THE OBJECT OF BELIEF

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The aim of this paper is to give a precise description of the object of belief. First, I will specify the general framework and place my proposal in relation with some of the other approaches one finds in the literature.

But before, it may be worthwhile to say a few words about what I do not want to do. This paper will not be concerned with an analysis of the different uses of belief. It has been argued that there are many different concepts of belief and that one of the problems about belief comes from some confusion between these different concepts. So one of the tasks which confronts someone who wants to write about belief is to explain what he means by "belief". But in doing so, he formulates a theory of belief. From this point of view the only thing I can say is that the concept of belief I want to treat is the one intuitively described in the first part of this paper. To someone who would say that this concept is not interesting, I have nothing more to say.

Moreover, this paper will not address itself to the very difficult and important questions of quantifying into belief-contexts and of the *de re-de dicto* occurrences of expressions in such contexts.

These problems are real ones, but I think that there is a more fundamental one that is to be solved before, *viz.* that of the object of belief. Certainly, any satisfactory theory of the nature of belief and of its object should be required to give a solution to these awkward puzzles, but this is another task, a second level one.

1. *The problem*

The general framework will be that of intensional logic, more specifically intensional logic as developed by R. Montague. Many attempts have been made to treat belief as an intensional operator.

Montague himself believed that this could be done (1). Some extensions of this framework have been proposed to solve more complicated situations. For example, the notion of propositional concept introduced by R. Stalnaker (2) (a propositional concept is a function from possible worlds into functions from possible worlds into truth values, that is, a function from possible worlds into propositions), to solve the problem of beliefs about necessarily false propositions.

An other example is the treatment of belief as an hyperintentional operator. This idea was first suggested by R. Carnap and has been afterwards more systematically worked out by D. Lewis, M. J. Cresswell, and J. C. Bigelow.(3)

All these approaches have something in common: to try to define finely enough the object of belief so that when two expressions do not receive the same value (i.e., when the two expressions do not stand for the same object) they are not substitutable *salva veritate* into belief contexts.

This condition is surely necessary but it is not sufficient.

The analysis proposed here started with the problem of belief about analytically false propositions. The problem with this kind of belief is not only that of solving the paradoxical situation that could be stated thus: if A believes that P, and P is analytically false than A believes any Q which is also analytically false. This problem, like those mentioned earlier, is important but, once again, there is another more fundamental. The basic problem is to propose an acceptable and intuitive account of what is believed when someone believes in

(1) Richard MONTAGUE, "Pragmatics and Intensional Logic", *Synthese*, 22 (1970), 68-94; "Universal Grammar", *Theoria*, 36, (1970), 373-98. Both are reprinted in R. H. Thomason (ed.), *Formal Philosophy. Selected Papers of Richard Montague*, New Haven and London: Yale University Press, 1974.

(2) Robert STALNAKER, "Propositions", in A. F. Mackay and D. D. Merrill (eds.), *Issues in the Philosophy of Language*, New Haven: Yale University Press, 1976, 79-91; "Assertion", in P. Cole (ed.), *Pragmatics*, New York: Academic Press, 1978, 315-332, (Syntax and Semantics, vol. 9); "Semantics for Belief", dittoed 1982.

(3) Rudolf CARNAP, *Meaning and Necessity, A Study in Semantics and Modal Logic*, 2nd ed., Chicago and London: The University of Chicago Press, 1956. David Lewis, "General Semantics", in D. Davidson and G. Harman (eds.), *Semantics for Natural Language*, Dordrecht: D. Reidel Publishing Company, 1972, 169-218. Max J. Cresswell, "Hyperintensional Logic", *Studia Logica*, 34, (1975), 25-38. John C. Bigelow, "Believing in Semantics", *Linguistics and Philosophy*, 2, (1978), 101-144.

analytically false propositions. More specifically, what one has to do here is to try to define a set-theoretical entity which play the role of the object of such beliefs. I do not pretend that approaches like those of Stalnaker or Cresswell could not give such an account, but that, at the present time, they do not. Why is the case of analytically false belief so important? Well, intuition tells me that there is no difference between this kind of belief and any other kind of belief, because we often don't even know if what we believe to be the case is analytic or not. In fact, if analytically false beliefs are of another species than beliefs that are not, then we would know *a priori* that the object of belief is analytic or at least we could come to know this just by contemplating the belief.

