Logique & Analyse **127-128** (1989), 215-239

EEEE: SET THEORY AND WHOLENESS

Wayne D. BLIZARD[1]

1. *Introduction and abstract*

The acronym EEEE abbreviates 'everything enfolds everything else" where the word 'enfolds' means 'contains within itself'. A theory in which some interpretation of the EEEE principle holds, is said to possess (the property of) *wholeness*; that is, each part contains, within itself, the whole (more precisely, the near-whole: every other part). We first discuss classical set theory and wholeness. We then interpret EEEE set theoretically and investigate whole set theory. A method is described by which classical set theory can be made to manifest a degree of wholeness, and further lines of investigation are suggested. A brief discussion of various examples of the EEEE principle is given in the Appendix.

2. *Classical set theory and wholeness*

To what extent, if any, does classical set theory manifest wholeness? By 'classical set theory' we mean the formal theory ZFC (Zermelo-Fraenkel set theory with the axiom of choice). We find that ZFC exhibits wholeness only in a very limited sense. However, in Section 5 we describe a procedure that introduces a much greater degree of wholeness into ZFC.

The formal theory ZFC that exists today has evolved slowly over the last 100 years. Two points of view about the nature of sets have influenced the evolution of ZFC: the *limitation of size* principle and the *iterative concept of set*. The concept of 'wholeness' would seem to be incompatible with both points of view. The *limitation of size* principle states that paradoxes arise in set theory if 'very large sets' are allowed to exist and, therefore, that set theory is safe from paradox if only 'modest sized' sets are permitted. The very idea that a set contain *everything* else (either as elements, or in its

[1] Research supported by the Natural Sciences and Engineering Research Council of Canada.

transitive closure) violates limitation of size.

In the *iterative concept of set*, every set appears at some stage of construction for the first time, and the elements of that set must appear at some earlier stage of construction. If a set is to contain *all* other sets (either as elements, or in its transitive closure) then it will contain some constructed at a later stage, thereby, violating the iterative concept of set. The idea that a set constructed at α, may contain a set constructed at β, where $\beta > \alpha$, is paradoxical only if one assumes (as the iterative concept demands) a sequential construction of sets; that is, that some sets are constructed 'before', or 'after', other sets. Clearly, the idea of wholeness would seem to run counter to the iterative concept of set. For a full discussion of limitation of size, see Hallett [10]; for alternative views of the iterative concept of set, see the papers by Boolos, Parsons and Wang in [4]; and for a concise account of how these two points of view shaped the selection of axioms of ZFC, see Maddy [12].

The cumulative hierarchy of classical sets (the class $\bigcup_{\alpha \in On} V_\alpha$ where $V_0 = \phi$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ and $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$, if λ is a limit ordinal) is the embodiment of both the iterative concept of set and the principle of limitation of size. By the very definition of the hierarchy, if one is located at level $\alpha \in On$, and one is constructing *new* sets, then the only candidates (the only sets available) for elementhood are sets constructed at levels $\beta < \alpha$. A set constructed at α cannot contain an element that is not constructed until level $\gamma > \alpha$. Classical sets, therefore, contain as elements only sets that are simpler in nature (constructed earlier in the hierarchy). In other words, every set in the transitive closure of a set must be of lesser rank. Classical sets do not 'contain' all other sets. The EEEE principle, therefore, does not hold in the universe of ZFC sets.

In classical set theory, we equate the class of all sets $V = \{x \mid x = x\}$ with the cumulative hierarchy $\bigcup_{\alpha \in On} V_\alpha$. In ZFC, therefore, $V = \bigcup_{\alpha \in On} V_\alpha$.

If we restrict our attention only to the class On of ordinals (a subclass of V; in fact, the 'backbone' of V), we find that every ordinal α is exactly the set of all other ordinals that are less than α; that is, for every $\alpha \in On$,

$$\alpha = \{\beta \in On \mid \beta < \alpha\} .$$

This is certainly closer to the EEEE principle than is the case for arbitrary sets in V. An ordinal contains exactly everything else less than itself, but it cannot contain an ordinal greater than or equal to itself. Thus, once again, EEEE fails for classical ordinals.

In ZFC, the *rank* of a set x (the least $\alpha \in On$ such that $x \in V_{\alpha+1}$, or $x \subseteq V_\alpha$) is a measure of the possible complexity of the set; that is, it determines the maximum finite length of any descending epsilon chain within the transitive closure, as well as the complexity of sets that may belong to the transitive closure. Such a hierarchy of complexity is Darwinian in nature: complexity builds from within, from earlier and simpler levels of complexity.

However, there is one sense in which one might argue that the EEEE principle holds in the universe V of ZFC. We note that the entire universe V is constructed from the empty set ϕ, using only the powerset operation $\mathcal{P}$ and the union operation $\bigcup$. Classical sets in ZFC are well-founded; that is, the axiom of foundation is an axiom of ZFC. Let x be an arbitrary but fixed set of ZFC. Beginning with any element y of x, one can move to any element z of y, and continue in this way. The result is a descending epsilon chain of the form $\ldots \in z \in y \in x$. If one moves down through the elements of x, the elements of elements of x, ...et cetera, one reaches the empty set ϕ after some finite number of steps. Well-foundedness is exactly the property that every descending epsilon chain is finite. No matter which epsilon chain one chooses, one always reaches ϕ (and, therefore, the chain stops) after a finite number of steps. What we have described here is *pure* set theory (for *pure* mathematicians): V is constructed from the empty set ϕ. In many *applications* of set theory, one needs *urelements* (individuals, atoms) that do not contain elements, but that can serve as elements of sets. We discuss set theory with urelements in Section 5. In the universe V of 'pure sets', every element of every set sits atop a network of finite epsilon chains, all with ϕ at their base. Within every set, at the base of every epsilon chain, lies the empty set ϕ. One could say, therefore, that 'within every set' lies the *potential* to construct the entire universe V ($V_0 = \phi$ is the foundation of V). Every reader will agree, however, that this is a rather unsatisfactory attempt to ascribe properties of wholeness to ZFC. We describe a method of 'injecting' wholeness into ZFC in Section 5.

When we come to investigate the properties of whole set theory, we find that whole sets are not well-founded. We allow infinitely descending epsilon chains, but we do not allow self-membership. At this point, non-well-foundedness of whole sets should come as no surprise. It is after all precisely the axiom of foundation that allows one to prove that the universe of ZFC (the class $V = \{x \mid x = x\}$) is equal to the cumulative hierarchy of sets (the class $\bigcup_{\alpha \in On} V_\alpha$). (In fact, the axiom of foundation is equivalent to the assertion $V = \bigcup_{\alpha \in On} V_\alpha$ ([9] pp. 94-95).) In other words, the axiom of foundation allows one to prove that the concept of set is identical to the iterative concept of set. Without foundation, one has only that $\bigcup_{\alpha \in On} V_\alpha$ is a subclass of V; that is, the concept of set includes the iterative concept of set. Without foundation, alternative (non-iterative) concepts of set are possible.

