ON PARACONSISTENT SET THEORY

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Introduction

A theory T is called inconsistent if it contains contradictory theorems, i.e. theorems such that one of them is the negation of the other; otherwise T is called consistent. T is called trivial when the class of its theorems coincides with the class of all formulas (or closed formulas) of its language; otherwise, T is called nontrivial. A logic is said to be paraconsistent if it can be the underlying logic of inconsistent but nontrivial theories. A theory based on a paraconsistent logic is called paraconsistent.

In previous works (for example [5], [8] and [9]), I formulated paraconsistent set theories, based on first-order paraconsistent logics. In these systems of set theory Russell's set, $\{x : x \notin x\}$, does exist, and as a consequence of this fact they are inconsistent, though apparently nontrivial. Moreover, I proved that if my systems are nontrivial, then the corresponding classical set theories, which served as partial motivations for my systems, are consistent. Naturally, a more significant result would be to prove the converse of this theorem, since the classical systems are more intuitive and at first sight more secure than the paraconsistent set theories.[1]

In the present paper, I demonstrate that if the classical systems of set theory correlated with my systems are consistent then the latter are nontrivial. So, certain inconsistent but apparently non-trivial systems of set theory are as trustworthy as the standard set theories, and conversely.

It is interesting to observe that my paraconsistent set theories are in a precise sense stronger than the corresponding classical theories.

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(1) The first formulations of my systems were trivial (for details, see [2] and [3]).

1. *The systems* NF_i

The basic language of all systems to be treated here has the following primitive symbols: 1) Connectives: $\supset$ (implication), & (and), $\vee$ (or) and $\neg$ (negation); the symbol of equivalence, $\equiv$, is introduced as usual. 2) Individual variables: a denumerably infinite collection of variables. 3) The quantifiers: $\forall$ (for all) and $\exists$ (there exists). 4) Two binary predicate symbols: $=$ (identity) and $\in$ (membership). 5) Parentheses. We define the concepts of formula, of bound variable, of closed formula, etc. as usual.

Now I proceed to define the hierarchy NF_i, $0 \leqslant i \leqslant \omega$, of set theories. I shall begin with NF_1. The postulates of NF_1, i.e. its axiom schemes and primitive rules of inference, are the following:

I) *Propositional postulates*, where A, B and C are formulas and A_o is an abbreviation for $\neg(A \,\& \neg A)$:

I_1) $\quad A \supset (B \supset A)$ $\qquad\qquad$ I_2) $\quad (A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$

I_3) $\quad \dfrac{A \quad A \supset B}{B}$

I_4) $\quad (A \,\& B) \supset A$ $\qquad$ I_5) $\quad (A \,\& B) \supset B$

I_6) $\quad A \supset (B \supset (A \,\& B)$ $\qquad$ I_7) $\quad A \supset (A \vee B)$ $\qquad$ I_8) $\quad B \supset (A \vee B)$

I_9) $\quad (A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$

I_{10}) $\quad B^o \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$

I_{11}) $\quad (A^o \,\& B^o) \supset ((A \supset B)^o \,\& (A \,\& B)^o \,\& (A \vee B)^o)$

I_{12}) $\quad A \vee \neg A$ $\qquad\qquad\qquad$ I_{13}) $\quad \neg\neg A \supset A$

II(*Postulates for the predicate calculus*, where A(x) is a formula, x is a variable, etc., subjected to the usual restrictions:

II_1) $\quad \forall x A(x) \supset A(y)$ $\qquad\qquad\qquad$ II_2) $\quad \dfrac{A \supset B(x)}{A \supset \forall x B(x)}$

II_3) $\quad A(y) \supset \exists x A(x)$ $\qquad\qquad\qquad$ II_4) $\quad \dfrac{A(x) \supset B}{\exists x A(x) \supset B}$

II_5) $\quad \forall x (A(x))^o \supset (\forall x A(x))^o$ $\qquad$ II_6) $\quad \forall x (A(x))_o \supset (\exists x A(x))^o$

II_7) If A and B are congruent formulas in the sense of [14], p. 153, or one is obtained from the other by the suppression of vacuous quantifications, then $A \equiv C$ is an axiom.