So if a proposed treatment of belief is unacceptable for analytically false belief, it is unacceptable *tout court*. If the foregoing considerations are kept in mind then Stalnaker's as well as Creswell's approaches can only be judged *ad hoc*. For instance, I see no justification whatsoever for the fact that the propositional concept of a logical constant, in Stalnaker's approach, could have a non-constant content, that is, that there is some world where the intension attached to a logical constant is not the same as in our world. In such a case, the natural analysis would be to say that what is believed is not what is expressed by the sentence used to express the belief according to the intended interpretation but some content expressed by the sentence according to some idiosyncratic interpretation. In other words, a theory of belief must give an account of the fact that analytically false beliefs are *really* analytically false beliefs.

Moreover, if the object of belief is some hyperintensional object, as in Cresswell's approach, i.e., not only an intension but a tree of intentions, how could we fail to see that the object we believe to be true is not? The analytic aspect of what is believed is at least as evident in the tree of intensions as in the intension itself. Rightly enough, one has to concede that this approach has a very rewarding and intuitive dimension inasmuch as it allows one to block undesired inferences between belief-sentences, but, for the reasons given above, the tree structure is not, in itself, a very intuitive support for belief. This brings us to another question.

What would an intuitive and acceptable solution to the problem of analytically false belief look like?

Let us consider the following points.
a) A believes that P is to be analyzed as a relation between a and an object which is determined by a and P (from now on and unless otherwise specified capital letters will be used to refer to expressions and the corresponding small letters will be used to refer to the corresponding semantic value).
b) This object must be such that if the sentences P receives the same value as the sentence Q, then to believe P will be the same as to believe Q.
c) There must be a certain notion of knowledge (we will call it "to have evidence for" or "to know that") which has the property that "A know that P" always implies "A believes that P".
d) If "A believes that P" is true, then a is committed to the truth of the proposition expressed by P. In particular if P materially implies Q then a is committed to the truth of the proposition expressed by Q.
e) If "A believes that P" is true than a is committed to the fact that a better knowledge (in the sense given above) of the situation would bring him to have evidence for the truth of P.
f) Finally, and this is a kind of consequence of the preceding points, the closer the bearer of a belief will get to omniscience, the closer the object of belief will get to being identical to the proposition determined by P, i.e., in the limiting case, the content of belief becomes "classical".

In order to satisfy these points the set of beliefs of an individual will have to be inconsistent. The problem will thus be to define a structure in spite of this inconsistency; we will base this structure on a certain notion of knowledge. Informally, the relation between knowledge and belief will be the following: one cannot believe P without believing Q if one cannot know P without knowing Q, and this notion of knowledge will be defined in such a way that it will be consistent.

These desiderata correspond intuitively to the conditions in which someone will be forced to give up a belief.

It is now time to define some formal tools in order to give a precise formulation to these intuitions. If someone does not agree with these intuitions or thinks that the notion of belief specified by these intuitions is not interesting, he should stop here, the rest of the paper is concerned with the development of these ideas in a more exact language.

2. *The framework*

As I said, the general framework will be that of intensional logic. The apparatus I will use is well known and is just a variant of that of Montague in "Universal Grammar". Let L be a language, that is the free monoid generated by concatenation over a finite set of words. Let Cat be a set of categories, a category being a set of words and two different categories being such that their intersection is empty. More precisely, Cat is defined in the following way:

a) $n \in Cat$, $s \in Cat$ where n is the category of nouns and s is the category of sentences
b) if $\mathrm{A} \in Cat$ and $\mathrm{B} \in Cat$ then $<\mathrm{A}, \mathrm{B}> \in Cat$
c) nothing else is in Cat

There is only one grammatical rule:

For any words x and y and for any A, $\mathrm{B} \in Cat$
if $x \in \mathrm{A}$ and $y \in <\mathrm{A}, \mathrm{B}>$ then $Conc(y, x) \in \mathrm{B}$ ($Conc$ is concatenation)

One can easily recognize the grammatical rule of categorial languages. The model structure for that language will be the following. Let T be the smallest set such that

a) $e \in T$
b) $t \in T$
c) if $\sigma \in T$ and $\tau \in T$ then $<\sigma, \tau> \in T$
d) if $\sigma \in T$ then $<s, \sigma> \in T$

The set T is the set of types, e is the type of entities or individuals and t the type of truth values. For each type, there is a set of possible denotations:

a) $D_e = E$
b) $D_t = \{0, 1\}$
c) $D_{<\sigma, \tau>} = D_\tau^{D_\sigma}$
d) $D_{<s, \sigma>} = D_\sigma^I$

where there is no general restrictions on the set E of entities and on the set I of possible worlds.

