q_2, \ldots Q_m$ be all the formulae such that $\neg\Box Q_i \in w_0$ and $\neg Q_i \notin w_0$.

Let $w' = \{P \mid \Box P \in w_0\}$. Since $\Box w' \subseteq w_0$, we know that $\Box w'$; $\neg\Box Q_i$ is *CS4*-consistent by (θ); hence so is each $X_i = \Box w'$; $\neg Q_i$, for $i = 1,\ldots,m$ by *(S4)*.

For each X_i we can find some *CS4*-saturated $v_i \supseteq X_i$, with $v_i \subseteq X^*_{CS4}$ by Lemma 1. Put $w_0 \prec v_i$, $i = 1,\ldots,m$ and call v_i the Q_i-successor of w_0. These are the immediate successors of w_0. Now repeat the construction with each v_i thus obtaining the nodes of level 2 and so on.

In general, the above construction of $\langle W, \prec\rangle$ runs ad infinitum. However, since $w \in W$ implies $w \subseteq X^*_{CL}$, a sequence $w_0 \prec w_1 \prec \ldots$ in $\langle W, \prec\rangle$ either terminates, or a node repeats. If in the latter case $n > m$ are minimal with $w_n = w_m$ we stop the construction and identify w_n and w_m in $\langle W, \prec\rangle$ thus obtaining a circle instead of an infinite path. One readily confirms that $\langle W, R\rangle$ is a model graph for X where R is the reflexive and transitive closure of $\prec$. Note that the clusters in $\langle W, R\rangle$ form a finite tree of finite nondegenerate clusters giving a *finite* model [Rau83].

Now $\langle W, R\rangle$ is an *S4*-model graph for X so by Lemma 2 (page 84), there exists an *S4*-model $\langle W, R, V\rangle$ which is an *S4*-model for X where $V : p \mapsto \{w \in W \mid p \in w\}$. ■

Proof for L = K45 and L = K45D: See [Shv89] for the original nonconstructive proofs based on a technique due to Mints. A constructive proof is given below based on Rautenberg's proofs.

Suppose X is *CK45*-consistent and create a *CK45*-saturated superset $w_0 \subseteq X^*_{CK45}$ of X according to the recipe given by the proof of Lemma 1. If no $\neg\Box P$ occurs in w_0 then $\langle\{w_0\}, \emptyset\rangle$ is the desired model graph since (i)-(iii) are satisfied.

Otherwise let $Y = \{Q_1, Q_2, \ldots, Q_k\}$ be all the formulae such that $\neg\Box Q_i \in w_0$ and let $w' = \{P \mid \Box P \in w_0\}$. Since X is *CK45*-consistent and $\Box w' \subseteq w_0$ then w'; $\Box w'$; $\neg\Box Y$; $\neg Q_i$ is also *CK45*-consistent by (θ) and *(45)* for each $i = 1,\ldots,k$.

By choosing one such set and applying all the static rules to this set we can form a *CK45*-saturated set $w_1 \subseteq X^*_{CK45}$ and put $w_0 \prec w_1$. We call w_1 a Q_i-successor of w_0 and denote this sequence by $S = w_0 \prec w_1$.

Now we can apply the same reasoning to obtain a successor for w_1. Continue construction of one such sequence $S = w_0 \prec w_1 \prec \ldots$ always choosing a successor w_{n+1} that fulfills an eventuality which is unfulfilled by the current S, and when this is impossible, choosing a successor that is new to the sequence. Since X^*_{CK45} is finite, we must sooner or later come to a node w_m such that the sequence $S = w_0 \prec w_1 \prec \ldots \prec w_m$ already contains *all* the successors of w_m. That is, it is not possible to choose a new successor. Choose the successor w_x of w_m that appears earliest in S and put $w_m \prec w_x$ giving $S = w_0 \prec w_1 \prec \ldots \prec w_x \prec \ldots \prec w_m \prec w_x$. There are two cases to consider depending on whether $x = 0$ or $x \neq 0$.

Case 1 : If $x = 0$, put R as the reflexive, transitive and symmetric closure of $\prec$ over $W = \{w_0, w_1, \ldots, w_m\}$. This gives a frame $\langle W, R\rangle$ which is a nondegenerate cluster of type 1.

Case 2 : If $x \neq 0$, put $W = \{w_0, w_x, w_{x+1}, \ldots, w_m\}$, discarding $w_1, w_2, \ldots, w_{x-1}$, and let R' be the reflexive, transitive and *symmetric* closure of $\prec$ over $W \backslash \{w_0\}$. That is, $R' = \{(w_i, w_j) \mid w_i \in W, w_j \in W, i \geq x, j \geq x\}$. Now put $R'' = R' \cup \{(w_0, w_x)\}$ and let R be the transitive closure of R''. The frame $\langle W, R\rangle$ is now of type 2.

Property (i) is satisfied by $\langle W, R\rangle$ by construction. We show that (ii) and (iii) are satisfied as follows.

