ALGEBRAIC THEORIES OF THE SYLLOGISM

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Summary

The present paper is divided in two chapters, each giving a theory and an algebraic method for calculating syllogisms. Chapter I compares the theory given in this issue of *Logique et Analyse* by Alfons Grieder[1], with the theory I presented[2] ten years ago, also in *L & A*. The two theories are *isomorphic*, and both lead to diverse interesting generalisations.

Chapter II is devoted to a new and original theory of the syllogism, based on the notion of *the tensor product* between elements of two "wefts", and allowing for very rapid and easy calculations.

Chapter I

THE GROSJEAN-GRIEDER THEORY

§ *1.1.- Flag-Diagrams and 2-Predicates Matrixes*

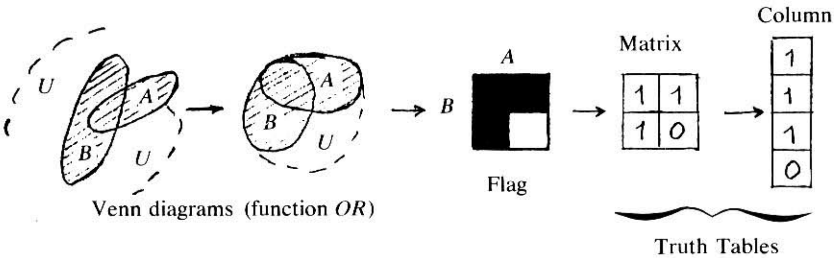
The 16 dyadic functions $f_i(A, B)$ of two predicates A and B can effectively be represented by their Euler-Venn diagrams, as done by Grieder in [1]. But through continuous deformations, such a diagram generates a square *flag-diagram*. If we valuate a flag according to the rule:

(1.1) $\qquad \mathrm{val(Black)} = 1 \qquad \mathrm{val(White)} = 0$

we get a *square truth-function*, i.e. a *matrix* which elements are elements of the *binary set* $\Phi = \{1,0\}$.

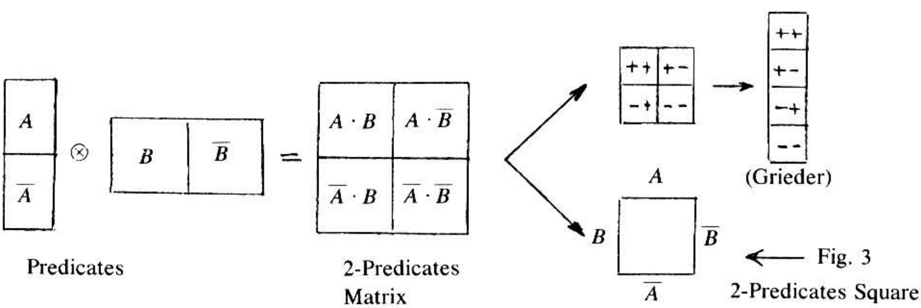
Grieder prefers column-vectors to matrixes for the truth-functions; that procedure is classical and correct, but it causes some losses of symmetries.

Fig. 1 gives as example the function OR (union).



(Fig. 1)

Let $\overline{A}$ be the negation of the predicate A. The tensor product (sign: $\otimes$) of two pairs $(A, \overline{A})$ and $(B, \overline{B})$ gives a *2-predicates matrix* (Fig. 2), which is presented as a column-vector by Grieder (Fig. 2).



(Fig. 2-3)

In each tile of a 2-predicates matrix, the product is the logical one (sign: a dot), i.e. the function AND.

§ *1.2.- Symmetry Group for the Square*

A *2-predicates square* (Fig. 3) is a shortened representation of a 2-predicates matrix. It is well known that such a square is invariant for a *symmetry group* of 8 operators; each of these operators permutes the letters A, $\overline{A}$, B, $\overline{B}$, with a restriction: A must remain opposite $\overline{A}$, and B opposite $\overline{B}$.

