ON LEMMON'S INTERPRETATIONS OF THE CONNECTIVE OF NECESSITY

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The reason of writing this paper is the discovery of some gaps and errors in Lemmon's paper [4], which again pose before us the problem of interpretation of the connective of necessity. We will show here that a gap which occurs in the consideration of S5 can be completed.

The term 'correctness under interpretation' plays the key role in Lemmon's considerations. I think it will be best to quote Lemmon's own explanation at this point. "... interpretation must involve ... the assignation of words belonginig to some language to the formal symbols, in such a way that formulae of the calculus are transformed into sentences of the language. ... There are conventionally understood ways of doing this: e.g., '... $\vee$ ...' becomes 'either ... or ...'. A set of such understood transformations I shall call an interpretational key. If interpretation into a natural language is envisaged, this key will very likely be exceedingly complex; think of specifying fully how brackets are to go over into punctation-marks (such as commas). Once this key is supplied, it seems natural to define a formula of a calculus as correct if the sentences obtained from it under the key is such that any statement made by using them is true, and as incorrect if some sentence obtained from it under the key is such that some statement made by using it is false. The calculus may then be said to be correct if its theorems and non-theorems coincide respectively with the correct and incorrect formulae."

Let us go into the details. Lemmon considers four modal logics: SO.5, M, S4 and S5. Let these symbols also denote the respective sets of theses of these logics. We focus our attention, for the present, on Lemmon's considerations connected with the calculus S4. He believes that with respect to a certain interpretation this calculus is correct, showing that:

(i) S4-axioms are correct under that interpretation,
(ii) The primitive rules of S4 are acceptable under that interpretation,

(iii) The formula $Lp \vee LNLp$ [1] (which added to S4-axioms gives the axiomatic of the next one of the four mentioned logics, i.e. S5-axiomatic) is incorrect under that interpretation.

The reader is now asked to compare the last sentence in the above quotation with (iii). If we denote by the symbol $S4^+$ the set of all formulas correct under his interpretation, then by showing (i) - (iii) we can say that the following disjunction is true:

$$S4^+ \not\subseteq S5 \text{ or } S4^+ - S5 \neq \emptyset$$

but not, as Lemmon whishes that $S4^+ = S4$.
It seems that he unconsciously assumed the following false assertion:

Every axiomatic extension of S4 contains S5

The equality $S4^+ = S4$ follows from this assertion and the above disjunction.
Similar objections can be made when Lemmon considers the calculi SO.5 and M. In considering S5 Lemmon proves only (i) and (ii) (for S5 obviously) and wrongly states that under his interpretation S5 is a correct calculus.

There is also another very disturbing point in Lemmon's paper. Considering the calculus SO.5 he reads the connective of necessity metalogically – 'it is tautologous (by truth table) that'; then he shows that the formula $L(Lp \supset p)$ is not acceptable under his interpretation and says: "though it may be a logical truth that what is tautologous is true, it is not a tautology that what is tautologous is true." Note that by the same argumentation we would have to accept the formula $LLp \supset q$ (that something is a tautology is not a tautology, then if we assert the contrary anything will follow). So the logic obtained in such a way contains SO.5 but is not contained in S5. Let us further note that in this argumentation there is a certain misunderstanding. If the connective L is read – 'it is tautologous (by truth table) that', then the interpretation of the formula La, where a contains the connective L, is not false but unfeasible or nonsense. The formula La can be interpreted only when a contains only classical connectives.

(1) The letters p, q, ... will denote sentential variables; the letters $a, a_1, a_2, \ldots$ formulas of modal language; symbols L, N, $\supset$, $\vee$ respectively the connectives of: necessity, negation, implication, disjunction.

A similar objection can be levelled against Gödel when he considers the formula $L(Lp \supset p)$(2) in [3]. In the same paper he writes up the S4-axiomatic and states ad hoc that the connective of necessity can be read as 'is provable'.

In spite of these objections, I think that one of Lemmon's, or strictly speaking Carnap's (cf. [1] pp. 174-5), interpretations can be accepted. Understanding Lp as – 'it is analytically the case that p' Lemmon shows that S5-axioms are correct formulas under this interpretation and the primitive rules of S5 (substitution, modus ponens and necessitation – if $\vdash a$ then $\vdash La$) are also acceptable under this interpretation.