III) *Postulates for identity* (x and y are distinct variables, etc.):

III$_1$) $x = x$ III$_2$) $x = y \supset (A(x) \equiv A(y))$

IV) *Specific postulates*:

IV$_1$) Extensionality: $\forall x(x \in y \equiv x \in z) \supset y = z$,
      where x, y and z are variables and x is dinstinct from y and z.

IV$_2$) Separation: $\exists y \forall x(x \in y \equiv F(x))$,
      where x and y are distinct variables, y does not occur free in
      F(x), and this formula is stratified (cf. [19] and [20]) or is of the
      form $x \notin x$ (i.e. $\neg(x \in x)$).

Adding to the postulates I$_1$ to III$_2$ the new axiom scheme
$\neg(A \,\&\, \neg A)$, we obtain the classical first-order predicate calculus
with identity. The system *NF* of Quine (see [19]) is obtained by
adjoining to this calculus extensionality (postulate IV$_1$) and separa-
tion, the latter postulate subject to the sole restriction that F(x) must
be stratified. I shall denote Quine's system by *NF*$_0$.

In order to introduce the set theories *NF*$_i$, $1 < i < \omega$, I need some
definitions. Given the formula A, A^i constitutes an abbrevitation for
A$^{oo \cdots o}$, where the symbol ^{o} appears i times. A$^{(i)}$ constitutes an
abbreviation for A$^1 \,\&\, A^2 \,\&\, \ldots \,\&\, A^i$. So, by definition, A^1 is the same
as A^o.

Then *NF*$_i$, $1 < i < \omega$, is formulated as follows: its postulates are
those of *NF*$_1$, but replacing I$_{10}$, I$_{11}$, II$_5$ and II$_6$ respectively by

I$'_{10}$) $B^{(i)} \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$

I$'_{11}$) $(A^{(i)} \,\&\, B^{(i)}) \supset ((A \supset B)^{(i)} \,\&\, (A \,\&\, B)^{(i)} \,\&\, (A \vee B)^{(i)})$

II$'_5$) $\forall x(A(x))^{(i)} \supset (\forall x A(x))^{(i)}$

II$'_6$) $\forall x(A(x))^{(i)} \supset (\exists x A(x))^{(i)}$.

We have:

Theorem 1. – Let us denote by $\neg^{(i)} A$ the formula $\neg A \,\&\, A^{(i)}$. Then
$\neg^{(i)}$ has all properties of classical negation (i.e. $\supset$, &, $\vee$, $\neg^{(i)}$, $\forall$, $\exists$ and
= satisfy all postulates of the underlying logic of *NF*$_0$.)

Proof. – See [7] and [8].

Theorem 2. – Suppose that A is a theorem of NF_0. If we replace in A all occurrences of $\neg$ by $\neg^{(i)}$, obtaining the formula Ã, then Ã is provable in NF_i, $1 \leqslant i < \omega$.

Proof. – Consequence of Theorem 1 and the form of the postulates of NF_i, $1 \leqslant i < \omega$.

Theorem 3. – NF_i, $1 \leqslant i < \omega$, contains NF_0.

Proof. – Consequence of the preceding results.

Theorem 4. – If NF_i, $1 \leqslant i < \omega$, is nontrivial, then NF_0 is consistent.

Proof. – Let us suppose that NF_i, $1 \leqslant n < \omega$, is not trivial and that NF_0 is inconsistent. Let A_1, A_2,..., A_k, where A_k is B & $\neg$B, be a derivation of an inconsistency in NF_0. So, employing the notations of Theorems 1 and 2, $Ã_1$, $Ã_2$,..., $Ã_k$ would be a derivation of $B̃$ & $\neg^{(i)}B̃$ in NF_i. But in this system $(C \& \neg^{(i)}C) \supset D$ is a valid scheme, and therefore NF_i would be trivial.

Theorem 5. – In the hierarchy NF_1, NF_2, NF_3,..., we have: for $j < k$, NF_j is stronger than NF_k.

Proof. – Immediate, taking into account the axiomatizations of NF_j and NF_k.

Theorem 6. – If NF_1 is nontrivial, then all NF_i, $1 < i < \omega$, are also nontrivial.

Proof. – Corollary to Theorem 5.

Theorem 7. – NF_i, $1 \leqslant i < \omega$, is inconsistent.