In other words, the square of Fig. 3 is supposed to be a rigid cardboard piece, with two faces; its 8 symmetries are schematized in Fig. 4, where the dotted lines are rotation axes. For that Fig. 4, we have used the Grieder's notations E, K, C, L, because the Grieder group is nothing else but the classical symmetry group of the square. The matrixes of Grieder belong to a *degree-4 representation* of that group (matrixes of order 4). A more classical representation of the same group is a degree-2 one,- without interest here.

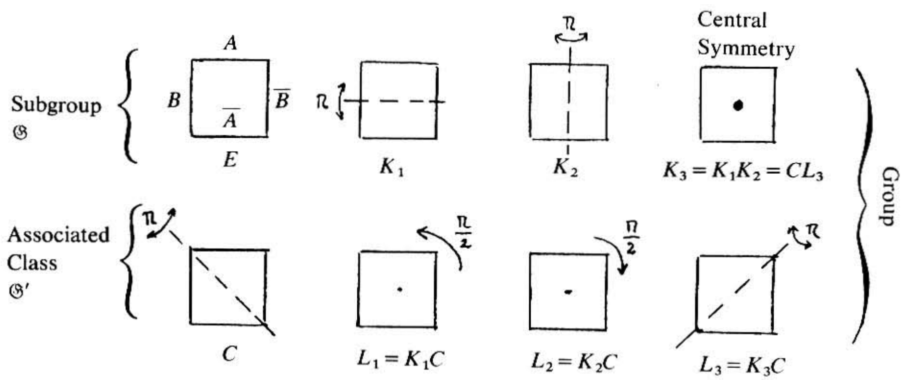
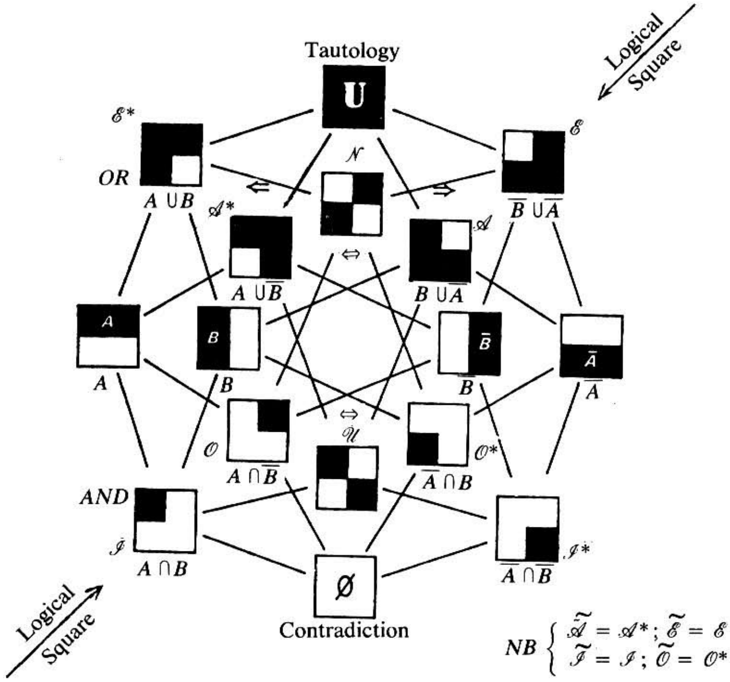


Fig. 4: Symmetry Operations for the Square

§ *1.3.- Boole Wefts*

It is well known that the 16 possible flags can be arranged in a *Boole lattice*, or, better said, in what I called in [2] a *Boole weft* $\mathfrak{W}(A, B)$. A Boole weft is a Boole lattice which basis is also a Boole lattice; the elements of $\mathfrak{W}(A, B)$ are called "events" $X(A, B)$. Fig. 5 is the *Hasse diagram* of that weft.