Giving up foundation (allowing infinitely descending epsilon chains, while disallowing self-membership) is not a serious difficulty. The status of the axiom of foundation has always been more 'tentative' than that of the other axioms of ZFC. As Fraenkel, Bar-Hillel and Levy note, each one of the other axioms of ZFC '...was taken up because of its essential role in developing set theory and mathematics in general; if any single axiom were left out we would have to give up some important fields of set theory and mathematics .... The case of the axiom of foundation is, however, different; its omission will not incapacitate any field of mathematics' ([9] p. 87). More specifically, Jech states 'This restriction on the universe of sets ...is irrelevant for the development of ordinal and cardinal numbers, natural and real numbers, and in fact of all ordinary mathematics' ([11] p. 70). Fraenkel, Bar-Hillel and Levy adopt the very sensible position of accepting the axiom of foundation '...not as an article of faith but as a convention for giving a more restricted meaning to the word 'set', to be discarded once it turns out that it impedes significant mathematical research' ([9] p. 89). We may now be at that point. Interesting set theories (and models of set theory) have been developed in the absence of the axiom of foundation and, in some cases, assuming a form of its negation. (See, for example, Section 5. 5, Chapter II (in particular, footnote 1, p. 101) of [9], Quine's set theory [9] pp. 161-167, Barwise and Etchemendy [2] Chapter 3, Aczel [1] and Parker-Rhodes [13].) A preliminary history of non-wellfounded set theory is given in Appendix A of [1]. Historical remarks can also be found in [2] p. 58. For our purposes at least, the axiom of foundation must be discarded if any reasonable notion of wholeness is to be investigated.

In giving up the axiom of foundation, we must give up the cumulative hierarchy of sets, the notion of rank and $\in$-induction and recursion. Indeed, we must give up the very idea of an ordered sequence of construction of sets in which some sets appear 'before' and 'after' other sets. For wholeness, all sets appear simultaneously, and every set is available for elementhood in every other set. In giving up foundation, however, we need not give up any of the other axioms of ZFC. The axiom of foundation and forms of its negation are consistent with, and independent of, the other axioms of ZFC ([9] pp. 98-102, [1] Introduction, p. 170 ff and [2] pp. 44-48). We can, therefore, assume that we are working against a background set theory

ZFC^- (ZFC without the axiom of foundation) in which we are free to use foundation, to work in the absence of foundation, or to introduce anti-foundation assumptions (certain non-wellfounded sets exist) without fear of introducing inconsistency.

3. *Whole set theory: a proposal*

We interpret EEEE set-theoretically in the simplest and most obvious way: every set is an element of every other set. We investigate whether it is meaningful to 'do' set theory subject to such a requirement. We assume throughout that we are working within the first-order predicate calculus with equality against a meta-set-theoretic background like ZFC^-. The symbol $=$ is, therefore, a logical symbol, and we assume the usual equality axioms ([9] p. 25).

Define a *universe* $U = \{x_i\}_{i \in I}$ of distinct elements ($x_i = x_j$ iff $i = j$) labelled by elements $i, j, k, \ldots$ of an arbitrary index set I. Elements of U are called wsets (whole sets). For simplicity, we assume $I = \{1, 2, \ldots, n\}$ for some arbitrary but fixed positive natural number n. The number n may be as large as required, but at least $n = 3$. The reason for this lower limit is explained below. We emphasize that, in principle, the index set I need not be a set of numbers, need not be finite, nor even countably infinite. All results apply equally well for arbitrary sets I. For our purposes, we have $U = \{x_1, x_2, \ldots, x_n\}$.

For EEEE to hold in U, we require that every wset contain every other wset as an element; that is,

$$\begin{aligned} x_1 &= \{x_2, x_3, \ldots, x_n\} \\ x_2 &= \{x_1, x_3, \ldots, x_n\} \\ &\;\,\vdots \\ x_n &= \{x_1, x_2, \ldots, x_{n-1}\} \end{aligned}$$

and, in general,

$$x_j = \{x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\}.$$

There are, therefore, exactly n distinct wsets each containing $(n - 1)$ distinct elements (exactly all *other* wsets). Since wsets and elements of wsets are elements of U, U is the *universe* $\{x \mid x = x\}$ of wsets. No other wsets are permitted. In particular, we do not allow arbitrary subsets of U like ϕ, $\{x_1\}$, $\{x_1, x_2, x_3\}$ or even U itself. The universe $U = \{x \mid x = x\}$ is exactly the set $\mathcal{P}_{n-1}(U)$ containing all $(n - 1)$-element subsets of U.

For arbitrary index sets I, we would not be able to display the elements of wsets as we did above. We would attempt to define wsets using class term definitions like

$$x_i = \{x_j \in U \mid x_j \neq x_i\} \text{ or}$$
$$y = \{x \in U \mid x \neq y\}$$

where we use variable symbols $x, y, z, \ldots$ to denote arbitrary wsets in U. However, these are *impredicative* definitions: that which is being defined (the term on the left) appears in the definition itself (the expression on the right). Such definitions are clearly unsatisfactory.

We, therefore, define a binary membership relation $\in$ on U as follows:

$$x \in y \text{ stands for } x \neq y.$$

In words, x belongs to y just in case x is distinct from y. In U, therefore, everything is an element of everything else. In our case, there are $n \cdot (n - 1)$ possible membership relationships. The relation $\in$ is the $n \cdot (n - 1)$-element subset of $U \times U$ containing all non-diagonal ordered pairs.

The definition

$$x \in y \text{ stands for } x \neq y$$

is the most straightforward way to interpret EEEE set theoretically — everything contains within itself everything else. It states that 'belonging' is coextensive with 'distinctness' (or membership is equivalent to difference). Taking the negation of both sides, we find that non-membership is equivalent to identity. In other words, x is not an element of y, if and only if, x equals y. The 'if' direction is the standard (classical) statement of non-self-membership. It is the 'only if' direction that makes distinctness and identity the *sole* criterion of membership and non-membership. Thus, wsets can be thought of as generalized 'ordinary sets' (non-self-belonging) since membership is totally determined by distinctness. At the same time, wsets are 'extraordinary' in the sense that infinitely descending epsilon chains exist.