Proof of (ii): The *(45)* rule carries *all* eventualities from the numerator to the denominator, including the one it fulfills. Therefore, for all $w_i \in W$ we have $\neg\Box P \in w_i$ implies $\neg\Box P \in w_m$. But we stopped the construction at w_m because no new successors for w_m could be found. Hence there is a P-successor for each eventuality $\neg\Box P$ of w_m. Since we have a cycle, and eventualities cannot disappear, these are all the eventualities that appear in the cycle. Furthermore, we chose w_x to be the successor of w_m that was earliest in the sequence S. Hence all of the eventualities of w_m are fulfilled by the sequence $w_x\ R \ldots R\ w_m$. All the eventualities of w_0 are also in w_m, hence (ii) holds.

Proof of (iii): The *(45)* rule carries all formulae of the form $\Box P$ from its numerator to its denominator. Hence $\Box P \in w$ and $w \prec v$ implies that P

$\in v$ and $\Box P \in v$. But we know that $w_x \prec \ldots \prec w_m \prec w_x$ forms a cycle, hence (iii) holds as well. ■

Proof for $L = K45D$: If the *(45D)* rule is ever used with no eventualities present then this can only happen when w_0 contains no eventualities [Shv89]. For if w_0 contained an eventuality then so would all successors. So if w_0 contains no eventualities and no formulae of the form $\Box P$ then $\langle\{w_0\}, \{(w_0, w_0)\}\rangle$ is the desired model graph. This gives a frame of type 1.

Otherwise, let $Y = \{Q_1,\ldots, Q_k\}$ be all the formulae such that $\neg\Box Q_i \in w_0$ and let $P_1,\ldots, P_m$ be all the formulae such that $\Box P_j \in w_0$. If Y is not empty then create a successor w_1 for w_0 using *(45D)* for some Q_i. Otherwise, create a successor w_1 using *(45D)* for some P_j. Continue creating successors in this exclusive-or manner using *(45D)*, always choosing a successor new to the sequence until no new successors are possible. Choose w_x as the successor nearest to w_0 giving a cycle $w_0 \prec \ldots \prec w_x \prec \ldots \prec w_m \prec w_x$ and discard $w_1, w_2, \ldots w_{x-1}$ as in the previous proof if necessary. As in the previous proof, $x = 0$ gives a frame of type 1 and $x \neq 0$ gives a frame of type 2. Properties (i)-(iii) can be proved in a similar manner. ■

Proof for S4R: Suppose X is $CS4R$-consistent, then no $CS4R$-tableau for X closes. Create a $CS4R$-saturated w_0 with $X \subseteq w_0 \subseteq X^*_{CS4R}$ as usual. If w_0 contains no eventualities then $\langle\{w_0\}, \{(w_0, w_0)\}\rangle$ is the desired model graph.

Otherwise let $E_0 = \{\neg\Box P \in w_0 \mid \neg P \notin w_0\} = \{\neg\Box P_1, \neg\Box P_2, \ldots, \neg\Box P_k\}$ be the eventualities of w_0 not fulfilled by w_0. Note that for any $CS4R$-saturated w with $\neg\Box P \in w$, we have either $\neg P \in w$ or $\Box\neg\Box P \in w$ by (R). In particular, we have $\Box\neg\Box P \in w_0$ for each member of E_0.

Let $w' = \{Q \mid \Box Q \in w_0\}$. Then $\Box w'$; $\neg P_i$ is $CS4R$-consistent by (θ) and $(S4)$ for each $i = 1\ldots k$. Create a $CS4R$-saturated successor w_1 for any $\neg\Box P \in E_0$ and put $S = w_0 \prec w_1$. Since w_1 contains $\neg P$ it must be different from w_0. Furthermore, $\Box\neg\Box P_i \in w_0$ so that $\Box\neg\Box P_i \in w_1$ and $\neg\Box P_i \in w_1$ by *(S4)* and *(T)* respectively for each $i = 1\ldots k$.

Since $CS4 \subseteq CS4R$ we can now create an $S4$-model-graph rooted at w_1 which is a finite tree of finite nondegenerate clusters. Consider any final cluster C of this $S4$-model-graph, discarding all other nodes, except w_0. If w_0 is duplicated in C then discard w_0 else put $S = w_0 \prec C$. Let R be the reflexive and transitive closure of $\prec$ over S or C as the case may be.

Any eventuality $\neg\Box P \in w_0$ is either fulfilled by w_0 itself because $\neg P \in w_0$, or it is carried into C by *(S4)* because $\Box\neg\Box P \in w_0$ by *(R)*. But if $\neg\Box P \in w_n \in C$ then the final cluster C fulfills $\neg\Box P$ within C. Properties (i)-(iii) then hold and hence $\langle W, R, V\rangle$ is a model graph for X where $V : p \mapsto \{w \in W \mid p \in w\}$.

■

4.3 *Completeness for Tableau Systems Containing Analytic Cut*

The tableau systems for *S4F* and *S4.2* require the analytic cut rule *(sfc)* of Smullyan for completeness. Adding analytic cut to the systems for *K45* and *K45D* simplifies the completeness proofs and gives a more direct satisfiability test, as shown below.

A set X is *subformula-complete* if $P \in Sf(X)$ implies that either $P \in X$ or $\neg P \in X$. A crucial consequence of adding analytic cut is the following lemma.

Lemma 3 If X is closed with respect to $\{(0), (\neg), (\wedge), (sfc)\}$ *or with respect to* $\{(0), (\neg), (\wedge), (sfcT), (T)\}$ *then X is subformula-complete.*

Proof: Obvious. ■

Also note that if $w \prec v$ then $v \subseteq \tilde{w}$ where $\tilde{w} = w \cup \neg w \cup \{0\}$; that is, the analytic cut rule can introduce new formulae of the form $\neg P$ into v only if $P \in Sf(w)$. This is crucial for the completeness proofs.