Symmetry elements and symmetries – { *figure centre*: negation of the function; *vertical axis*: negation of the 2 predicates; *horizontal axis*: order inversion in the lattice

De Morgan duality: $\overline{\mathscr{E}^*} = \mathscr{I}^* = \mathscr{N} \circ \mathscr{I} \circ \mathscr{N}$, i.e.: $\overline{\mathrm{A} \cup \mathrm{B}} = \bar{\mathrm{A}} \cap \bar{\mathrm{B}}$

(Fig. 5)

A Boole weft is also a *vector space* built on the binary set Φ; its 4 basis vectors are the four *AND*'s, and these are also the basis events of the weft. The predicates A and B are the *generators* of the weft $\mathfrak{W}(A, B)$.

If we apply the valuation (1.1), any particular weft $\mathfrak{W}(A, B)$ gets into the same *general weft* $\mathfrak{W}$, the events of which are the 16 matrixes truth-functions. These events are noted, here and in [2], by *cursive capital letters*: $\mathscr{A}$, $\mathscr{B}$,..., $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$.

The Hasse diagram of $\mathfrak{W}$ has been given in [2] (Fig. I, page 549); it is isomorphic to the above diagram of Fig. 5.

The vowels $\mathscr{A}$, $\mathscr{E}$, $\mathscr{J}$, $\mathscr{O}$, are the ones of the famous scholastic *logical square* (Fig. 6).

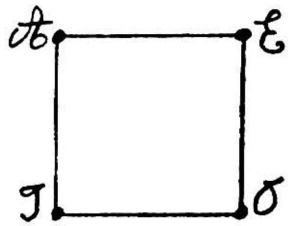


Fig. 6

The basis vectors are noted f_i in [1], with i = 1,2,3,4, and $e_{\alpha\beta}$ in the present paper, with $\alpha, \beta \in \Phi$. We have, in various notation systems:

$$(1.2)\begin{cases} f_1 = e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \mathscr{I} = \overline{\mathscr{E}}\,; \quad f_2 = e_{10} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathscr{O} = \overline{\mathscr{A}} \\ f_3 = e_{01} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathscr{O}^* = \overline{\mathscr{A}}^*\,; f_4 = e_{00} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathscr{I}^* = \overline{\mathscr{E}}^* \end{cases}$$

Notations: a) $\tilde{\mathscr{X}}$ is the *transposed* matrix of $\mathscr{X}$; b) $\mathscr{I}$ and $\mathscr{E}^*$ are respectively the truth-functions of *AND* and of *OR*.

For any $\mathscr{Z} \in \mathfrak{B}$, we have the vectorial decomposition:

$$(1.3) \qquad \mathscr{Z} = \sum_i z_i f_i = \sum_\alpha \sum_\beta z_{\alpha\beta} e_{\alpha\beta} \begin{cases} i = 1,2,3,4 \\ \alpha, \beta \in \Phi \end{cases}$$

At every flag $X(A, B)$ is associated a *normal disjunctive form* (abbreviated notation: NDF), which is the logical sum of all the black tiles of the flag. More mathematically said, we have for the NDF associated to $Z(A, B)$:

$$(1.4) \qquad Z(A, B) \stackrel{\text{def}}{=} (A, \overline{A}) \circ \mathscr{Z} \circ \begin{pmatrix} B \\ \overline{B} \end{pmatrix} =$$

$$(1.5) \qquad = z_{11} \cdot (A \cdot B) + z_{01} \cdot (\overline{A} \cdot B) + z_{10} \cdot (A \cdot \overline{B}) + z_{00} \cdot (\overline{A} \cdot \overline{B})$$

For instance, the NDF of the function OR is $(A \cdot B + A \cdot \overline{B} + \overline{A} \cdot B)$; the equivalent forms $(\overline{A} \cdot B + A)$ and $(A\overline{B} + B)$ are not "normal". The *non-NDF's* shall never be used hereafter.

The weft of the 16 NDF's on A and B will also be noted $\mathfrak{W}(A, B)$.

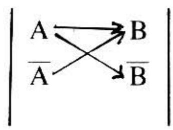
§ *1.4.- Relation Products*

I pointed out in 1970[3] that every dyadic function is at the same time a *binary composition* on Φ and a *relation* on Φ. More generally and more precisely, the matrix of a dyadic function is the *characteristic function* of a relation from $(A, \overline{A})$ to $(B, \overline{B})$.