It still needs to be proved that under this interpretation all non-theses of S5 are incorrect. In [5] it is shown that we can reject all non-theses of S5 admitting the following rule:

$$\frac{\dashv a \supset a_1, \ldots, \dashv a \supset a_k}{\dashv La \supset La_1 \vee \ldots \vee La_k} \quad R$$

where the formulas $a, a_1, \ldots, a_k$ do not contain the connective L, and the symbol $\dashv$ is understood as the symbol of rejection in S5.

We will prove that the above rule is acceptable under the mentioned way of understanding neccesity. But first we must introduce some additional material. Lemmon has some trouble in understanding the term 'analytic' and the comments that the expression 'it is analytically the case that p' can be understood as follows – "it is the case that p, solely in virtue of the meanings of the words in the sentence used to make the statement that p." From this explanation follows the following theorem:

(1) There are infinitely many analytically false independent sentences, i.e. every complex sentence formed from them by means of classical connectives is analytically true iff it is a substitution of a classical thesis.

E.g. for $n = 3$ the following are such sentences: 'John sleeps', 'Some dog runs' and 'Every building has two floors'. It is obvious that

(2) Basing on the technique of arithmetization and treating L as a term defined in a certain manner, Gödel's considerations in the final part of [3] can be regarded as correct (see also [4] pp. 32-3).

without empirical verification we are unable to decide about the logical values of these sentences, or of complex sentences formed from them by means of classical connectives, unless they are substitutions of classical theses or counter-theses. I think that it is not necessary to show the truth of (1) for every natural number.

Let us denote by the symbols e, s, Cn respectively: any substitution (in the formulas of modal language) of formulas for sentential variables, any substitution (in the formulas of modal language) of sentences for sentential variables, the usual classical consequence operation determined by classical theses and modus ponens. In [2] a theorem is proved from which follows the following implication:

(2) For all formulas $a, a_1, \ldots, a_k$ not containing the connective L, if $\{a_1, \ldots, a_k\} \cap Cn(\{a\}) = \emptyset$ then there exists such a substitution e that formulas not containing L are substituted for variables and the formulas $ea_1, \ldots, ea_k$ are not classical theses and ea is a classical thesis.

Now, we are ready to prove the acceptability of the rule R. Assume that $\dashv a \supset a_1, \ldots, \dashv a \supset a_k$. By the assumptions of the rule R, the formulas $a, a_1, \ldots, a_k$ do not contain the connective L. Then the formulas $a \supset a_1, \ldots, a \supset a_k$ are classical non-theses. This can be written as follows:

$$a \supset a_1 \notin Cn(\emptyset), \ldots, a \supset a_k \notin Cn(\emptyset)$$

Hence and by the deduction theorem we have:

$$\{a_1, \ldots, a_k\} \cap Cn(\{a\}) = \emptyset$$

Hence and by (2) there exists such a substitution e that:

(3) The formulas $ea_1, \ldots, ea_k$ are not classical theses and ea is a classical thesis.

Let the same letter e also denote a substitution for formulas containing the connective L consistent with the previous one for every variable, and let s be a substitution under which sentences satisfying (1) are put for variables appearing in the formulas $ea, ea_1, \ldots, ea_k$ with different sentences substituted for different variables (for the other variables s is arbitrary). By the above assumptions and (1) and (3) we have:

> Sentences $sea_1, \ldots, sea_k$ are analytically false and sentence sea is analytically true.

Hence the sentence se($La \supset La_1 \vee \ldots \vee La_k$) is false. Then the formula $La \supset La_1 \vee \ldots \vee La_k$ is rejected. So the proof of acceptability is completed.
In this way we have proved that the calculus S5 is correct under interpretation of L as 'it is analytically the case that'.

Finally one more question concerning the interpretation of the connective L in S4. Lemmon proposes to read it as 'it is informally provable in mathematics that'. Assume that the calculus obtained under this interpretation, i.e. $S4^+$ in our notation, is $\subseteq S5$. If so, the rule R should be acceptable in $S4^+$. In our proof of acceptability of this rule (for S5) an essential part is played by theorem (1). It is a question whether such an assumption is satisfied under the above reading of L, i.e.:

> There are infinitely many provably independent mathematical sentences, i.e. every complex sentence fromed from them by means of classical connectives is provable iff it is a substitution of a classical thesis.

I think that the above hypothesis is true.

To sum up, the questions of interpretation of the connective L in the logics SO.5, M and S4 are open. A different interpretation must be choosen for SO.5, while in the case of M and S4 one can tray to fill the gaps in Lemmon's considerations.

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