Theorem 3 (completeness) If X is a finite set of formulae and X is CL†-consistent then there is an L-model for X on a finite L-frame where $L \in \{K45, K45D, S4.2, S4F\}$.

Proof for K45: Suppose X is *CK45*†-consistent and create a *CK45*†-saturated superset w_0 with $X \subseteq w_0 \subseteq X^*_{CK45\dagger}$ as usual. If no $\neg\Box P$ occurs in w_0 then $\langle\{w_0\}, \emptyset\rangle$ is the desired model graph since (i)-(iii) are satisfied.

Otherwise, let $Q_1, Q_2, \ldots, Q_m$ be all the formulae such that $\neg\Box Q_i \in w_0$ and create a Q_i-successor v_i for each Q_i using the *(45)* rule. This gives all the nodes of level 1, so put $w_0 \prec v_i$, for each $i = 1 \ldots m$, and stop!

Consider any two nodes v_i and v_j with $i \neq j$. Using the facts that each node is subformula-complete and there are no building up rules, we show that (a) $\Box P \in v_i$ implies $\Box P \in w_0$ implies $P \in v_j$ and $\Box P \in v_j$; and (b) $\neg\Box P \in v_i$ implies $\neg\Box P \in w_0$ implies there exists a v_k such that $\neg P \in v_k$. Properties (iii) follows from (a) and property (ii) follows from (b).

Proof of (a): Suppose $\Box P \in v_i$. Then $\Box P \in Sf(w_0)$ and so $\Box P \in w_0$ or $\neg\Box P \in w_0$ since w_0 is subformula-complete. If $\neg\Box P \in w_0$ then $\neg\Box P \in v_i$ by *(45)*, contradicting the *CK45*†-consistency of v_i. Hence $\Box P \in w_0$. Note that this holds only because the *(45)* rule carries $\neg\Box P$ into its denominator along with $\neg\Box Y$.

Proof of (b): Suppose $\neg\Box P \in v_i$. Then $\Box P \in Sf(w_0)$ and so $\Box P \in w_0$ or $\neg\Box P \in w_0$ since w_0 is subformula-complete. If $\Box P \in w_0$ then $\Box P \in v_i$ by *(45)*, contradicting the *CK45*†-consistency of v_i. Hence $\neg\Box P \in w_0$. The crux of the proof is that *(45)* preserves all formulae of the form $\Box P$.

Hence we can put $v_i \; R \; v_j \; R \; v_i$ for all v_i and v_j giving a reflexive, transitive and symmetric nondegenerate cluster. If we also put $w_0 \; R \; v_i$ for all $i = 1 \ldots m$, then we obtain a *K45*-frame of type 2. If some $v_k = w_0$ then we obtain a *K45*-frame of type 1. In either case, (i)-(iii) are satisfied giving a model graph and hence a *K45*-model for X. ■

Proof for K45D: Similar to previous proof except that we create one single successor v using *(45D)* if w_0 contains $\Box$-formulae but contains no eventualities. It is easy to show $\neg\Box P \in v$ implies $\neg\Box P \in w_0$ which in turn implies that v also contains no eventualities. It is also easy to show that $\Box P \in v$ implies $\Box P \in w_0$ implies $P \in v$ so we can let $W = \{w_0, v\}$ and let $R = \{(w_0, v), (v, v)\}$. ■

Proof for S4.2: Suppose X is *CS4.2*†-consistent, then no *CS4.2*†-tableau for X closes. Construct some *CS4.2*†-saturated set w_0 containing X as usual.

Let $P_1, P_2, \ldots, P_k$ be the formula such that $\neg\Box P_i \in w_0$ and $\neg P_i \notin w_0$. These are the unfulfilled eventualities of w_0. Create a tree of successors using the (*S4*) rule as for *S4*, giving a finite tree of finite nondegenerate clusters. The tree is not an *S4.2*-model graph since it does not have a last cluster. We show that all the final clusters can be merged into one last cluster.

Every final cluster fulfills all its own eventualtities by construction so we need not worry about property (ii). Suppose C_1 and C_2 are two arbitrarily chosen final clusters and suppose that some node $c_1 \in C_1$ contains $\Box P$. We show that there exists some node $c_2 \in C_2$ with $\Box P \in c_2$. If this holds then we can form $C = C_1 \cup C_2 \cup \ldots \cup C_n$, where each C_i is a final cluster and let $wRw'Rw$ for all $w, w' \in C$ giving a last cluster.

Fact 1: If $w \prec v$ and $\neg\Box Q$ is marked in w then it is also marked in v. That is, any marked formula $\neg\Box Q \in w$ is the result of (T) on $\Box\neg\Box Q \in w$. Since $(S4)$ preserves $\Box$-formulae, we must have $\{\Box\neg\Box Q, \neg\Box Q\} \subseteq v$ where $\neg\Box Q$ is also marked.(1)

Fact 2: If $w \prec v$ and $\Box P \in v$ then $\Box P \in Sf(w)$.

Proof of Fact 2: The only way for $\Box$-formulae from outside $Sf(w)$ to appear in v is via the $(S4.2)$ rule. This rule is driven by formulae of the form $\neg\Box P$.