Remember that a relation $\mathfrak{R}$ from a set $\mathfrak{M}$ to a set $\mathfrak{N}$ is a subset of the cartesian product $\mathfrak{M} \times \mathfrak{N}$. The characteristic function takes the value 1 if $x \in \mathfrak{R}$ and 0 if $x \notin \mathfrak{R}$.

A relation $\mathfrak{R}$ can also be represented by a *graph* (arrows diagram). We have, f.i., for the relation OR:

$$(1.6) \qquad \begin{array}{c|cc} \nearrow & \mathrm{B} & \overline{\mathrm{B}} \\ \hline \mathrm{A} & 1 & 1 \\ \overline{\mathrm{A}} & 1 & 0 \end{array} \quad \leftarrow \text{matrix ; graph} \rightarrow$$



The *relation product* (sign: a little circle) of two relations $\mathfrak{R}_1$ and $\mathfrak{R}_2$ is also a relation $\mathfrak{R}$. The characteristic matrix $\mathscr{R}_3$ of the result is the matricial product of $\mathscr{R}_1$ and $\mathscr{R}_2$; but, in that calculation, the multiplication is the logical one (function AND) and the addition is also the logical one (function OR), which is *idempotent*: $1 + 1 = 1$, $\mathrm{A} + \mathrm{A} = \mathrm{A}$. (The binary addition is nilpotent: $\mathrm{A} + \mathrm{A} = 0$). We have thus:

$$(1.7) \qquad \mathfrak{R}_1 \circ \mathfrak{R}_2 = \mathfrak{R}_3 ; \qquad \mathscr{R}_1 \circ \mathscr{R}_2 = \mathscr{R}_3$$

The graph of $\mathfrak{R}_3$ is the set of *all resultants of the arrows* of $\mathfrak{R}_1$ and of $\mathfrak{R}_2$. For instance, we have for $\mathscr{A}$ (implication), which is idempotent:

$$\mathscr{A} \circ \mathscr{A} = \mathscr{A}$$

$$(1.8)\left\{\begin{array}{cc} \text{matrixes} & \text{graphs} \\ \hline \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \left|\begin{tikzcd} \mathrm{A} \arrow[r] & \mathrm{B} \arrow[r] & \mathrm{C} \\ \overline{\mathrm{A}} \arrow[r] \arrow[ur] & \overline{\mathrm{B}} \arrow[r] \arrow[ur] & \overline{\mathrm{C}} \end{tikzcd}\right| = \left|\begin{tikzcd} \mathrm{A} \arrow[r] & \mathrm{C} \\ \mathrm{A} \arrow[r] \arrow[ur] & \overline{\mathrm{C}} \end{tikzcd}\right| \end{array}\right.$$

In (1.4), the product between the matrix $\mathscr{Z}$ and the vectors $(A, \overline{A})$ or $(B, \overline{B})$ were already relations products.

In the paper[1] of Grieder, the Pythagoras table at the end of its Section 5, is a table of relation products.

§ *1.5.- Syllogisms as Relation Products*

Definition: The *conclusion* $Z(S, P)$ of a *syllogism* is the NDF of an element of $\mathfrak{W}(S, P)$, the matrix $\mathscr{Z}$ being the relation product of $\mathscr{X} \in \mathfrak{W}$ and $\mathscr{Y} \in \mathfrak{W}$. The *premisses* of that syllogism are the *minor* one $X(S, M)$ and the *major* one $Y(M, P)$.

Shortened definition: *A syllogism is a relation product in* $\mathfrak{W}$

$$(1.9)\qquad \begin{array}{ccccc} \mathscr{X} & \circ & \mathscr{Y} & = & \mathscr{Z} \\ \text{(minor)} & \circ & \text{(major)} & = & \text{(conclusion)} \end{array}$$

For instance, (1.8) is the famous "Barbara" of the scholastic logic. For other details or developments, see[2].