The denotations of type $<s, \sigma>$ are also called intentions of type σ.

To each expression belonging to the language is associated an intension. In particular, to sentences are associated intensions of type t.

Let us consider now the following subset L' of L such that L' contains all the expressions that generated L and all the expressions which belong to some category, i.e. expressions formed by application of *Conc* and respecting the grammatical rule.

Let $g: Cat \rightarrow T$ be a function which assigns a type to each category and such that

a) $g(s) = t$

b) $g(<\mathrm{A}, \mathrm{B}>) = <g(\mathrm{A}), g(\mathrm{B})>$

An interpretation of L' is a 3-uples $<M, f, g>$ such that

$$M = \bigcup_{\sigma \in T} D_{\sigma}^{I}$$

i.e., M is the set of intensions, g is as above, and f is such that for any $x, y \in L'$ *and any* $i \in I$

$$f(Conc(x, y))(i) = (f(y)(i)(f(x)(i)))$$

which means that the intension of the expression resulting from the concatenation of x with y is that function which at i takes the value of the function associated to y at i when the argument is the value associated to x at i.

To this well known model structure we will add some more entities. Informally, the idea is as follows. When someone uses an expression, this expression refers to an intension. The user, let us call him(her, it) a cognitive agent, has a *representation* of this intension. In the best cases this representation will coincide with the intension itself. In most situations this will not be the case, but this is not relevant to semantics for the very reason that the truth value of a sentence depends on its intension and on what is the case in the world and *not* on the representation of the user.

But is this always the case? It seems that when a sentence expresses a relation between a cognitive agent and the semantic content of an other sentence, the representation of that semantic content is relevant. Does this mean that any representation of a semantic content, here of an intension, of a user is acceptable as semantically relevant? To give an affirmative answer to this question would be to fall into idiosyncraticism. To avoid this, we need a criterion about the acceptability of representations. Such a criterion could be stated thus: a representation is acceptable if it does not induce its user into error. This brings us to the notion of a *good representation*.

3. *The notion of good representation*

Let us consider the following definition which gives a precise content to the idea of *not inducing the user into error*, when considering a representation of an intension or of a denotation.

a) The only non-null good representation (*gr*) of an individual $a \in E$ is a itself. So for any x which gives itself as a representation of $a \in E$, there are three possibilities: x is a and x is a *gr* of a, x is not a and a is not a *gr* of a, and finally, x is not defined. This third case is abbreviated by writing $x = \Phi$, and we will call Φ the null-representation (the usefulness of this convention will soon appear).
b) The only non-null good representation of 0 and 1 (the false and the true) are respectively 0 and 1. The same convention as above will hold for an undefined representation of 0 and 1.
c) If $f \in D_\tau^{D_\sigma}$ then f' is a *gr* of f iff for any a' which is a *gr* of $a \in D_\sigma$, $f'(a')$ is a *gr* of $f(a)$ when $f'(a')$ is defined. When $f'(a')$ is not defined, we will use our convention and write $f'(a') = \Phi$.
d) If $f \in D_\sigma^I$ then f' is a *gr* of f iff for any $i \in I, f'(i)$ is a *gr* of $f(i)$, when $f'(i)$ is defined. When $f'(i)$ is not defined, we will write $f'(i) = \Phi$.

Let us remark that this last part of the definition gives us a definition of a *gr* of an intension in terms of *gr*'s of denotations.

Thus a *gr* is something which coincides with what it represents when it is defined. In functional terms, a *gr* of a function from D_σ into D_τ is a partial function from *gr*'s of members of the domain D_σ into *gr*'s of members of the codomain D_τ.

In particular, we can remark that if S is a sentence and f_S is its intension, then if there is some *gr* of f'_S such that $f'_S(i) = 1$ (or $f'_S(i) = 0$) for some i then $f_S(i) = 1$ (or $f_S(i) = 0$).

This idea of a good representation calls for further comments. At the zero level, i.e., at the level concerned with the good representations of individuals, the definition of a *gr* expresses the very intuitive idea that our knowledge of an *individual* cannot be partial: an individual is either known or unknown. For predicates, the notion is even more intuitive: a *gr* of a predicate is a partial predicate.