So suppose that some $\neg\Box P \in v$ is lifted by $(S4.2)$ to give $\Box\neg\Box P \in v$ or $\Box\neg\Box\neg\Box P \in v$. The $(S4.2)$ rule is not applicable to its own creations so $\neg\Box P$ itself was not the result of a building up operation. Hence $\neg\Box P \in \tilde{w}$ and $\Box P \in Sf(w)$, so either $\Box P \in w$ or $\neg\Box P \in w$. As usual, the former implies that $\Box P \in v$ giving a contradiction, so we must have $\neg\Box P \in w$. Now by Fact 1 we know that $\neg\Box P$ is non-starred in w, hence by $(S4.2)$ we have either $\Box\neg\Box P \in w$ or $\Box\neg\Box\neg\Box P \in w$.

Fact 3: If $w \prec v$ and $\neg\Box P \in v$ then $\Box P \in Sf(w)$. The proof is similar to the previous proof.

Fact 4 : By applying Facts 2 and 3 repeatedly we can show that *any* descendant v of w_0 must satisfy $v \subseteq \tilde{w}_0$.

So suppose $\Box P \in c_1 \in C_1$. Then by Fact 4 we must have $\Box P \in Sf(w_0)$, and hence either $\Box P \in w_0$ or $\neg\Box P \in w_0$. If $\Box P \in w_0$ then we are done because all the final cluster nodes are descendants of w_0 and hence will contain $\Box P$. Otherwise, $\neg\Box P \in w_0$. If $\neg\Box P$ is non-starred then by $(S4.2)$ we must have $\Box\neg\Box P \in w_0$ or $\Box\neg\Box\neg\Box P \in w_0$. If it is starred then we must have $\Box\neg\Box P \in w_0$.

If $\Box\neg\Box P \in w_0$ then $\Box\neg\Box P \in c_1$ and hence $\neg\Box P \in c_1$ by (T). But this contradicts the supposition that $\Box P \in c_1$; hence $\Box\neg\Box P \notin w_0$. This means that $\neg\Box P \in w_0$ cannot be starred and then by $(S4.2)$ that $\Box\neg\Box\neg\Box P \in w_0$.

Therefore $\{\Box\neg\Box\neg\Box P, \neg\Box\neg\Box P\} \subseteq c_i$ for all worlds $c_i \in C_2$. Since

(1) This aspect complicates the completeness proofs for multisets because we then have to check that this hold for this particular occurrence of $\neg\Box Q$.

C_2 fulfills all its own eventualities there must be some node $c_2 \in C_2$ that fulfills $\neg\Box\neg\Box P$; that is $\neg\neg\Box P \in c_2$ and hence $\Box P \in c_2$. But then $\Box P \in c_i$ for all $c_i \in C_2$ since C_2 is a final cluster, and we are done. ■

Proof for S4F: The proof is an amalgamation of the proofs for *CK45†*, *CS4R* and *CS4.2†* since the right denominator of the (*S4F*) rule is similar to the *(45)* rule and *CS4F†* also contains (*S4.2*). The first cluster is created using an argument similar to the *CK45†* proof and the second cluster is created as in the *CS4R* proof. There is no complication about last clusters; refer to Figure 5.

Suppose X is *CS4F†*-consistent. Create a *CS4F†*-saturated w_0 with $X \subseteq w_0 \subseteq X^*_{CS4F\dagger}$ as usual. If no $\neg\Box P$ occurs in w_0 then we are done since $\langle\{w_0\}, \{(w_0, w_0)\}\rangle$ is the desired model graph. Otherwise, let $w' = \{P \mid \Box P \in w_0\}$ and let $Y = \{P \mid \neg\Box P \in w_0\}$. As in the proof for *CS4.2†*, we know that $\neg\Box P \in w$ implies $\Box\neg\Box P \in w$ or $\Box\neg\Box\neg\Box P \in w$ by (*S4.2*) where $\neg\Box\neg\Box P$ is a starred formula. By (θ) and (*S4F*) we know that for each $\neg\Box P \in w_0$:

(a) $\Box w'$; $\neg P$; $\neg\Box Y$ is *CS4F†*-consistent or
(b) w_0 ; $\Box\neg\Box P$ is *CS4F†*-consistent.

Let $E_0 = \{\neg\Box P_i \in w_0 \mid \Box\neg\Box P_i \notin w_0 \text{ and } \neg P_i \notin w_0\}$. We restrict our attention to these eventualities because these are the unfulfilled and non-invariant eventualities of w_0. That is, these are the eventualities for which (b) fails so they *must* be fulfilled by the first cluster. Now do Step 1.
Step 1: If $E_0 = \{\neg\Box P \in w_0 \mid \Box\neg\Box P \notin w_0 \text{ and } \neg P \notin w_0\}$ is empty then go to Step 2. Otherwise, let $w = \{P \mid \Box P \in w_0\}$ and let $Y = \{P \mid \neg\Box P \in w_0\}$. By ($\theta$) and (*S4F*) for each formula P in Y the set $\Box w$; $\neg\Box Y$; $\neg P$ is also *CS4F†*-consistent via option (a) above. Create a *CS4F†*-saturated successor w_j for each member of E_0 giving the nodes $w_1, w_2, \ldots, w_m$ of level 1 where m is the number of formulae in E_0; this is shown in Figure 5. Let $C_1 = \{w_0, w_1, w_2, \ldots, w_m\}$ be the members of the first cluster. If all eventualities in w_0 are fulfilled then put $C_2 = \varnothing$ and go to Step 3, else do Step 2.
Step 2: Let $Z = \{\neg\Box P \in w_0 \mid \Box\neg\Box P \in w_0 \text{ and } \neg P \notin w_0\}$ and let $w' = \{P \mid \Box P \in w_0\}$. Since $(S4) \in CS4F\dagger$ the set $\Box w'$; $\neg\Box P$ must be *CS4F†*-consistent by (θ) and (*S4*) for each $\neg\Box P \in Z$. That is, we can create a successor v_1 for w_0 using (*S4*) instead of (*S4F*). Since $\neg P \in v_1$ we know that $v_1 \neq w_0$. By repeatedly using (*S4*) and mimicking the complete-