To a given syllogism (1.9), are associated 3 equivalent ones; those 4 *expressions* for a same syllogism correspond to the 4 Aristotelician "figures", namely I, II, III, IV:

$$(1.10)\quad \left\{\begin{array}{cccc} \mathscr{X} \circ \mathscr{Y} & \mathscr{X} \circ \widetilde{\mathscr{Y}} & \widetilde{\mathscr{X}} \circ \mathscr{Y} & \widetilde{\mathscr{X}} \circ \widetilde{\mathscr{Y}} \\ & \mathrm{II} & \mathrm{III} & \mathrm{IV} \end{array}\right\}$$

Recall here that the matrix $\widetilde{\mathscr{X}}$ is the transposed one of the matrix $\mathscr{X}$; so, the *relation* $\widetilde{\mathscr{X}}$ is the *converse* of $\mathscr{X}$.

§ *1.6.- Algebra on $\mathfrak{W}$ – Symmetry Group*

The relation product is distributive on the addition in $\mathfrak{W}$. Consequently, its existence gives to the vector space $\mathfrak{W}$ the *structure of an algebra*, the *syllogistic algebra* $\mathfrak{W}$:

$$\left[(\mathscr{X} \in \mathfrak{W}) \text{ and } (\mathscr{Y} \in \mathfrak{W})\right] \Rightarrow \left[(\mathscr{X} \circ \mathscr{Y}) \in \mathfrak{W}\right] \tag{1.11}$$

Remark: "Syllogism" is taken here in a far more general sense than in the Greek logic. Indeed, in that logic, the conclusion *must be only one of the 4 Aristotelician relations* $\mathscr{A}$, $\mathscr{E}$, $\mathscr{I}$, $\mathscr{O}$, while different classical rules do restrict the choice of the premisses. Moreover, 9 of the 24 valid classical syllogisms are "hidden sorites", as can be seen in [2]. The classical syllogistic compositions between the 8 classical relations $\mathscr{A}$, $\widetilde{\mathscr{A}}$, $\mathscr{E}$, $\widetilde{\mathscr{E}}$, $\mathscr{I}$, $\widetilde{\mathscr{I}}$, $\mathscr{O}$, $\widetilde{\mathscr{O}}$, do not generate a complete subalgebra. For instance $\mathscr{I} \circ \mathscr{O} = \mathscr{O}$ is not classical; the tautologic conclusion of $\mathscr{A} \circ \widetilde{\mathscr{A}}$ is not classical.

Group: $\mathfrak{W}$ is provided with one *unit element* $\mathscr{U}$ (the *affirmation*) and with only one other *regular* element, which is the involutive *negation* $\mathscr{N}$ (see Fig. 5). Both form a *group* $\mathfrak{S}$:

$$\mathscr{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \mathscr{N} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \left\{ \begin{array}{l} \mathscr{U} \circ \mathscr{U} = \mathscr{N} \circ \mathscr{N} = \mathscr{U} \\ \mathscr{U} \circ \mathscr{N} = \mathscr{N} \circ \mathscr{U} = \mathscr{N} \end{array} \right\} \mathfrak{S} \tag{1.12}$$

The four order-4 matrixes E, K_1, K_2, K_3 of the Grieder group $\mathfrak{G}$ are tensor products between the matrixes (1.12):

$$\left\{ \begin{array}{ll} E = \mathscr{U} \otimes \mathscr{U} \, , & K_1 = \mathscr{N} \otimes \mathscr{U} \\ K_2 = \mathscr{U} \otimes \mathscr{N} \, , & K_3 = \mathscr{N} \otimes \mathscr{N} \end{array} \right\} \mathfrak{G} \tag{1.13}$$

Consequently, there exists an *isomorphism between the Grieder theory and mine of 1972*. We have, f.i.:

$$\underset{\text{(Grieder)}}{K_1 \cdot f = g} \quad \overset{\text{is}}{\Leftrightarrow} \quad \underset{\text{(Grosjean)}}{\mathscr{N} \circ \mathscr{F} \circ \mathscr{U} = \mathscr{G}} \tag{1.14}$$