Proof. – In fact, Russell's set does exist in NF_i, $1 \leqslant i < \omega$. That is, we have in this system: $\vdash \exists y \, \forall x (x \in y \equiv x \notin x)$. Denoting by $\mathbb{R}$ the set $\{x : x \notin x\}$, we easily prove that $\mathbb{R} \in \mathbb{R}$ & $\mathbb{R} \notin \mathbb{R}$. (^{2}) (In NF_i, $1 \leqslant i < \omega$, $\cup\mathbb{R}$ is the universal set (cf. [4]).)

Now I proceed to show that if NF_0 is consistent, then NF_1 is nontrivial. Therefore, by Theorem 6, the consistency of NF_0 entails the nontriviality of any NF_i, $1 \leqslant i < \omega$.

Lemma 1. – The postulates I_1 to I_{13} constitute an axiomatization of the propositional calculus called C_1 (cf. [7]). If the scheme $\neg(A \ \& \ \neg A)$, where A is not an atomic formula, is added to C_1, we obtain a propositional logic which may be axiomatized as follows:

$I_1^*)$ $\ A \supset (B \supset A)$ $I_2^*)$ $\ (A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$

$I_3^*)$ $\ \dfrac{A \ \ A \supset B}{B}$ $I_4^*)$ $\ ((A \supset B) \supset A) \supset A$

$I_5^*)$ $\ (A \ \& \ B) \supset A$ $I_6^*)$ $\ (A \ \& \ B) \supset B$

$I_7^*)$ $\ A \supset (B \supset (A \ \& \ B))$ $I_8^*)$ $\ A \supset (A \lor B)$

$I_9^*)$ $\ B \supset (A \lor B)$ $I_{10}^*)$ $\ (A \supset C) \supset ((B \supset C) \supset ((A \lor B) \supset C))$

$I_{11}^*)$ $\ (\neg A \supset B) \supset ((\neg A \supset \neg B) \supset A)$, where B is not atomic.

Proof. – Immediate, taking into account results of [16] and [17].

Lemma 2. – Let NF_1^* be the system resulting from NF_1 when postulates I_1 to I_{13} are replaced by I_1^* to I_{11}^*. Then, in NF_1^* postulates II_5 and II_6 are provable.

Proof. – In effect, A° is provable when A is not atomic.

Lemma 3. – NF_1 is weaker than NF_1^*.

Proof. – Immediate.

Lemma 4. – The consistency of NF_0 implies the nontriviality of NF_1^*.

Proof. – Let f be the function whose domain is the collection of formulas of NF_1^* and whose range is the collection of formulas of NF_0, defined as follows:

1. $f(x = y) = x = y$
2. $f(x \in y) = x \in y$
3. $f(x \in y) = x \in y$
4. $f(x \notin y) = x \in V \ \& \ y \in V$, where V is the universal set

5. $f(A \supset B) = f(A) \supset f(B)$
6. $f(A \& B) = f(A) \& f(B)$
7. $f(A \vee B) = f(A) \vee f(B)$
8. $f(\forall x A) = \forall x f(A)$
9. $f(\exists x A) = \exists x f(A)$

Then, using lemmas 1 to 3, it is not difficult to see that $f(A)$ is a theorem of NF_0 whenever A is an axiom of NF_1^*. Since the rules of inference of NF_1^* are valid in NF_0, given any theorem A of NF_1^*, $f(A)$ is also a theorem of NF_0. (For example, representing, as above, Russell's set by $\mathbb{R}$, $f(\mathbb{R} \in \mathbb{R} \& \mathbb{R} \notin \mathbb{R}) = f(\mathbb{R} \in \mathbb{R}) \& f(\mathbb{R} \notin \mathbb{R}) = V \in V \& (V \in V \& V \in V) \equiv V \in V$, which is a theorem of NF_0.)

Therfore, supposing that NF_0 is consistent, NF_1^* cannot be trivial (for instance, $\emptyset \in \emptyset$ is not a theorem of NF_1^*, since $f(\emptyset \in \emptyset) = \emptyset \in \emptyset$ is not provable in NF_0).

Theorem 8. – If NF_0 is consistent, then NF_1 is nontrivial.

Proof. – The consistency of NF_0 implies the nontriviality of NF_1 because the latter system is weaker than NF_1^*.

Theorem 9. – If NF_0 is consistent, then all inconsistent systems NF_i, $1 \leqslant i < \omega$, are nontrivial.