In the meta-set theory, the universe U is *transitive* — elements of elements of U are elements of U, or every element of U is also a subset of U. But the relation $\in$ itself is not a transitive relation on U since $x \in y \wedge y \in z$ does not, in general, imply $x \in z$ (since $x = z$ is possible). However, since $=$ is transitive, $\notin$ is transitive on U; that is,

$$\forall x\, \forall y\, \forall z\, ((x \notin y \wedge y \notin z) \rightarrow x \notin z).$$

Since we have assumed the equality axioms of reflexivity, symmetry, transitivity and substitutivity ([9] p. 25) as part of the underlying logic of wset theory, we also have immediately

$$\left.\begin{array}{c} \forall x\, (x \notin x) \\ \forall x\, \forall y\, (x \in y \leftrightarrow y \in x) \\ \forall x\, \forall y\, (x \neq y \leftrightarrow (x \in y \wedge y \in x)) \end{array}\right\} \qquad (*)$$

and, for every formal statement $\Phi(x)$, if $\Phi(x)$ holds and $x \notin y$, then $\Phi(y)$ holds:

$$\forall x\, \forall y\, ((\Phi(x) \wedge x \notin y) \rightarrow \Phi(y)).$$

In other words, all the properties of wset membership follow directly from the properties of equality in the underlying logic.

The property of wsets expressed in the formulae (*) can be summarized in words: distinct wsets are mutual members or co-belong. This is simply a formal manifestation of EEEE — everything belongs to everything else, or everything is an element of everything else.

The two predicates $=$ and $\notin$ (or, equivalently, $\neq$ and $\in$) are co-extensive; that is,

$$\forall x\, \forall y\, (x = y \leftrightarrow x \notin y) \text{ or}$$
$$\forall x\, \forall y\, (x \neq y \leftrightarrow x \in y).$$

We do not really need both predicates. In particular, we do not really need the binary predicate $\in$. However, we use $\in$ to get a set-theoretic 'feel' for the nature of wsets. Either predicate can be defined as the negation of the other. We use both, however, to assist us in conceptualizing exactly what EEEE means for wsets. We need $\in$ to interpret EEEE set-theoreticall-y.

We require that U be a *set* of *n distinct* elements where $n \geq 3$. We must be able to interpret the notions of 'everything' and 'everything else' in a meaningful way. If U were a *multiset* (a set with repeated elements) containing indistinguishable elements, then the notion of 'else' breaks down. For example, if $U = [x]_n$ which contains n copies of x and nothing else, there is nothing else in U other than x itself. Elements of U must be distinguishable. We, therefore, require that U be a set of *distinct* elements. If U is empty, then the concepts 'everything' and 'everything else' are vacuous. If $n = 1$, say $U = \{x\}$, then the concept 'else' breaks down.

The classical axiom of foundation is

$$\forall y\,(y \neq \phi \rightarrow \exists x\,(x \in y \wedge x \cap y = \phi)).$$

Every non-empty classical set contains an element from which it is disjoint. To understand why this axiom fails for wsets, is to understand why we require $n \geq 3$. If one scans the elements x_i of an wset y in search of an element that is disjoint from y, one fails. Every element x_i of y has an element in common with y; namely, any *other* element x_j in y. For this to be possible, y must contain *at least two* elements and, therefore, U must contain at least three elements. With $n \geq 3$, foundation fails for every wset.

In the case $n = 2$, where $U = \{x_1, x_2\}$, we have $x_1 = \{x_2\}$ and $x_2 = \{x_1\}$. There *are* infinitely descending epsilon chains; namely, $\ldots \in x_1 \in x_2 \in x_1$ and $\ldots \in x_2 \in x_1 \in x_2$. However, the statement of the axiom of foundation holds: each wset is disjoint from its single element (otherwise, either $x_1 \in x_1$ or $x_2 \in x_2$). This is unsatisfactory. In the limiting case (for $n = 3$ where $i \neq j \neq k$), we have typically $x_i = \{x_j, x_k\}$. In this case, x_i and x_j are not disjoint since x_k belongs to both, and x_i and x_k are not disjoint since x_j belongs to both. Therefore, x_i is not disjoint from any of its elements. We, therefore, require $n \geq 3$.

For wsets, we have a strong anti-foundation property: every wset is non-empty and has elements in common with all of its elements. Stating this in the formal meta-set theoretic language, we have

$$\forall y\,(y \neq \phi \wedge \forall x\,(x \in y \rightarrow x \cap y \neq \phi)).$$

This is stronger than simply the negation of the axiom of foundation —

$$\exists y\,(y \neq \phi \wedge \forall x\,(x \in y \rightarrow x \cap y \neq \phi)).$$

Not only does there exist a non-empty non-wellfounded wset, but every wset is non-empty and non-wellfounded.

In classical set theory, there exists a unique set ϕ such that $\forall x\ (x \notin \phi)$. For wsets, every wset is non-empty. Further, for every wset, there is a unique wset that does not belong to it (the wset itself):

$$\forall y\ \exists!\ x\ (x \notin y).$$

This is, of course, trivial. For any y, put $x = y$. Then $x \notin y$. For any other wset x' such that $x' \notin y$, we have $x' = x = y$.

For distinct wsets $x \neq y$, since $x \in y \wedge y \in x$ holds, there is no concept of 'rank'. A wset y cannot come into existence 'for the first time' since y itself is an element of each of its elements. All wsets exist simultaneously: there is no ordered sequence of construction, no hierarchy of wsets. This is in the very nature of non-wellfoundedness. Epsilon chains

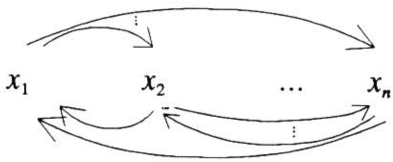
$$\ldots \in x \in y \in z \in \ldots$$

continue infinitely in both directions. The only restriction is that neighbours in the chain must be distinct ($x \in y \rightarrow x \neq y$). For wsets in $U = \{x_1, x_2, \ldots, x_n\}$, the longest finite epsilon chains of distinct elements are of length n. They are $x_1 \in x_2 \in \ldots \in x_n$, or any chain obtained from $x_1 \in x_2 \in \ldots \in x_n$ by a permutation of the labels $1, 2, \ldots, n$. Chains of greater length contain repeated elements, but never as neighbours.

Using Mirimanoff's definition of 'extraordinary' ([1], epigraph), wsets are extraordinary in the extreme — *every* descending epsilon chain is infinite. Many authors confuse the notions of non-wellfoundedness (the existence of infinitely descending epsilon chains) and self-membership (the existence of an x such that $x \in x$). It is important to emphasize that the two notions are not equivalent. Self-membership certainly implies non-wellfoundedness, but non-wellfoundedness need not imply self-membership (the wsets being a case in point).