And the operator C of Grieder is nothing else but the *transposition* for the matrixes of $\mathfrak{W}$:

(1.15) $\quad C \cdot f = h \quad \overset{\text{is}}{\Leftrightarrow} \quad \mathscr{F} \to \widetilde{\mathscr{F}} = \mathscr{H}$

Combined with C, the group (1.13) gets into the symmetry 8-group of the square. But that group is too rich, because only a few of its subgroups are usefull here. These subgroups divide $\mathfrak{W}$ in 6 subsets (or categories), hereafter enumerated following the number of black tiles (see Fig. 5):

0) The subset $\{\emptyset\}$: invariant for all subgroups.
1) The basis (the four *AND*'s set): invariant for $\mathfrak{G}$.
2a) The group $\mathfrak{S}$: invariant for $\mathfrak{S}$ itself.
2b) The set of the 4 degenerate functions: invariant for $\mathfrak{G}$.
3) The dual basis (the four *OR*'s set): invariant for $\mathfrak{G}$.
4) The subset $\{U\}$: invariant for all subgroups.

NB: The 4 degenerate dyadic functions are $A, \overline{A}, B, \overline{B}$ (Fig. 5); they are mere monodic functions.

"Classical" subalgebra: At the end of Section 5 in [1], one can see the Pythagoras table of a *subalgebra* of $\mathfrak{W}$. According to Grieder, this is the table of the classical syllogisms of the first Aristotelician figure, with *two universal premisses*. That interpretation thus admits that the tautology may be considered as a classical premiss or conclusion (compare with the remark in § 1.6 above).

Basical products: It is well known that an algebra is fully determined by the products between its basis vectors. Here, the basical products are the elements of the S_1-matrix of Grieder, [1] Section 5:

(1.16) $\quad S_{(1)} = \widetilde{f} \otimes f$, where f is the column-vector of the f_i's.

We have:

(1.17) $$\mathscr{X} \circ \mathscr{Y} = \Big(\sum_I x_i f_i\Big) \otimes \Big(\sum_J y_j f_i\Big) = \sum_I \sum_J x_i S_{(1)ij} y_i$$

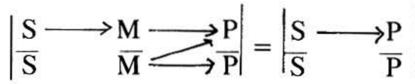
For the other Aristotelician figures, we have the 3 other $S_{(k)}$ matrixes of Grieser. F.i: $\mathscr{X} \circ \widetilde{\mathscr{Y}} = \sum \sum x_i S_{(2)ij} y_j$

§ *1.7.- Some examples*

1.7.1.- **Classical examples**:

a) *Barbara*: see (1.8)

b) *Darii*: $\mathscr{I} \circ \mathscr{A} = \mathscr{I}$

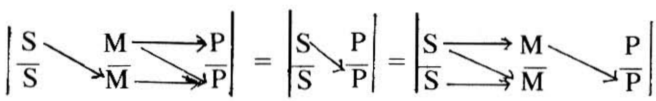
(1.18) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; 

c) *Baroco-Bocardo*: $\mathscr{O} \circ \tilde{\mathscr{A}} = \mathscr{O} = \tilde{\mathscr{A}} \circ \mathscr{O}$

Baroco (2^{nd} figure) – Bocardo (3^{rd} figure)

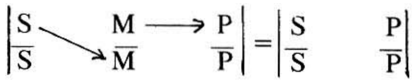
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(1.19)

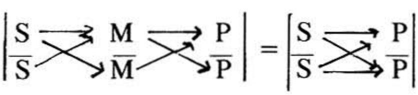


1.7.2.- **Non classical examples**

d) with a void conclusion: $\mathscr{O} \circ \mathscr{I}$ = contradiction = $\mathscr{I} \circ \mathscr{O}$

(1.20) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; 

e) with a tautologic conclusion: $(OR) \circ (OR)$ = Tautology

(1.21) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; 

f) similar calculations and graphs can be made about the syllogism hereafter:

(1.22) $\mathscr{I} \circ \mathscr{I} = \mathscr{I}$

Remarks about that example (f): i) the *implication* $\mathscr{A}$ is an *order relation*, and the *AND* $\mathscr{I}$ is a *projector*, see[2]; thus they are *idempotent*, see (1.8) and (1.22). ii) See the important remark in 2.7.3 hereafter.