To attach *gr*'s to expressions is in a certain way to give them much finer semantic contents than classical intensions. But since these objects are user-dependent, one can think that this brings us back to idiosyncraticism. In fact, the notion of a *gr* should be interpreted as a description of the limits in which idiosyncraticism is acceptable. One is free to associate to an expression any semantic content whatsoever inasmuch as this content does not induce him into error, and this is exactly the requirement that our notion of a *gr* is designed to obey.

We will now examine some interesting properties of the sets of all the *gr*'s of a given intension or of a given denotation.

4. *The lattice of gr's*

Let us call GR_f the set of all the *gr*'s of $f \in D_\sigma$. The recursive definition of *gr*'s based on the hierarchy of types permits us to define an ordering relation on each GR_f. We will call this relation "$\succcurlyeq_f$" which is to be read "at least as good as". The index will be dropped because no confusion is possible: if the domains of two different relations intersect, the relations coincide on the common part of their domains.

a) If a and b are *gr*'s of the same individual, or of 0, or of 1, $a \succcurlyeq b$ iff $a \neq \Phi$.
b) If f and g are *gr*'s of $h \in D_\tau^{D_\sigma}$ then $f \succcurlyeq g$ iff for any x, if $g(x) \neq \Phi$ then $f(x) \succcurlyeq g(x)$
c) If f and g are *gr*'s of $h \in D_\sigma^I$ then $f \succcurlyeq g$ iff for any $i \in I$ if $g(i) \neq \Phi$ then $f(i) \succcurlyeq g(i)$

It can be easily proved that this relation is reflexive, antisymetric and transitive. The intuitive interpretation of this relation is that f is at least as good as g if and only if f is more often defined than g. Recursively this is expressed by the clause that when g is defined, f is defined too and its value is at least as good as the value of g.

Now we are going to show that each GR_f is a lattice for $\succcurlyeq$.

Proposition: The family of functions $(h_f)_{f \in D}$ (where $D = \bigcup_{\sigma \in T} D_\sigma$) satisfying (i)-(iii) below is the family of joins for the family $<GR_f, \mathrel{\epsilon\!\!-}_f>_{f \in D}$ (From now on, we will drop the indexes of the h_f's.)

i) If a and b are gr's of the same individual or of 1 or of 0, then
$h(a,b)$
$= a$ if $a \neq \Phi$,
$= b$ otherwise

ii) For any f_1, f_2 that are gr's of $f \in D_\tau^{D_\sigma}$, and for any a' that is gr of $a \in D_\sigma$, $h(f_1, f_2)(a') = h(f_1(a'), f_2(a'))$

iii) For any f_1, f_2 that are gr's of $f \in D_\sigma^I$, and for any $i \in I$ $h(f_1, f_2)(i) = h(f_1(i), f_2(i))$

(The dual definition for meet is obvious).

a) The uniqueness and the existence of h are easily verified.
b) Let us prove that h is the join for $\mathrel{\epsilon\!\!-}$.

First, we have to check that for any a and b

$h(a,b) \mathrel{\epsilon\!\!-} a$ and $h(a,b) \mathrel{\epsilon\!\!-} b$

i) if a and b are gr's of the same individual, or of 1, or of 0, our proposition is true in virtue of both the definition of h and of $\mathrel{\epsilon\!\!-}$.

ii) Let us suppose that h has this property for all denotations of all types up to τ and σ and let us show that h has this property for any $f \in D_\tau^{D_\sigma}$. Let f_1 and f_2 be two gr's of f, and a' be any gr of $a \in D_\sigma$. We have

$h(f_1, f_2)(a') = h(f_1(a'), f_2(a'))$ definition of h

$h(f_1(a')) \mathrel{\epsilon\!\!-} f_1(a')$ induction hypothesis

so $h(f_1, f_2) \mathrel{\epsilon\!\!-} f_1$

For f_2 the proof is the same.

iii) For any $f \in D_\sigma^I$ and for any f_1, f_2 that are gr's of f the proof is similar.

Next, we have to prove that if a, b, c are three gr's such that $c \mathrel{\epsilon\!\!-} a$ and $c \mathrel{\epsilon\!\!-} b$ then $c \mathrel{\epsilon\!\!-} h(a,b)$

i) If a, b, c are gr's of the same individual, or of 1 or of 0, the property follows from the definitions of h and $\mathrel{\epsilon\!\!-}$.

ii) Let us suppose that h has this property for all denotations of all types up to τ and σ and let us show that h has this property for f_1, f_2 and f_3 of type $<\sigma, \tau>$.