ness proof of *CS4* we can create an *S4*-model-graph F rooted at v_1 where F is a finite tree of finite nondegenerate clusters. Let C_2 be any final cluster of F; this is shown in Figure 5. Note that if $\neg\Box P \in Z$ then $\neg\Box P \in w$ for all $w \in C_2$ since $\Box\neg\Box P \in w_0$ and the *(S4)* rule preserves $\Box$-formulae. Furthermore, C_2 fulfills all its eventualities within C_2 by the fact that it is a final cluster of an *S4*-model-graph. In particular, it fulfills all the eventualities in Z. Now go to Step 3.

Step 3: Put $R = (C_1 \times C_1) \cup (C_2 \times C_2) \cup (C_1 \times C_2)$ giving an *S4F*-frame. We have to show that this frame satisfies properties (i), (ii) and (iii). We show these as follows. Property (i) is satisfied by construction.

Proof of Property (iii): Property (iii) is satisfied for most worlds by construction because both the *(S4F)* and *(S4)* rules preserve $\Box$-formulae and because *(T)* is present. The only exceptions are the nodes $w_1, w_2, \ldots, w_m$ of C_1 constructed in Step 1. For these nodes we show that $\Box P \in w_i$ implies $\Box P \in w_0$, from which (iii) follows.

Now, suppose $\Box P \in w_i$, for some $1 \leq i \leq m$. If $\Box P$ is not a result of a building up operation in w_i then $\Box P \in Sf(w_0)$ and hence either $\Box P \in w_0$ or $\neg\Box P \in w_0$ since w_0 is subformula-complete. If $\neg\Box P \in w_0$ then $\neg\Box P \in w_i$ since the *(S4F)* rule carries *all* the eventualities of its numerator into its (right hand) denominator. But then w_i contains both $\Box P$ and $\neg\Box P$ contradicting its *CS4F*†-consistency. Hence $\Box P \in w_0$. This part of the proof does not go through for *CS4.2*†; hence the footnote.

If $\Box P$ is built up in w_i then $\Box P$ must be of the form $\Box P = \Box\neg\Box Q$ or $\Box P = \Box\neg\Box\neg\Box Q$ for some non-starred $\neg\Box Q \in w_i$. Exactly as for *CS4.2*† we can prove that either $\Box\neg\Box Q \in w_0$ or $\Box\neg\Box\neg\Box Q \in w_0$. As usual $\Box P \in w_i$ implies $\Box P \in w_0$ regardless of the form of $\Box P$; hence $\Box P \in w_0$.

Proof of Property (ii): First, all the eventualities in C_2 must be fulfilled by members belonging to C_2 itself by its construction. So we can restrict attention to members of C_1. These are w_0 itself and the sets $w_1, w_2, \ldots, w_m$ created in Step 1.

Consider the eventualities in the nodes $w_1, w_2, \ldots, w_m$ of C_1 constructed in Step 1. We first prove that if $\neg\Box P \in w_i$ then $\neg\Box P \in w_0$ allowing us to restrict attention to the eventualities in w_0.

Suppose $\neg\Box P \in w_i$, is not built up. Then $\Box P \in Sf(w_0)$ and hence either $\Box P \in w_0$ or $\neg\Box P \in w_0$ since w_0 is subformula-complete. If $\Box P \in w_0$ then $\Box P \in w_i$ since *(S4F)* preserves $\Box$-formulae. Contradiction, hence

$\neg\Box P \in w_0$.

If $\neg\Box P$ is built up then it must be starred and of the form $\neg\Box P = \neg\Box\neg\Box Q$ for some non-starred $\neg\Box Q \in w_i$ with $\Box Q \in Sf(w_0)$. Hence $\neg\Box Q \in w_0$ or $\Box Q \in w_0$. The latter implies $\Box Q \in w_i$, giving a contradiction; hence $\neg\Box Q \in w_0$. Furthermore, $\neg\Box Q$ is non-starred in w_0 by Fact 1; hence by (*S4.2*) either $\Box\neg\Box Q \in w_0$ or $\Box\neg\Box\neg\Box Q \in w_0$. The former implies that $\Box P = \Box\neg\Box Q \in w_i$ giving a contradiction; hence $\neg\Box P = \neg\Box\neg\Box Q \in w_0$ by (*T*).