Chapter II

TENSORIAL THEORY OF THE SYLLOGISM

§ *2.1.- Generators Independence*

When drawing the navy flags of the weft Fig. 5, we had made an implicit assumption: *The generators A and B are independent.*

Definition: A and B will be independent iff none of the four predicates $A, \overline{A}, B, \overline{B}$ implicates another. In the sets language, the definition is:

$$(2.1) \qquad X \not\subset Y \ , \ \forall X \neq Y \ , \ \forall X, Y \in \{A, \overline{A}, B, \overline{B}\}$$

In other words, no X is the "cause" of any $Y \neq X$.

An equivalent definition is:

$$(2.2) \qquad (A \cdot B \neq 0) \text{ and } (A \cdot \overline{B} \neq 0) \text{ and } (\overline{A} \cdot B \neq 0) \text{ and } (\overline{A} \cdot \overline{B} \neq 0)$$

If that condition is not verified, then at least one of the basis vectors of $\mathfrak{W}(A, B)$ will be zero, and the weft will be degenerated.

Remark: The condition (2.2) is necessary to the usual *probabilisation* of the weft. Indeed, if we have the probabilities $p(A)$ and $p(B)$, then we shall have[3]:

$$(2.3) \qquad p(A \cdot B) = p(A) \cdot p(B)$$

§ 2.2.- *Syllogisms as restricted Tensor Products*

Vector spaces: Let us introduce the 3 *independent predicates* S, M, P, and the 3 wefts (vector spaces) hereafter:

$$(2.4)\quad \begin{cases} \mathfrak{W}(S, M), \text{ the minor-premisses space} \\ \mathfrak{W}(M, P), \text{ the major-premisses space} \\ \mathfrak{W}(S, P), \text{ the conclusions space} \end{cases}$$

Fundamental definition: *A syllogism is a restricted tensor product* between an element of $\mathfrak{W}(S, M)$ and an element of $\mathfrak{W}(M, P)$, the result being an element of $\mathfrak{W}(S, P)$. The composition sign of such a product will be *a star*. Thus:

$$(2.5)\qquad X(S, M) * Y(M, P) = Z(S, P)$$

iff:

$$(2.6)\qquad \mathscr{X} \circ \mathscr{Y} = \mathscr{Z}$$

That star-product will be called here a *syllogistic product*. It is generally *not commutative*, with a notable exception, the classical Baroco-Bocardo, see § 1.6 and 2.6.

Remarks: i) All these elements of the three wefts are NDF, and *never non-NDF* in the calculations. ii) In the same calculations the sign (+) is always the sign of the logical addition, idempotent (recall: $A + A = A$).

NB: In the tensor theories, the dimension of the product-space is the product of the dimensions of the two factor-spaces. Here, the dimension of our three spaces (2.4) is the same, and equal to 4. Therefore, our tensor product is a "restricted" one. In a Euclidian vector space f.i., the usual vector product $\vec{a} \times \vec{b} = \vec{c}$ is a restricted tensor product.

§ 2.3.- *Basical syllogistic products*

A tensor product (restricted or not) is always *distributive on the vector addition*, i.e. it is *bilinear*. Consequently, such a product is fully determined by the products between the basis vectors of the two factor-spaces.

We will give here as an *axiom*:

$$(2.7)\qquad \boxed{(\mathrm{A}\cdot\mathrm{X}) \geqq (\mathrm{Y}\cdot\mathrm{B}) = \begin{cases} (\mathrm{A}\cdot\mathrm{B}) \text{ iff } & \mathrm{X}=\mathrm{Y} \\ 0 \quad \text{ iff } & \mathrm{X}\neq\mathrm{Y} \end{cases}}$$

where (A · X) and (Y · B) are *basis vectors* in two different wefts respectively, and where A and B are independent.