Since wsets are non-wellfounded, they do not have rank. To have a rank within the iterative concept of set means that there is a stage α in *On* at which every set x comes into being. Before α, the set x simply does not exist and after α, the set x occurs at every level of the hierarchy. For wsets, on the other hand, every wset x coexists with every other wset y, and they coexist simultaneously, always. There is no sequential time concept in wset theory. For wsets, every epsilon chain is an infinitely ascending and descending epsilon chain. Thus, the notion of 'wholeness' is expressed not only in terms of the parts and the whole, but also in terms of the simultaneity of the existence of the entire universe U.

Using the notation of Aczel [1] (see also Chapter 3 in [2]), we can represent the membership patterns of wsets as follows:

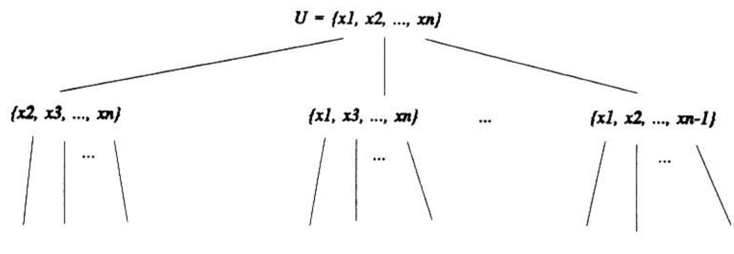


There is an arrow from every x_i to every *other* x_j. Here (as in [1]), $x_i \rightarrow x_j$ (for $i \neq j$) represents the relationship $x_j \in x_i$, but (unlike [1]) all $x_i \rightarrow x_i$ are not allowed. The notation $x_i \rightarrow x_j$ for $x_j \in x_i$ may appear unnatural. However, it is derived from the notation of 'epsilon trees' in set theory in which, quite naturally, the set in question rests at the top (the root of the tree), and downward arrows represent upward membership:

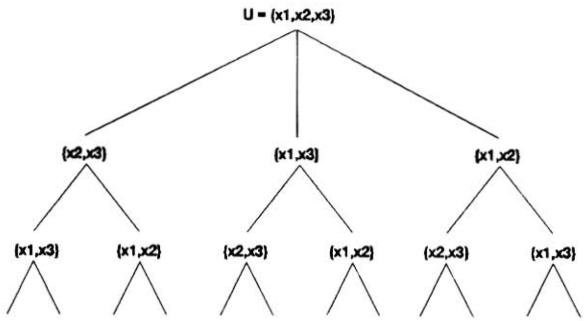
$$\begin{array}{cc} \{\,\phi, & \{\,\phi\,\}\,\} \\ \downarrow & \downarrow \\ \phi & \{\,\phi\,\} \\ & \downarrow \\ & \phi \end{array}$$

where $\phi \in \{\phi, \{\phi\}\}$, $\{\phi\} \in \{\phi, \{\phi\}\}$ and $\phi \in \{\phi\}$.

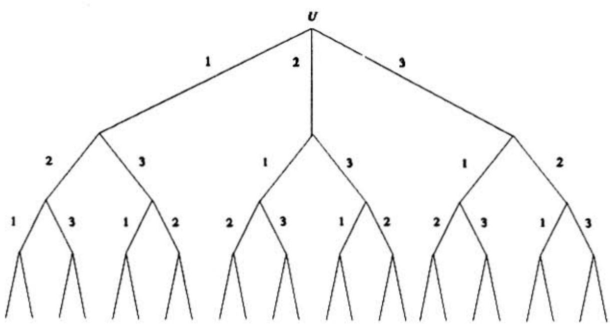
Although Aczel's arrow diagrams are elegant and compact, they are less perspicuous than the corresponding epsilon trees. By replacing each element in U by the explicit set of its elements, we obtain a tree structure —



where the nodes of the tree are wsets (elements of U) and the edges of the tree represent equality of wsets (the name of the wset equals the explicit list of its elements). The simplest tree structure when $U = \{x_1, x_2, x_3\}$ is

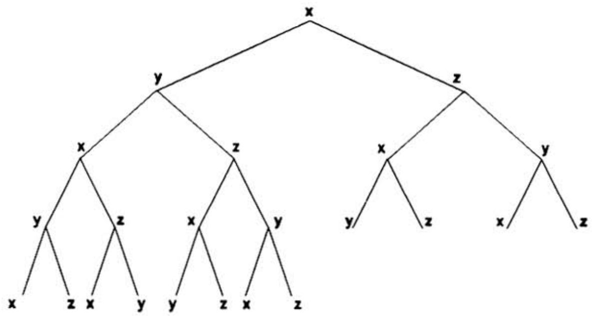


Since each edge is equality of wsets, we can simplify the above tree by labelling the edges rather than the nodes. The result is the binary tree structure



The 'element-structure' of these wsets (each wset contains the other two — typically, $x_i = \{x_j, x_k\}$) is reflected in their tree structure (the tree structure below each node is exactly the tree structures of the other two nodes). In the meta-set theory, the transitive closure of any element of U is U; that is, $\forall x \; TC(x) = U$.

If we rewrite $U = \{x_1, x_2, x_3\}$ as $\{x, y, z\}$ where $x = \{y, z\}$, $y = \{x, z\}$ and $z = \{x, y\}$, and look at the tree structure of the transitive closure of any element of U (using x, for example), we obtain the epsilon tree:



If one collects together the elements at each level of this tree structure, one obtains a *multiset* with elements x, y and z. The theory of multisets is described in [5]. Moving from top to bottom in the above tree, the successive levels give the multisets $\{x\}$, $\{y, z\}$, $[x, y, z]_{2,1,1}$, $[x, y, z]_{2,3,3}$, $[x, y, z]_{6,5,5}$, $[x, y, z]_{10,11,11}$, ... The multiplicity of an element at level $n + 1$ of the tree equals the sum of the multiplicities of the other elements at level n. This makes sense, since $x \notin x$ but $x \in y$ for all $y \neq x$. Therefore, x will occur at level $n + 1$ exactly the number of times all $y \neq x$ occur at level n. Since we began arbitrarily with x, there will always be one more x than y's or z's (the number of y's and z's is always equal) followed by one less x than y's or z's. Therefore, the number of x's relative to the number of y's at level $n \geq 2$ is simply $(-1)^n$. Had we started with y instead of x, then the number of x's and z's would be equal and the number of y's would alternate from one less to one more than the number of x's and z's.