We have	$f_3(a') \sqsubseteq f_1(a')$	hypothesis
	$f_3(a') \sqsubseteq f_2(a')$	hypothesis
	$f_3(a') \sqsubseteq h(f_1(a'),f_2(a'))$	induction hypothesis
	$f_3(a') \sqsubseteq h(f_1,f_2)(a')$	Definition of h
	so $f_3 \sqsubseteq h(f_1,f_2)$	

iii) For any $f \in D_\sigma^I$ and for any f_1, f_2, f_3 that are gr's of f, the proof is similar.

This completes the demonstration.

5. *Cognitive agent*

Up to now we have defined the notion of good representation; they are entities which, when defined, will behave like denotations and intensions. We now have to describe how gr's are ascribed to users of the language.

Let $f: I \to D_\sigma$ be an intension of type σ, C be a set which will be called the set of cognitive agents (C must be large enough to represent all the *possible* users of the language).

Consider a function

$f' : C \times I \to \mathrm{GR}_{f(i)}$

$(c, i) \to f'(c, i)$ such that $f'(c, i)$ is a gr of $f(i)$

The derived function

$f_c : I \to \mathrm{GR}_{f(i)}$ such that for any $i \in I$, $f_c(i) = f'(c, i)$ will be called the cognitive intension that c attaches to f.

We will also suppose that C is large enough in order that the following property will always be satisfied: if f' is a good representation of f, there is a $c \in C$ such that $f' = f_c$.

Moreover, we will use the following convention. If $p \in \{0, 1\}^I$ and $p_a(i) = 1$, we will say that "A knows that P" is true at i or that "A has evidence for the truth of P" at i. We will symbolize these expressions by writing "know (A, P)" is true at i, i.e., $(K\ a\ p)(i) = 1$

So the situation is the following. To each possible cognitive agent in any world is attached, for each intension, a cognitive intension which is a gr of that intension. Let's call $C \times I$ the set of all *possible points of view* (*ppv*). In order to eliminate redundant *ppv*'s, let's consider the following equivalence relation on $C \times I$.

$$(c, i) \equiv (c', i') \text{ iff for any } f, f_c(i) = f_c'(i')$$

From now on, we will work with equivalence classes. The relation between gr's induces a similar relation on the equivalence classes, a relation defined by: $(c, i) \sqsubseteq (c' i')$ iff for any f, $f_c(i) \sqsubseteq f_c'(i')$. One can easily prove that the quotient set is a meet-semilattice.

It easily follows from this definition that if we fix $i \in I$, this ordering relation induces on the set of cognitive agents the following relation, which will be called "to be cleverer than":

$$c \text{ Cl } c' \text{ iff } (c, i) \sqsubseteq (c', i).$$

We also clearly have for any c, c', that join (c, c') and meet (c, c') exist.

6. *Belief*

Let us now apply these notions to the analysis of belief.

First, we will reduce belief to an other relation which will be extensional with respect to its object. Consider the following definition:

(Bel a p)(i) = 1 iff $a(i)$ is committed to the truth of p_c for some c
(Bel $a \sim p$)(i) = 1 iff $a(i)$ is committed to the falsity of p_c for some c.

In order to obtain an extensional relation, we simply define $Com(a, i)$ as a set of pairs (p_c, t) which is to be interpreted as
$(p_c, t) \in Com(a, i)$ iff the cognitive agent a is committed at i to the t-value of p_c where t is the true or the false (4).

(4) More precisely, we would have the following definition of the intension of Bel:
(Bel a p) $\in \{0, 1\}^I$ is such that
(Bel a p)(i) = 1 iff $(p_c, 1) \in$ Com(a, i)

this suggest that

(Bel a) $\in \{0, 1\}^{\{0, 1\}^I}$ is such that
(Bel a)(i)($p(i)$) = (Bel a p)(i)

which in turn gives

Bel $\in \{0, 1\}^{\{0, 1\}^{E^I}}$ is such that
Bel(i)($a(i)$) = (Bel a)(i)

The question is now to define constraints on the set of commitments. The following constraints seem reasonable.

a) If $p_a(i) = 1$ (or $p_a(i) = 0$) then $(p_a, 1) \in Com(a,i)$ and $(p_a, 0) \notin Com(a,i)$ (or $(p_a, 0) \in Com(a,i)$ and $(p_a, 1) \notin Com(a,i)$).