Proof. – The set theories NF_i, $1 \leqslant i < \omega$, are weaker than NF_1^*.

Remarks. – 1) Changing a little the proof of Theorem 8, it is possible to prove the following proposition: Let us suppose that NF_0 is consistent; then the system obtained from NF_i, $1 \leqslant i < \omega$, by adjoining axioms guaranteeing the existence of the sets of all non-k-circular sets, $k = 1,2,...$, is not trivial (for example, the set of all non-3-circular sets is the following set: $\{x: \neg \exists y_1 \exists y_2 \exists y_3 (x \in y_1 \& y_1 \in y_2 \& y_2 \in y_3 \& y_3 \in x)\}$). 2) The relations $\{<x_1,..., x_n>:<x_1,..., x_n> \notin x_1\}$, $\{<x_1,..., x_n>:<x_1,..., x_n> \notin x_2\}$, ..., $\{<x_1,..., x_n>:>x_1,..., x_n> \notin x_n\}$ are called Russell's relations. Given the consistency of NF_0, if we add to NF_i, $1 \leqslant i < \omega$, new postulates guaranteeing the existence of all Russell's relations, the resulting system if also nontrivial. 3) Since the system NF_ω, investigated for instance in [5]

and [8], is weaker than NF_1, it is nontrivial in case that NF_0 is consistent. ([2])

2. The systems ZF_i

Starting with ZF (Zermelo-Fraenkel system (cf. [13], pp. 274-275)), it is natural to introduce a hierarchy ZF_i, $0 \leqslant i \leqslant \omega$, similar to the hierarchy NF_i, $0 \leqslant i \leqslant \omega$. Instead of ZF, one could employ any other classical system of set theory or of type theory as the first system of the hierarchy (cf. [11] and [12]).

Among the several versions of ZF, it is better for my purposes to use that of Church (see [6]), in which there exists the universal set, and which I shall denote here by ZF_0.

I shall begin with the description of ZF_1, whose underlying language is the same as that of NF_1. The logical postulates of ZF_1 are I_1 to III_2 above. We define in ZF_1 the notions of inclusion, empty set, ordered pair, relation, function, transistive set, connected set, well-founded set, ordinal and finite ordinal as in [6], but employing strong negation, $\neg^{(1)}$, instead of the primitive negation $\neg$. The notations wf(x) and finord(x) mean, respectively, that x is well-founded and that x is a finite ordinal.

The specific postulates of ZF_1 are the following (see [6]):

1) Extensionality: $\forall x(x \in y \equiv x \in z) \supset y = z$
2) Pair set: $\exists u \forall x(x \in u \equiv (x = y \lor x = z)$
3) Sumset: $\exists u \forall x(x \in u \equiv \exists y(y \in z \,\&\, x \in y)$
4) Product set: $y \in z \supset \exists u \forall x(x \in u \equiv \forall y(y \in z \supset x \in y))$
5) Infinity: $\exists u \forall x(x \in u \equiv finord(x))$
6) Choice: Every well-founded set is equivalent to an ordinal.
7) Complement: $\exists u \forall x(x \in u \equiv x \notin y)$
8) Russell's set: $\exists u \forall x(x \in u \equiv x \notin x)$
9) Separation: $wf(v) \supset \exists u \forall x(x \in u \equiv (x \in v \,\&\, F(x)))$, with the common restrictions.

(2) Introducing in NF_1 sets which may be called 'strong Quine individuals', i.e. sets x such that $x = \{x\} \,\&\, (x \in x)^\circ$, it is possible to show that $\mathbb{R}$ is distinct from $\{x : x = x\}$. As a corollary, neither $\mathbb{R} = \{x : x = x\}$ nor $\mathbb{R} \neq \{x : x = x\}$ is provable in NF_1, if NF_1 is nontrivial (and the same is true in connection with the theories NF_i, $1 < i \leqslant \omega$, and ZF_i, $1 \leqslant i \leqslant \omega$, the latter studied in section 2).

10) Replacement: $((\forall x \forall y(A(x,y) \supset (\forall z(A(x,z) \supset y = z))$ & $\forall x \forall y(A(x,y) \supset (\forall z(A(z,y) \supset . x = z))$ & $\forall y(y \in v \equiv \exists x A(x,y))$ & $wf(v))) \supset \exists u \forall x(x \in u \equiv \exists y A(x,y))$), with obvious restrictions.
11) Power set: $wf(v) \supset \exists u \forall x(x \in u \equiv x \subset v)$.