Wsets can be thought of as generalizations of classical ordinals. An ordinal is the collection of all other ordinals 'less than itself'. There is an assumption of ranking and hierarchy. For wsets, one simply drops the expression 'less than itself' since there is no notion of ranking or hierarchy for wsets. The result is that a wset is the collection of all other wsets.

We can use our meta-set theoretic formalism to investigate the properties of wsets. For every wset x in U, $x = U - \{x\}$. For distinct wsets $x \neq y$, $x \cup y = U$ and $x \cap y = U - \{x, y\}$. For wsets x, $\bigcup x = U$ and $\bigcap x = \{x\}$ (that is, x is the only wset that belongs to every one of its own elements). We also have that $x \subseteq y \leftrightarrow x = y$ holds. There are no empty wsets since $\forall x\ (x \neq \phi)$, and there are no disjoint wsets since $\forall x\ \forall y\ (x \cap y \neq \phi)$. For all wsets x, $x \cup \{x\} = U$; that is, each wset is very large in the sense that it differs from U by a single element (in its lack of self-membership). For classical sets, if x has rank α, then $x \cup \{x\}$ has rank $\alpha + 1$.

For ordinals, $\alpha \cup \{\alpha\} = \alpha + 1$. For wsets, $x \cup \{x\}$ is the universe U. More colourfully, add anything to itself and the result is everything. We, therefore, say that wsets are 'near-universal'. If U is infinite, then every wset is infinite with one less element than U. Wsets are 'very large' also in the sense that they are 'very far away from' ϕ. Wsets although distinct, are also 'very much alike' in the sense that they differ from each other only by single elements (each other). One might say that wsets are as much alike as they can be without being identical. This similarity is evident in the fact that their intersection is 'large' ($x \cap y = U - \{x, y\}$).

It is important to emphasize that meta-set theoretic expressions like ϕ, U, $\{x\}$, $x \cup y$, $x \cap y$, $\bigcup x$, $\bigcap x$, $x - y$, $\{x, y\}$, ...do *not* denote wsets. There are no wsets that correspond to these expressions. In our case, every wset contains exactly $(n - 1)$ elements, no less, no more. To develop a simple algebra of wsets, one must allow wsets containing n, $n-1$, $n-2$, ..., 2, 1 and 0 elements. One could define, for example,

$$\begin{aligned} x \cup y &= \{z \in U \mid z \in x \vee z \in y\}, \\ x \cap y &= \{z \in U \mid z \in x \wedge z \in y\}, \text{ or} \\ \phi &= \{z \in U \mid z \in z\}. \end{aligned}$$

To do this would expand the universe of wsets from $\mathcal{P}_{n-1}(U)$ to the full powerset $\mathcal{P}(U)$. Such a system would still possess 'wholeness' in the sense that every non-empty wset would contain at least one element of U, and that one element contains all other elements of U. However, in such a system, the equivalence $x \in y \leftrightarrow x \neq y$ breaks down. Wsets would not contain all other wsets; for example, U does not contain any $\{x\}$. We do not take this course, and we define wsets as before.

For wsets there is a symmetry of belonging: just as every element of a wset belongs to that wset, the wset itself belongs to every one of its elements (in fact, the only wset to do so). Such an unusual property of mutual membership is similar to that in the Theory of Sorts (a *sort* is a collection of indistinguishables) of A. F. Parker-Rhodes ([13], axiom 4. 1, p. 57), although sorts do admit limited self-membership (elementary sorts are their own members [13] p. 68).

4. *Whole sets as process*

We noted earlier that not only is every wset one step removed from the universe U, but every wset is also very similar to every other wset. Wsets are as much alike as they can be without becoming identical. We may think of the universe U not as a *static* collection of wsets, but rather as a *dynamic* universe in which wsets are constantly 'becoming' each other. T1 3 4ry, and more as a system in a constant *process* of becoming.

In U, for x to become something *else*, it must become one of its elements, since it consists of exactly everything else. To become one of its elements, x simply takes the place of that element (substitutes itself for that element) and the resulting wset is exactly the wset that is substituted for.

By 'x becomes y', which we denote by $x \to y$, we mean the following effective procedure:

1. scan the elements of x,
2. locate the element y,
3. substitute x for y.

The input to this effective procedure is the wset x and the output is the wset y. Steps 1 and 2 will terminate because $y \in x$. Step 3 is: delete y and add x in its place, or even more simply, write x over y. Since x differs from y only in this single element, the result must be the wset y.

We can also denote the process $x \to y$ by an equation (a substitution rule):

$$y = x\,[x/y]$$

where x $[x/y]$ is the result of substituting x for y in x.

It is interesting to note that one mathematical aspect of wholeness has found expression in Grassmann's early 'algebra of becoming' ([8] pp. 140-142) and Prigogine's 'science of becoming' ([8] pp. 208-209). The universe described by Briggs and Peat is a '...universe where everything affects everything else — an alive, multidimensional, creative reality where the observer is the observed, the laws of nature evolve, and wholeness is a flowing' ([8] p. 209).

We also note that Aczel's interest in non-wellfounded set theory and anti-foundation properties began with the modelling of concurrent *processes* in computer science, and a graduate mathematics course at Stanford called 'Sets and Processes' ([1] Forward, Preface, p. 111). In his words, 'The original stimulus for my own interest in the notion of a non-wellfounded set came from a reading of the work of Robin Milner in connection with his development of a mathematical theory of concurrent processes. This topic in theoretical computer science is one of a number of such topics that are generating exciting new ideas and intuitions that are in need of suitable mathematical expression' ([1] p. xix). In Chapter 8 of [1], Aczel applies his non-wellfounded set theory to communicating systems, incorporating Milner's ideas on concurrent processes.

5. *Classical sets made whole*

We now give a brief description of a procedure that introduces a degree of wholeness into classical set theory. We begin with the theory ZF. An *urelement* is an object that can serve as an element of sets, but that has no elements itself. In these two respects, urelements are similar to the empty set ϕ. In pure set theory, there is only one such individual — the empty set ϕ itself. In set theory with individuals, one allows any number of individuals, in addition to ϕ itself. There are a variety of formal techniques to generalize ZF set theory into ZFU set theory with urelements (see, for example, [3] pp. 7-11, [9] pp. 23-25, especially footnote 1, p. 25, [14] pp. 19-56 and [11] pp. 198-199). We are not concerned here with the formal details. The universe V^U (a cumulative hierarchy of sets with individuals) of ZFU is constructed as follows: let U be a *set* of urelements, and define

$$V_0 = U$$
$$V_{\alpha+1} = \mathcal{P}(V_\alpha) \cup U$$
$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha, \text{ if } \lambda \text{ is a limit ordinal, and}$$
$$V^U = \bigcup_{\alpha \in On} V_\alpha.$$

Thus, urelements appear as elements of sets at every stage $\alpha \in On$ of the hierarchy. An element of a set, therefore, is either an urelement, or a set of lesser rank.