Therefore we can restrict attention to w_0. Then either $\Box\neg\Box P \in w_0$ or $\Box\neg\Box P \notin w_0$ for each eventuality $\neg\Box P \in w_0$. If $\neg\Box P \in w_0$ and $\Box\neg\Box P \notin w_0$ then either $\neg P \notin w_0$ and there exists a $w_i \in C_1$ such that $\neg P \in w_i$ by Step 1, or $\neg P \in w_0$. So all such eventualities of w_0 are fulfilled. Otherwise, if $\neg\Box P \in w_0$ and $\Box\neg\Box P \in w_0$ then we must have $\Box\neg\Box P \in v_i$ for all $v_i \in C_2$ and hence $\neg\Box P \in v_i$ by (*T*). But we already know that C_2 fulfills all its own eventualities and hence C_2 must contain some v such that $\neg P \in v$. ∎

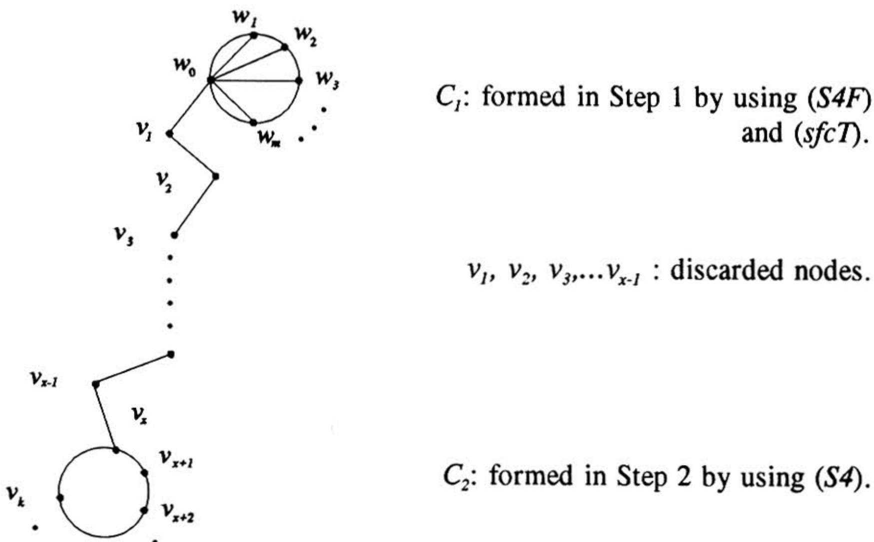

Figure 5: Countermodel construction of *S4F* model graph.

5. *Decision Procedures*

Each *CL* gives a nondeterministic procedure to decide whether or not some given (finite) formula A is a theorem of logic L. To test whether a formula

A is L-valid, we simply have to run a CL-tableau construction for $X = \{\neg A\}$. Since X^*_{CL} is finite, there are only a finite number of such CL-tableaux. If one of them is closed then, by soundness, $\{\neg A\}$ has no L-models, and hence A is L-valid. If none of these CL-tableaux closes then, by completeness, we can construct a finite L-model $\langle W, R, V\rangle$ which satisfies $\neg A$, hence A is not L-valid. We already know that the axiomatically formulated logic L is characterised by L-frames. That is, we know that a formula A is L-valid iff $\vdash_L A$. Therefore, each CL is a highly nondeterministic decision procedure for theoremhood in L.

Furthermore, since each completeness proof is constructive, each proof gives a *deterministic* procedure to test whether an arbitrary (finite) set of formulae X is L-satisfiable. However, it is known that the satisfiability problem is PSPACE-complete for propositional *S4* and NP-complete for propositional *S5* [HM85]. A similar argument shows that the satisfiability problem for *K45, K45D, S4R* and *S4F* are also NP-complete whilst the satisfiability problem for *S4.2* is PSPACE-complete.

6. *Eliminating Thinning and Contraction*

In all of our modal tableau systems, the only explicit structural rule is the rule of thinning (θ). The thinning (or weakening) rule introduces a form of nondeterminism where we have to guess which formula to throw away. That is, we have to guess which formulae are really essential to the proof. The (θ) rule can be eliminated from all our systems by building the effects of (θ) into the transitional rules. For example, in *CK45* we change *(45)* from

$$(45)\ \frac{\Box X;\ \neg\Box Y;\ \neg\Box P}{X;\ \Box X;\ \neg\Box Y;\ \neg\Box P;\ \neg P} \quad \text{to} \quad \frac{Y;\ \neg\Box P}{Y';\ \neg P;\ \neg\Box P}$$

where $Y' = \{Q \mid \Box Q \in Y\} \cup \{\Box Q \mid \Box Q \in Y\} \cup \{\neg\Box Q \mid \neg\Box Q \in Y\}$ and simultaneously change the basic axiomatic tableau rule from (0) to (0') as shown below

$$(0)\ \frac{P;\ \neg P}{0} \qquad\qquad (0')\ \frac{X;\ P;\ \neg P}{0}$$

This technique is used by Fitting [Fit83] and also works for all the other modal calculi we have presented.