This simply expresses the fact that anybody is committed to the truth (or falsity) of what he (or she or it) knows to be the case (or not the case), and that belief is consistent inasmuch as knowledge is concerned.

b) If $(p_c, t) \in Com(a,i)$ and c' Cl c, then $(p_{c'}, t) \in Com(a,i)$

The idea is simply that if you are committed to the fact that a better knowledge of a situation will bring a cognitive agent to see that *P* is true (or false) then you are committed to the fact that all cleverer agents will also see that *P* is true (or false).

c) If $(p_c, t) \in Com(a,i)$ and $(q_c, t) \in Com(a,i)$ then $((p \,\&\, q)_c, t) \in Com(a,i)$

This condition only means that $Com(a,i)$ is closed under "and".

As I announced at the beginning, we have here a kind of inconsistency for *Com* in the following sense. For example, it may be the case that

$$(p_c, 1) \in Com(a,i) \text{ and so } (p, 1) \in Com(a,i)$$
$$(q_c, 0) \in Com(a,i) \text{ and so } (q, 0) \in Com(a,i)$$

But if $p(i) = q(i)$ then we have $(p, 0) \in Com(a,i)$, i.e., $a(i)$ is committed to the truth and the falsity of *P*.

This situation is to be interpreted as "*A* believes that *P*" and "*A* believes that $\sim P$". This paradoxical situation is the price we have to pay for the extensional status of *Com*.

But how can the notion of an inconsistent set be of any usefulness? Well, eventhough belief is inconsistent in our previous sense knowledge is not. What we have to do now is to define a relation, in fact a kind of implication, that will be such that if this relation holds between *P* and *Q* then to know *P* implies to know *Q*. This relation will affect belief in the following way: if someone believes *P* we can bring him to believe $\sim P$ just by bringing him to know $\sim Q$. This, it seems to me, is the crux of the logic of belief.

Let us consider the following relation between sentences ($\rightarrow_{cg}$ must be read as *material cognitive implication*):

For any $i \in I$, "$P \rightarrow_{cg} Q$" is true at i iff for any $c \in C$ it is not the case that:

$$p_c(i) = 1 \text{ and } q_c(i) = 0$$
$$p_c(i) = 1 \text{ and } q_c(i) = \Phi$$
$$p_c(i) = \Phi \text{ and } q_c(i) = 0$$

This definition may seem paradoxical but it is not more paradoxical than material implication and it is in the same way.

We can now prove that $\rightarrow_{cg}$ has the following property:

Proposition:

For any $i \in I$

$$P \rightarrow_{cg} Q \text{ iff for any } c \in C \text{ (if } p(i) = 1 \text{ then } q_c(i) \mathrel{\varepsilon\!\!-} p_c(i) \text{ and if } q(i) = 0 \text{ then } p_c(i) \mathrel{\varepsilon\!\!-} q_c(i)).$$

This proposition expresses the following property of material cognitive implication: to say that P materially cognitively implies Q is to say that it is not possible to have evidence for the truth of P without having also evidence for the truth of Q when P is the case, and that it is not possible to have evidence for the falsity of Q without having also evidence for the falsity of P when Q is not the case.

The demonstration is straightforward. Consider the following truth table (where, for simplicity, the i's have been dropped).

	p_c	q_c	$p_c \mathrel{\varepsilon\!\!-} q_c$	$q_c \mathrel{\varepsilon\!\!-} p_c$	p	q	
1	1	1	1	1	1	1	
2	1	Φ	1	0	1	?	x
3	1	0	0	0	1	0	x
4	Φ	1	0	1	?	1	
5	Φ	Φ	1	1	?	?	
6	Φ	0	0	1	?	0	x
7	0	1	0	0	0	1	
8	0	Φ	1	0	0	?	
9	0	0	1	1	0	0	

Let us suppose that $P \rightarrow_{\mathrm{cg}} Q$. Then for any $c \in C$ we are not on lines 2, 3 and 6. It is easy to check that on any other line $p(i) = 1$ is false or $q_c(i) \mathrel{\varepsilon\!\!-} p_c(i)$ and that $q(i) = 0$ is false or $p_c(i) \mathrel{\varepsilon\!\!-} q_c(i)$. The only problematic cases are the lines where "?" appears. Let's check each of them separately.