Within ZFU, one establishes the following conventions: arbitrary elements of V^U are denoted by x, y, z ...; arbitrary urelements in U are denoted by p, q, r, ...; and arbitrary *sets* (elements of $V^U - U$) are denoted by a, b, c, ... The axiom of extensionality of ZFU, the universal closure of

$$\forall x\,(x \in a \leftrightarrow x \in b) \rightarrow a = b,$$

establishes *set* equality. Nothing is said about equality of urelements. Had the axiom read

$$\forall x\,(x \in y \leftrightarrow x \in z) \rightarrow y = z$$

then all urelements would be equal to the empty set ϕ, since they contain no elements. The axiom of foundation of ZFU is the universal closure of

$$a \neq \phi \rightarrow \exists x\,(x \in a \wedge \ \sim\exists y\,(y \in a \wedge y \in x)).$$

Thus, urelements may serve as $\in$-minimum elements of non-empty sets. In ZFU, the relation = (equality of sets) behaves classically, and the relation $\in$ is wellfounded. The variable symbols p, q, r, ...may appear to the left of $\in$, but never to the right of $\in$ (except in the case $\sim x \in p$).

We now have the theory ZFU in place with a *set* U of urelements. The *set* U may be a finite or infinite collection of *distinct* urelements. We can, therefore, extend the = predicate from sets in V^U to all of V^U. Therefore, = is well-defined on U and $p = q$ and $p \neq q$ make sense in U. We define a binary predicate E on U as follows:

$$p \mathrm{\ E\ } q \text{ stands for } p \neq q.$$

This makes U a universe of whole sets with respect to the membership relation E. Every urelement belongs to every other urelement, as in Section 3. There is, therefore, a form of E-extensionality for urelements; namely,

$$\forall r\,(r \mathrm{\ E\ } p \leftrightarrow r \mathrm{\ E\ } q) \rightarrow p = q.$$

All the results of Section 3 apply to the set U, and the predicates E and =.

We call the result of these modifications the theory ZFU*. The theory ZFU* contains the binary predicates $\in$ and =, and a defined binary predicate E (logically equivalent to $\neq$). The predicate $\in$ is well-founded (on V^U), but the predicate E is non-wellfounded (on U).

Since we want 'pure sets' (sets containing no urelements in their $\in$-transitive closure) to exhibit wholeness, we extend the predicate E such that $\forall p\,(p \mathrm{\ E\ } \phi)$ holds. The variable symbols p, q, r, ...refer *only* to urelements and not to ϕ.

From the point of view of $\in$, ϕ is the empty set ($\forall x\,(x \notin \phi)$) and U is the universe of urelements ($\forall p\,(p \in U)$). From the point of view of E, the universe U is empty (since $\forall p\,(p \mathrm{\ E\ } U)$, or strictly, $p \mathrm{\ E\ } U$ is not defined), but ϕ is the 'universe' of urelements (since $\forall p\,(p \mathrm{\ E\ } \phi)$). The predicate $\in$ is still wellfounded (every non-empty *set* has an $\in$-minimal element) where-as E is not (no urelement has an E-minimal element). In ZFU*, ϕ and U play dual roles with respect to $\in$ and E. This is not surprising since U has replaced ϕ as the foundation of the cumulative hierarchy of sets.

We claim that ZFU* exhibits wholeness. Let x be an arbitrary element of the universe of ZFU*. If x is an urelement (if $\exists p\ (x = p)$ holds), then x exhibits wholeness since it contains, as E-elements, every other urelement. If, on the other hand, x is a set (if $\exists a\ (x = a)$ holds), then either x contains an urelement in its $\in$-transitive closure ($\exists p\ (p \in TC\ (x))$ holds), or it is a pure set ($\forall p\ (p \notin TC\ (x))$ holds). If there is an urelement in $TC\ (x)$, then x 'contains within itself' all other urelements. Every descending $\in$-chain is finite, ending with either ϕ, or some $p \in U$. Every descending E-chain is infinite. Every such set x 'contains' all of U (in its E-transitive closure, which equals $TC\ (x) \cup U$). Therefore, every such x contains the foundation ($V_0 = U$) from which the entire universe V^U is constructed.

If, on the other hand, $TC\ (x)$ is free of urelements, then $x \in V = \bigcup_{\alpha \in On} V_\alpha$ where $V_0 = \phi$. In this case, every $\in$-chain is finite and ends with ϕ. With respect to E, however, ϕ 'contains' all the urelements in U. Therefore, x itself 'contains' the foundation from which the entire universe V^U is constructed.

We have grafted wholeness into ZFU by using a universe U of urelements in which properties of wholeness apply. Every downward $\in$-chain is finite, ending with ϕ or some urelement $p \in U$. However, below ϕ and every $p \in U$, every E-chain is infinite. Every set in ZFU* contains (the foundation of) every other set in ZFU*. Since we have incorporated the theory of whole sets (developed in Section 3) into the theory ZFU*, it exhibits a greater degree of wholeness than either ZFU or ZF.

6. *Other set-theoretic interpretations of EEEE*

In Section 3, we interpreted EEEE strictly as 'every set contains (as an element) every other set'. In sections 2 and 5, we loosened our interpretation of EEEE to include 'every set contains (within its transitive closure, the foundation of, or the potential for) every other set. In this section, we briefly consider closely related and alternative set-theoretic interpretations of EEEE. As such, these are proposals for possible future investigation. When one first considers set-theoretic interpretations of EEEE, the idea of interpreting 'enfolds' as 'contains as a subset' suggests itself. This is equivalent to interpreting 'enfolds' as 'contains as an element of its power-set'.

Writing this formally we have

$$\forall x\, \forall y\, (x \neq y \rightarrow x \in \mathcal{P}(y)).$$

Unfortunately, any reasonable notion of a subset relation $\subseteq$ includes anti-symmetry:

$$\forall x\, \forall y\, ((x \subseteq y \wedge y \subseteq x) \rightarrow x = y).$$

To require that everything 'contain' everything else as a subset, is to claim that everything is equal to everything else; that is, under EEEE,

$$\forall x\, \forall y\, (x \subseteq y \leftrightarrow y \subseteq x).$$

In other words, the universe consists of a single set. This is clearly unsatisfactory.