Some of the tableau rules we have given are not standard; for example, the (T) rule is usually given as:

$$\frac{X;\ \Box P}{X;\ P}\quad (T')$$

where $\Box P$ is not carried from the numerator into the denominator [Rau83]. It is well known that the rule of contraction, which is implicit in the set formulation, then becomes *essential* for completeness. It is also well known that although contraction becomes essential, it is required *only* for $\Box$-formulae in most normal modal logics, and on both $\Box$-formulae and $\Diamond$-formulae in some symmetric normal modal logics [Fit88]. We have deliberately built contraction into our rules to highlight this fact. We believe that if we interpret ";" as multiset union, and rework our formulation using multisets instead of sets, then all the proofs will still go through with appropriate modifications. That is, the rule of contraction appears to be *eliminable* from our systems as long as the static rules build in contraction as given by our rules. Unfortunately, the proofs become very messy.

Therefore these systems are amenable to the Prolog implementation technique of Fitting [Fit88].

7. *Deducibility and Strong Completeness*

The notion of deducibility, $\Gamma \vdash_L A$, which we use is stronger than the usual one in modal logic where the weaker notion of theoremhood, $\vdash_L A$, is standard. The semantic notion of characterisation by a class of frames must be strengthened to mirror this change. A modal logic L is *strongly characterised* by a class of frames C if for every set of formulae Γ and every formula A,

$$\Gamma \vdash_L A \text{ iff } \forall V, \text{ if } \langle W, R\rangle \in C \text{ and } \langle W, R, V\rangle \models \Gamma \text{ then } \langle W, R, V\rangle \models A.$$

It is known that the weak characterisation results of modal logics can be strengthened to strong characterisation results if the modal logic L is canonical and the class of frames C contains the canonical frame for L [McD82, MST91]; see Hughes and Cresswell [HC84] for the meanings of canonicity and canonical frame. The logics we study are known to be strongly characterised by the class of frames shown in Figure 2 because these logics and the corresponding L-frames satisfy these two conditions.

We can extend our tableau systems to decide whether Γ deduces A, that is, if $\Gamma \vdash_L A$. The trick is to start a tableau for $\Gamma \cup \{\neg A\}$ and then to simply add all members of Γ to each new tableau node as it is created. The set Γ now has "global" status, the proofs of soundness and completeness still go through, and the completeness proofs remain constructive. We are now searching for a countermodel for A where every world satisfies Γ. If we find a closed tableau then we know that no such countermodel is possible and hence, by soundness, that any L-model validating Γ must validate A; that is, $\Gamma \vdash_L A$. If we find no closed tableau then, by completeness, we can construct an L-model validating Γ and containing a world w_0 which falsifies A. Hence $\Gamma \nvdash_L A$.

An easier way to achieve the same effect is to utilise the fact that all of our transitional rules preserve $\Box$-formulae. That is, for $L \in \{K45, K45D\}$ we run a normal tableau for Γ; $\Box\Gamma$; $\neg A$ to decide whether $\Gamma \vdash_L A$. For $L \in \{S4, S4R, S4.2, S4F\}$ we run a normal tableau for $\Box\Gamma$; $\neg A$. In each case, the members of Γ will be added to each node automatically by CL.

Let $\hat{\Gamma}$ denote the conjunction of all formulae in Γ. Then, the fact that our tableau rules extend to handle (strong) deducibility indicates that a form of the deduction theorem goes through because: $\Gamma \vdash_L A$ iff $\vdash_L (\widehat{\Box\Gamma} \wedge \hat{\Gamma}) \Rightarrow A$ for $L \in \{K45, K45D\}$ and $\Gamma \vdash_L A$ iff $\vdash_L \Rightarrow \widehat{\Box\Gamma}\, A$ for $L \in \{S4, S4R, S4.2, S4F\}$.

A caveat is in order because our tableaux procedures terminate only because we deal with *finite* sets. Consequently, we cannot handle first order nonmonotonic modal theories where the infinite set Γ contains all ground instances of the formulae in the (real) first order theory Γ'.

8. *Sequent Systems*

For each tableau rule there is an analogous sequent rule associating the tableau set X with the sequent $\Sigma \longrightarrow \Delta$ by putting $X = (\Sigma \cup \neg\Delta)$ and turning the tableau rule upside down. For example, the sequent analogue of the tableau rule *(45)* is

$$\frac{\Box\Gamma \longrightarrow \Box\Delta, \Box A, A}{\Box\Gamma \longrightarrow \Box\Delta, \Box A}\,(\rightarrow \Box\text{: } K45)$$

Thus, each tableau system CL has an analogous sequent system SL.

These sequent systems then give a Gentzen characterisation of theoremhood $\vdash_L$ via the equivalence: $\vdash_L A$ iff the sequent $\longrightarrow A$ is provable in *SL*. In order to handle (strong) deducibility Fitting [Fit83] introduces the notation $\Gamma \Vdash \Sigma \longrightarrow A$ where $\Sigma \longrightarrow A$ takes the role mentioned above and $\Gamma \Vdash \Sigma \longrightarrow A$ takes the role of $\Gamma \vdash_L \hat{\Sigma} \Rightarrow A$. Note that the sequent arrow changes to classical implication in this transformation and $\hat{\Sigma}$ denotes the conjunction of all formulae in Σ. We can obtain $\Gamma \vdash_L A$ by putting Σ to be empty.

Some sort of distinction must be made between Γ and Σ because the former are like "global assumptions" while the latter are like "local assumptions" [Fit83]. If we work with transitive logics then we can use the same trick as in our tableau method by asking whether the sequent Γ, $\Box\Gamma$, $\Sigma \longrightarrow A$ is provable, allowing us to drop the distinction between Γ and Σ. This trick will not work for non-transitive logics because the tableau/sequent rules no longer remain sound with the intended semantics.