On line 4, $q_c(i) = 1$, so the second part of the conjunction is satisfied.

But $q_c(i) \mathrel{\varepsilon\!\!-} p_c(i)$ and so the first part is also satisfied.

On line 5, both members of the conjunction are true because both are implications whose consequents are true.

On line 8, $p(i) = 1$ is false, and so the first member of the conjunction is true. The second member of the conjunction is also true because its consequent is true.

Let us now prove converse. Let us suppose that the two members of the conjunction are true and let us show that we cannot be on lines 2, 3 or 6. On line 2 the first member of the conjunction is false. On line 3 both members are false. On line 6 the second member of the conjunction is false. This ends the demonstration.

It is easy to see that if $(\mathrm{K}\ a\ p)(i) = 1$ and $P \rightarrow_{\mathrm{cg}} Q$ then $(\mathrm{K}\ a\ q)(i) = 1$ and that

if $p(i) = 1$ and $P \rightarrow_{\mathrm{cg}} Q$ then for any $a \in C$,
$(p_c, 1) \in Com(a, i)$ implies $(q_c, 1) \in Com(a, i)$

because in that case $p_c(i) \mathrel{\varepsilon\!\!-} q_c(i)$.

So belief is closed under $\rightarrow_{\mathrm{cg}}$ for true beliefs. But the cases of false beliefs are not relevant; in fact we do not want to put restrictions on material cognitive consequences of false beliefs, so we can put the following restriction on *Com*:

For any $a \in C$ and any $i \in I$

if $(p_c, 1) \in Com(a, i)$ and $P \rightarrow_{\mathrm{cg}} Q$ then $(q_c, 1) \in Com(a, i)$ and
if $(p_c, 0) \in Com(a, i)$ and $P \rightarrow_{\mathrm{cg}} Q$ then $(q_c, 0) \in Com(a, i)$.

This simply means that *Com* is closed under $\rightarrow_{\mathrm{cg}}$.

We will now take a look on some applications.

7. *Some properties of belief and knowledge*

We have defined a kind of entailment between sentences, an entailment based on relations between *gr*'s. But restrictions on *gr*'s are not very strong and it would be difficult to find a non-trivial situation where we have $P \rightarrow_{cg} Q$. We need stronger restrictions on *gr*'s.

Consider, for example, the following restrictions on the *gr*'s of propositional connectives. &, $\vee$, $\rightarrow$ are maximally defined for any c, i.e.,

a) if $p_c(i) = 1$ and $q_c(i) = 1$ then $(p \,\&\, q)_c(i) = 1$
   and if $p_c(i) = 0$ or $q_c(i) =$ then $(p \,\&\, q)_c(i) = 0$
b) if $p_c(i) = 1$ or $q_c(i) = 1$ then $(p \vee q)_c(i) = 1$
   and if $p_c(i) = 0$ and $q_c(i) = 0$ then $(p \vee q)_c(i) = 0$
c) if $p_c(i) = 0$ or $q_c(i) = 1$ then $(p \rightarrow q)_c(i) = 1$
   and if $p_c(i) = 1$ and $q_c(i) = 0$ then $(p \rightarrow q)_c(i) = 0$

These restrictions are to be interpreted as the faculty of any cognitive agent to draw the logical consequences of what he (she or it) knows. This is related to the extensional view of cognitive content and to the specific character of logical constants.

Under these restrictions, knowledge and, thus, belief appear to have very interesting properties in relation with introduction and elimination rules for the connectives in natural logic ([5]). In fact, for each of these rules except one, there is a corresponding property in terms of knowledge or belief. We give the proof for belief only. For the introduction and elimination rules for & we have the corresponding property.

([5]) These introduction (I) and elimination (E) rules are:

$$\&\mathrm{I})\ \frac{A \qquad B}{A \,\&\, B} \qquad\qquad \&\mathrm{E})\ \frac{A \,\&\, B}{A} \qquad \frac{A \,\&\, B}{B}$$

$$\vee\mathrm{I})\ \frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad\qquad \vee\mathrm{E})\ \frac{A \vee B \qquad \begin{matrix}(A)\\ C\end{matrix} \quad \begin{matrix}(B)\\ C\end{matrix}}{C}$$

$$\rightarrow\mathrm{I})\ \frac{\begin{matrix}(A)\\ B\end{matrix}}{A \rightarrow B} \qquad\qquad \rightarrow\mathrm{E})\ \frac{A \qquad A \rightarrow B}{B}$$

See D. Prawitz, *Natural Deduction. A Proof Theoretical Study*. Stockholm, Göteborg and Uppsala: Almqvist & Wiksell, 1965.