If, however, we read 'else' strictly as in Section 3, then we must use proper subset $\subset$ instead of $\subseteq$. In this case, $x \not\subset x$ holds, and $\subset$ is clearly not anti-symmetric. However, the fact that EEEE requires that

$$\forall x\, \forall y\, (x \subset y \leftrightarrow y \subset x) \text{ hold,}$$

means that the relation $\subset$ does not correspond to any classical notion of proper subset. However, in multiset theory, it is possible to define a subset relation such that unequal multisets are proper subsets of each other. Meyer and McRobbie define 'subset' such that $[x, y]_{1,4} \neq [x, y]_{2,3}$ but $[x, y]_{1,4} \subset [x, y]_{2,3}$ and $[x, y]_{2,3} \subset [x, y]_{1,4}$ since every element of one multiset is also an element of the other multiset (see [5] for details).

If one attempts to translate EEEE as 'every set contains every other set in the sense that the transitive closure of the second is a subset of the transitive closure of the first', one obtains

$$\text{'}x \text{ contains } y\text{' means } TC\,(y) \subseteq TC\,(x).$$

Since EEEE requires that this hold for all distinct sets, one is forced to the conclusion that all sets must have the same transitive closure:

$$\forall x\, \forall y\, (x \neq y \rightarrow TC\,(x) = TC\,(y)).$$

Since this is trivially true when $x = y$, we have

$$\forall x \, \forall y \, (TC\,(x) = TC\,(y)).$$

The idea here is that every set is made up of exactly the same parts, but those parts may be arranged differently into distinct sets. For example, in classical set theory we have the distinct sets $\{\{\phi\}\}$ and $\{\phi, \{\phi\}\}$ with the same transitive closure. More striking still, the infinite ordinal $\omega^+ = \{0, 1, 2, \ldots, \omega\}$ and the finite singleton set $\{\omega\}$ although *very* different, have the same transitive closure.

Interpreting 'enfolds' as 'contains as an element', suggests that we may also interpret 'enfolds' as 'contains as an element, or as an element of an element'. Thus, we have

$$\text{`}x \text{ contains } y\text{' means } y \in x \cup \bigcup x.$$

Such loosening of the interpretation can be pushed deeper and deeper into the transitive closure of x,

$$\text{`}x \text{ contains } y\text{' means } y \in x \cup \bigcup x \cup \bigcup^2 x$$
$$\text{`}x \text{ contains } y\text{' means } y \in x \cup \bigcup x \cup \bigcup^2 x \cup \bigcup^3 x$$
$$\vdots \qquad\qquad\qquad \vdots$$

until we reach

$$\text{`}x \textit{ contains } y\text{' means } y \in TC\,(x)$$

where $y \in TC\,(x)$ means $y \in \bigcup^n x$ for some natural number $n \geq 0$. We have thus generalized 'contains as an element' to 'contains as an element of some finite iterated union'.

One could, therefore, interpret EEEE set-theoretically, as

$$\forall x \, \forall y \, (x \neq y \rightarrow x \in TC\,(y)).$$

In other words, every set contains every *other* set as an element of its transitive closure. Every set contains every other set at some level of its construction. In the creation of such sets, everything else is used, and nothing is wasted. Thus, for $x \neq y$, the transitive closures $TC\,(x)$ and $TC\,(y)$ are almost equal but their 'structure' may be very different. The transitive closure of every such set is a class ($TC(x) = V - \{x\}$, $TC(y) = V - \{y\}$). Each such set sits atop a transitive closure that contains everything else. For $x \neq y$, $TC(x)$ and $TC(y)$ are 'very large' and differ from each other only in a single element. Although $TC(x)$ and $TC(y)$ are very similar, the sets x and y themselves may be very different (the transitive closures are structured differently). This can all be done without contradicting the classical requirement $x \notin TC(x)$. However, there is no notion of 'rank' since within each set is found every other set.

Such alternative attempts at a set-theoretic characterization of wholeness, require a much deeper understanding of the structure and properties of the transitive closure of a set, as well as the very nature of well-foundedness. For example, in ZFC, self-membership contradicts the axiom of foundation, since $x \in x$ allows the $\in$-chain

$$\ldots \in x \in x \in x.$$

Further, a set cannot belong to its transitive closure, since $x \in TC(x)$ (or, $x \in \bigcup^n x$ for some $n \geq 0$) implies the existence of a finite $\in$-chain $x_1 \in x_2 \in \ldots \in x_m$ for which $x_1 = x_m = x$ (that is, an $\in$-loop). Such a finite $\in$-loop allows the $\in$-chain

$$\ldots \in x_{m-1} \in x_1 \in x_2 \in \ldots \in x_m$$

which contradicts the axiom of foundation.

Further, for any $y \in TC(x)$, y can occur at most once along any $\in$-chain (since otherwise a finite $\in$-chain with y at its end points exists). The set y may occur, however, along different $\in$-chains 'within' x (as long as it occurs at most once in any single $\in$-chain 'within' x).

In classical set theory, the requirement $x \notin TC(x)$ (or, $\forall n\, (x \notin \bigcup^n x)$) is much more restrictive than simply $x \notin x$. It may be possible (in ZFC$^-$, for example) to disallow extended self-membership (to require $x \notin \bigcup^n x$ for $0 \leq n \leq N$ where N is some fixed depth), while allowing 'deep self-membership' in the transitive closure (that is, $x \in \bigcup^n x$ for some $n > N$, need not be inconsistent). In such a scheme, a $y \in TC(x)$ could occur more than once along a single $\in$-chain 'within' x as long as the resulting finite $\in$-loop $x_1 \in x_2 \in \ldots x_m$ where $y = x_1 = x_m$ is such that $m > N$. If this is the case, then $y \in TC(y)$ satisfies the requirement above: $y \notin \bigcup^n y$ for $0 \leq n \leq N$. The theory ZFC$^-$ with these modifications could be said to possess 'limited foundation'. This permits a step toward wholeness.

How well does the classical transitive closure of a set characterize that set? If $TC(x) = \phi$, we know that $x = \phi$. If $TC(x) = x$, we know that x is a transitive set. If $y \in TC(x)$, then $TC(y) \subseteq TC(x)$. If $x \subseteq y$, then $TC(x) \subseteq TC(y)$. However, the set $TC(x)$ says nothing about how often its elements are used in the construction of x, and nothing about the $\in$-order in which they are used in the construction of x. The assertion $y \in TC(x)$ says nothing more than the rank of y is less than the rank of x, and that y is involved somewhere in the construction of x. The converse of the logical truth

$$\forall x \, \forall y \, (x = y \rightarrow TC(x) = TC(y))$$

is false, as we have seen. Two *very* different sets (one, a very finite singleton set; the other, an infinite ordinal) may have the same transitive closure. It is, therefore, fair to say that the set $TC(x)$ does not characterize the set x very well. $TC(x)$ tells us every constituent part of x down to ϕ, but little else.