Theorem 10. – In ZF_1 we have: $\cup \mathbb{R} = V$, where $\mathbb{R} = \{x : x \notin x\}$ and $V = \{x : x = x\}$.

Proof. – See [4].

Theorem 11. – ZF_1 contains Church's system ZF_0.

Proof. – Analogous to the proof that NF_1 contains NF_0.

Theorem 12 – If ordinary ZF, described in [13], pp. 274-275, is consistent, then so is ZF_0.

Proof. – See [6], pp. 305-307.
   The propositions below may be proved without difficulty, precisely as the corresponding ones of section 1;

Theorem 13. – If ordinary ZF is consistent, then ZF_1 is nontrivial.

Theorem 14. – Let as suppose that ZF_i, $1 < i \leqslant \omega$, is obtained from ZF_1 as NF_i, $1 < i \leqslant \omega$, from NF_i. ZF is consistent if, and only if, ZF_i, $1 \leqslant i < \omega$, is nontrivial. The consistency of ZF entails the nontriviality of ZF_ω.
   Remarks 1 and 2 of the preceding section apply, *mutatis mutandis*, to the set theories ZF_i, $0 \leqslant i \leqslant \omega$.

3. *The paraconsistent programme.*

   In this part of my paper I make some remarks on the paraconsistent programme. The main concern to paraconsistent set theory is not to make possible the existence, and thereby the investigation, of some sets which cause trouble in naive set theory, such as Russell's set, Russell's relations and the set of all non-k-circular sets $(k = 1, 2, ...)$. On the contrary, the most important characteristic of paraconsistent

set theories is that they allow us to handle the extensions of 'inconsistent' predicates which may exist in the real world or are inherent in some universes of discourse in the fields of science and philosophy. According to several dialecticians, for example, there exist real contradictions in the world, and we need paraconsistent logic to handle them (cf. [10] and [18]). Analogously, contradictions must be taken into account in some psychoanalytic theories: the so-called analytic discourse is envisaged as inconsistent or the 'metatheory' of this discourse is considered as getting necessarily involved in contradictions (cf. [1] and [15]). In philosophy, some reconstructions of Meinong's theory of objects do also require a paraconsistent logic (see [21]). Of course, paraconsistent logic by itself can not prove that such theoretical constructions are legitimate and that some domains of knowledge are in fact involved in unsurmountable contradictions. The contribution of paraconsistent logic is more modest, though of great importance: it shows that inconsistencies can not always be considered as apparent difficulties, eliminable in principle as fallacies or errors, by an appeal to logic alone. In other words, if contradictions can always be overcome without undesirable residues, then it is impossible to establish this fact relying solely on logical grounds.

What I am trying to say is that the paraconsistent programme should not be judged solely by the mathematico-formal features of the paraconsistent set theories (for example, if they allow one to demonstrate the existence of infinitely many 'pathological' sets, if Russell's set does exist and, supposed its existence, if it is identical or not to the universal set), but above all by their aptness to cope with *concrete* problems. That is, problems originated from the vicissitudes of inquiry, in the domains of science and of philosophy, such as those mentioned above.

Another important comment to be made is the following: since several systems of paraconsistent set theory are stronger than classical set theory, everything that can be done with the help of the latter, can *ipso facto* be done by means of the former. Thus, the best choice of a logic for coping with a given domain of knowledge is not to be made by purely logical norms, but depends on considerations of various categories: pragmatic, philosophical, etc. In this way, paraconsistent set theories contribute for the better understanding of the true nature of logic.

It seems also worth while to observe that one may construct paraconsistent set theories without Russell's set and any other well-known sets which caused troubles in naive set theory. Notwithstanding this circunstance, they are strong and interesting, especially as a formalization of some fuzzy concepts. However, this topic is out of the scope of the present paper.

Summarizing, the paraconsistent programme, at least in its connection with set theory, has two kinds of motivation: one mathematico-formal, related to 'abstract' problems, and another 'concrete', linked to actual scientific and philosophic issues. Perhaps the second kind of motivation is more fruitful than the first, as a source of relevant paraconsistent insights.

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