Proposition:

For any a, p and q

$$(\text{Bel } a\, p)(i) = 1 \text{ and } (\text{Bel } a\, q)(i) = 1 \text{ then } (\text{Bel } a\ (p \,\&\, q))(i) = 1$$

From right to left, the implication follows from the fact that

$$(P \,\&\, Q)(i) \rightarrow_{\text{cg}} P(i)$$
$$\text{and } (P \,\&\, Q)(i) \rightarrow_{\text{cg}} Q(i)$$

From left to right, the properties of $\rightarrow_{\text{cg}}$ cannot be used, because we do not have a proposition of the form if (Bel $a\, r)(i)$ then (Bel $a\, r')(i)$. Let us proceed directly. Let us suppose that (Bel $a\, p)(i) = 1$ and (Bel $a\, q)(i) = 1$. So there is a c and a c' such that

$$(p_c, 1) \in Com(a, i)$$
$$\text{and } (q_c, 1) \in Com(a, i)$$

Let c'' be such that for any intension f,
$f_{c''} = \text{join}\,(f_c, f_{c'})$

In particular, we have for P and Q that

$$p_{c''}(i) \mathrel{\varepsilon\!\!-} p_c(i)$$
$$q_{c''}(i) \mathrel{\varepsilon\!\!-} q_{c'}(i)$$

Then

$$(p_{c''}, 1) \in Com(a, i)$$

and

$$(q_{c''}, 1) \in Com(a, i)$$

and so

$$((p \,\&\, q)_{c''}, 1) \in Com(a, i)$$

i.e., $\quad (\text{Bel } a\ (p \,\&\, q))(i) = 1$

As rules corresponding to the introduction and elimination of $\vee$ we have:

Proposition:

(Bel $a\ p)(i) \rightarrow$ (Bel $a\ (p \vee q))$(i), (Bel $a\ q)(i) \rightarrow$ (Bel $a\ (p \vee q))(i)$
and if (Bel $a\ p)(i) \rightarrow$ (Bel $a\ r)(i)$ and (Bel $a\ q)(i) \rightarrow$ (Bel $a\ r)(i)$ then
(Bel $a\ (p \vee q))(i) \rightarrow$ (Bel $a\ r)(i)$

The proofs are straightforward.

Finally, concerning the elimination of $\rightarrow$ we have:

Proposition:

If (Bel a p)(i) = 1 and (Bel a $p \rightarrow q$)(i) = 1 then (Bel a q)(i) = 1

The proof is the following. First, it follows from the property of belief concerning introduction of & that the antecedent of the implication to prove implies

$$(\text{Bel } a\ (p \,\&\, (p \rightarrow q)))(i) = 1$$

But, it can be easily verified that

$$(P \,\&\, (P \rightarrow Q)) \rightarrow_{cg} Q$$

which completes the proof.

As I said, there is one rule for which there is no corresponding belief rule. It is the introduction rule for $\rightarrow$. Such a rule would be

$$\text{if } (\text{Bel } a\ p)(i) \rightarrow (\text{Bel } a\ q)(i)) \text{ then } (\text{Bel } a\ (p \rightarrow q))(i)$$

This proposition is not true. In fact, it is possible, for some agent in some world, for *Com* to be empty, although this implies that this agent in that world has only null representations, i.e., he (or she or it) knows nothing. But besides this limiting case, it might be interesting for our theory of belief to have no universal belief. Moreover, the introduction rule for $\rightarrow$ is not realized for knowledge while it is in the other cases.

These consequences of my definition of belief may be generalized by introducing cognitive implication which would be the intensional counterpart of material cognitive implication: P implies cognitively Q, iff for any $i \in I$, $P \rightarrow_{cg} Q$ is true at i.

In fact this last notion brings us much closer to the kind of implication that must hold between sentences when belief is forced.

By way of conclusion we might add that these considerations suggest that the usefulness and power of our treatment of belief and knowledge rely on our capacity to define restrictions on *gr*'s in order to prove material cognitive implication and cognitive implication where we want to force belief and knowledge. To define such restrictions is yet another task, a task which I believe the present framework will be successfully applied to.

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