Since $TC(x)$ is a *set* (elements occur at most once), it cannot *count* the number of times a set $y \in TC(x)$ is used in the construction of the set x (that is, the number of different $\in$-chains 'within' x containing y). A *multiset* (a set with repeated elements) transitive closure could keep track of the multiplicity of y's in $TC(x)$ (see [5] for details of multiset theory). Denote such a multiset transitive closure of x by $MTC(x)$. If $y \in^m MTC(x)$, then m equals the number of different $\in$-chains 'within' x containing y (equals 'the number of times' $TC(y)$ is a subset of $TC(x)$). Such a multiset $MTC(x)$ could be defined as a cardinal-valued function in ZFC with domain $TC(x)$. The root set $MTC(x)^*$ equals $TC(x)$ (in other words, $MTC(x)$ and $TC(x)$ have exactly the same elements). $MTC(x)$ characterizes x better than does $TC(x)$ since the plurality of y's in $TC(x)$ is taken into account. The multiset $MTC(x)$ is still *transitive* with respect to the classical $\in$. The multiset $MTC(x)$ could be useful in the characterization of classical sets.

APPENDIX : THE EEEE PRINCIPLE

The EEEE principle has a distinguished history and possesses (for many) a deep aesthetic appeal. Most recently, the quantum physicist, David Bohm, has espoused the notion of 'the implicate order' in which any element contains enfolded within itself the totality of the universe (see, for example, his [6]). In Bohm's own words, '...each separate and extended form in the explicate order is enfolded in the whole and ..., in turn, the whole is enfolded in this form ...The way in which the separate and extended form enfolds the whole ...is essential to what that form *is* and how it acts, moves and behaves quite generally' ([7] p. 41). An introduction to David Bohm's ideas on quantum mechanics, hidden variables and the implicate order can be found in [6], [7] and [8]. For our purpose, the central idea in Bohm's thought is the notion that every element contains, enfolded within itself, the totality (every other element).

The idea enunciated in the EEEE principle is, of course, a very old one. Poets, philosophers, mystics, theologians, cosmologists...have stated similar ideas since ancient times. We have, for example, Jorge Luis Borges: '...each thing implies the universe' and '...everything is an infinity of things'. Blake wrote, 'To see the world in a grain of sand ...And eternity in an hour.' Leibniz's monads, as infinite conjunctions of predicates, contain within themselves all possible contingencies. Leonardo da Vinci wrote, 'Every body placed in the luminous air spreads out in circles and fills the surrounding space with infinite likenesses of itself and appears all in all and all in every part.' St. Paul said, 'We are members of one another.' Schopenhauer held that any individual is all individuals: whatever one person does, it is as if all persons do the same thing. Stated in the first person, 'I am all other men'. Most striking of all, we find in Buddhism, *The Flower Garland Sutra*: 'In the heaven of Indra, there is said to be a network of pearls, so arranged that if you look at one you see all the others reflected in it. In the same way each object in the world is not merely itself but involves every other object, and in fact *is* every other object.' ([8] pp. 275, 276, 279). Doubtless, the reader can supply other examples.

Since antiquity, there has been a mystical view of reality which holds that what we observe as a vast variety of differentiated things, is in actuality, simply the various manifestations of the same thing; that is, that there exists an underlying 'oneness' hidden beneath the observed surface of 'manyness'.

More recently, the EEEE principle has manifested itself in two interesting ways. It is known that a DNA molecule, taken from an organism, contains within it all the genetic information for the entire organism. This fact makes cloning possible — a complete organism can be manufactured, or grown, from a single cell. The other example is the hologram (or, what has come to be known as the holographic paradigm). Unlike a photographic negative, a holographic plate contains the complete image in any one of its parts. If any part of a holographic plate is removed (or, indeed, if any part of the plate is used), the complete image still appears ...only with diminished intensity. In both examples, it is accurate to state that every part contains every other part.

We note an apparent similarity between properties of whole sets (discussed in Section 3 and 4) and those of *bootstrap hadrons* proposed by Geoffrey Chew ([8] p. 203). Each hadron (a strongly interacting particle like a proton or a neutron) is said to *contain* (consists of, involve) all other hadrons. In Chew's words, "Each hadron plays three different roles: it may be a 'constituent' of a 'composite structure', it may be 'exchanged' between constituents and thus constitute part of the force holding the structure together, it may itself *be* the entire composite." (*Science*, Vol. 161 (August 23, 1968), 762-765). The word 'particles' applied to such hadrons is misleading since bootstrappers think of hadrons as 'intermediate states in ongoing process webs of energy' ([8] p. 203).

Wholeness is also found in the new mathematics of chaotic dynamics, or complex dynamics: the study of fractal geometry. One of the central concepts of fractal geometry is the property of *self-similarity* (or, *scaling*). Self-similarity is the characteristic property of fractals that a small portion, when magnified, can reproduce exactly a large portion (or, in other words, the parts are similar to the whole, only on a reduced scale). This is clearly another example of the EEEE principle. The fact that many aspects of the natural world have been successfully modelled using fractals, indicates that reality itself may possess hidden properties of self-similarity.

A final example of wholeness may be found in the surprising properties of Penrose tilings. A periodic tiling repeats a pattern of tiles over and over again like wallpaper (there exists a smallest region that generates the whole tiling by periodic repetition without rotation). Periodic tilings, therefore, manifest wholeness. In 1974, Roger Penrose showed that the plane can be tiled in infinitely many different ways (all of which are non-periodic) using just two different types of tiles (called 'kites' and 'darts' because of their shape) that fit together only in certain allowable ways. Each such tiling has a high degree of symmetry, but never repeats itself as do periodic tilings. What is remarkable is that given any finite region of a Penrose tiling of the plane, that same finite region occurs infinitely often elsewhere in the same tiling *and* infinitely often in every other Penrose tiling of the plane. Therefore, an examination of any finite region (no matter how large) of any of these tilings can never determine which of the infinitely many tilings is being examined. Every Penrose tiling of the plane contains any finite region of any other Penrose tiling of the plane. This is a very nice example of the

EEEE principle in modern mathematics. If one interprets Penrose tilings cosmologically, then infinitely many universes can be constructed from two simple types of 'atoms', and any finite region of any one such universe repeats infinitely often in any of the other universes.

Department of Pure Mathematics
and Mathematical Statistics,
University of Cambridge.

Current address:
Department of Mathematics and Statistics
University of Victoria
P.O. Box 3045
Victoria B.C. V8W 3P4
Canada.

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