9. *Related Work*

The cut-free tableau systems *CK45* and *CK45D* are the tableau versions of Schwarz's sequent systems for *K45* and *K45D* [Shv89]. The systems *CK45*† and *CK45D*† are based on the work of Rautenberg [Rau83]. The advantage of *CK45*† and *CK45*† is that the associated deterministic satisfiability test is purely local. That is, we need only a two level graph representation, and there is no need to check for repeated nodes.

Apparently Serebriannikov has also obtained a sequent system for *S4.2* but I have been unable to trace this work, let alone determine if this system is cut-free. I know of no other systems for *S4F* or *S4.2* although the idea of the (*S4.2*) rule is due to Rautenberg [Rau83].

Zeman [Zem73] gives a tableau system for *S4R* which he calls *S4.4* but Zeman's system is not cut-free, requiring analytic cut. Zeman also gives a sequent system for *S4R* which is an amalgamation of his sequent systems for *S4* and *S5*. Note however that Zeman explicitly uses an *S5* system and his system for *S5* is not cut-free whereas we simply add one extra static rule to *CS4* to obtain *CS4R*.

Fitting [Fit83] gives strongly complete analytic tableau systems for many modal logics including *S4* but does not give (strongly complete) systems for the logics we have considered. Incidentally, Rautenberg's technique also gives a strong *analytic* tableau system (containing an analytic cut rule) for

S5.

Catach [Cat88] has implemented an automatic theorem prover in Prolog for many modal logics. His method is essentially the "method of semantic diagrams" of Hughes and Cresswell [HC68] where the reachability relation is represented explicitly and the various constraints like reflexivity and transitivity are also enforced explicitly. We have *deliberately* avoided this method because it does not yield a proof theory for the logic in question. However, such "explicit" methods can be used to implement the deterministic decision procedures obtained from our completeness proofs.

10. *Further Work*

Segerberg [Seg71] shows that *S4F* can also be axiomatised as *S4.3* plus the axiom schema $\Diamond (\Box P \land \Diamond \Box Q \land \neg Q) \Rightarrow P$ where *S4.3* itself is *S4* plus the axiom schema $\Box(\Box P \Rightarrow Q) \lor \Box(\Box Q \Rightarrow P)$. Elsewhere we have given cut-free tableau systems for *S4.3* and other Diodorean modal logics [Gor92]. We believe that a much more elegant system for *S4F* is possible based on the system for *S4.3*.

Note that some of the completeness proofs (and hence the corresponding L-satisfiability tests) can be made more efficient; that is, a more careful choice of successor using global considerations may lead to a smaller counter-model. Our counter-model constructions for *CS4R* and *CS4F*† are parasitic on the basic *CS4* counter-model construction because, in both cases, we have to wait until "eventually this process cycles". Are there more direct counter-model constructions for *CS4F*† like the ones for *CK45*†? Does adding analytic cut to *CS4R* help?

The (weak) interpolation theorem for modal logics states that if $\vdash_L A \Rightarrow B$ then there exists some formula C such that $\vdash_L A \Rightarrow C$; and $\vdash_L C \Rightarrow B$; and $vars(C) \subseteq vars(A) \cap vars(B)$ where $vars(F)$ means the set of propositional variables that appear in formula F.

Rautenberg [Rau83] gives a very general (weak) interpolation theorem for a wide range of logics with tableau rules of a specific form. Our tableau rules for *CK45*, *CK45D*, *CS4*, *CS4.2*† fall into the scope of his theorem giving proofs that the (weak) interpolation theorem holds for these logics. We conjecture that the interpolation theorem does not hold for *S4F* because of its intimate relationship to the Diodorean modal logic *S4.3* which is known not to have interpolation. It is known that there are connections between the interpolation theorem and circumscription via Beth definability.

This may shed light on the connections between modal nonmonotonic logics and circumscription.

11. *Conclusions*

We have given (semi)analytic tableau systems for the propositional (monotonic) modal logics *K45*, *K45D*, *S4R*, *S4F* and *S4.2*. Each system provides a nondeterministic decision procedure for theoremhood $\vdash_L A$ and each completeness proof gives a deterministic L-satisfiability test. Each tableau system has a sequent analogue giving a Gentzen cut-elimination theorem for *S4R* from the fact that *CS4R* is cut-free. These systems extend trivially to handle (strong) deducibility, $\Gamma \vdash_L A$, as long as (strong) deducibility allows necessitation on members of the theory Γ. This is usually forbidden in modal logic but is essential for nonmonotonic modal logics.

Recent results indicate that the *nonmonotonic* modal logics based upon *K45*, *K45D*, *S4R*, *S4F* and *S4.2* elegantly generalise various *nonmodal* nonmonotonic formalisms like autoepistemic logics and default logics. Indeed, the currently known algorithms for computing the nonmonotonic consequences of a theory Γ depend on decision procedures for (strong) deducibility in the underlying monotonic modal logic. Therefore our systems are important for automating nonmonotonic deduction in both modal and nonmodal formulations. The limitation is that Γ must be finite, but when this holds, our tableau calculi are directly implementable in Prolog.

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