According to (1.4), we have:

$$(2.8)\qquad Z(S,P) = \left[(S,\bar{S}) \circ \mathscr{X} \circ \begin{pmatrix} \mathrm{M} \\ \bar{M} \end{pmatrix} \right] * \left[(M,\bar{M}) \circ \mathscr{Y} \circ \begin{pmatrix} \mathrm{P} \\ \bar{P} \end{pmatrix} \right]$$

Postulating that the products (°) and (*) are *jointly associative* in (2.8), we can formulate an axiom equivalent to (2.7):

$$(2.9)\quad \begin{pmatrix} M \\ \bar{M} \end{pmatrix} * (M,\bar{M}) = \mathscr{U} \text{ i.e: } \boxed{\begin{pmatrix} M*M & M*\bar{M} \\ \bar{M}*M & \bar{M}*\bar{M} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

Introducing (2.9) in (2.8) we obtain immediatly $Z(S, P)$ as in (1.4).

§ *2.4.- Sorites*

A usual tensor product (restricted or not) is *associative*. Consequently, we can define the *sorite*:

A sorite of order n (shortly: a *n-sorite*) is a chain of (n-1) syllogisms, i.e. an (associative) syllogistic product between n NDF's:

$$(2.10)\qquad X_1(M_0, M_1) * X_2(M_1, M_2) * \ldots * X_{n+1}(M_n, M_{n+1}) = Z(M_0, M_{n+1})$$

The product between a n-sorite and a m-sorite is a $(m+n)$-sorite. A

2-sorite is a syllogism; a 1-sorite is a mere logical proposition (WFF) built on two predicates.

§ *2.5.- The Aristotle Group*

Since there exist $16^2 = 256$ relation products on $\mathfrak{B}$, there exist also 256 syllogisms in the above admitted sense of the word. But every one of them admits 4 distinct *expressions*, corresponding to the 4 *Aristotle figures*. Indeed, for a same NDF $F(A, B)$ we have the two equivalent expressions hereafter:

$$(2.11) \qquad \widetilde{F}(B, A) = F(A, B)$$

The 4 figures are generated by what we can call the *Aristotle group* $\mathfrak{A}$, which distributes on the symbols $\mathscr{X}$ and $\mathscr{Y}$ in $\mathscr{X} \circ \mathscr{Y}$, either 0 or 1 or 2 *transposition tildes*, see (1.10).

That group $\mathfrak{A}$ has 4 elements; consequently, there exist $256 \times 4 = 2^{10} = 1024$ syllogistic expressions.

§ *2.6.- The De Morgan Group*

In 1847, Auguste De Morgan introduced a further generalisation of the syllogism notion: For him, each of the 3 predicates of a syllogism may be either affirmative or negative, [4].

So we have to consider the effects of the "De Morgan group" $\mathfrak{D}$, which distributes *negation bars* on the 4 symbols S, M (1^{st} factor), M (2^{nd} factor) and P. Such a group has thus $2^4 = 16$ elements and it generates 16 syllogistic expressions associated to each of the former 1024 expressions (§ 2.5).

The De Morgan syllogisms are mere "expressions" in our theory. Indeed, if we let the group $\mathfrak{G}$ act on a given relation $\mathscr{F}$, we obtain 3 other elements of $\mathfrak{B}$:

$$(2.12) \qquad \mathscr{F}' = \mathscr{N} \circ \mathscr{F} \;,\; \mathscr{F}'' = \mathscr{F} \circ \mathscr{N} \;,\; \mathscr{F}^* = \mathscr{N} \circ \mathscr{F} \circ \mathscr{N}$$

For instance (Fig. 5): $\mathscr{A}'' = \mathscr{E}$, $\mathscr{A}' = \mathscr{N} \circ \mathscr{E} \circ \mathscr{N} = \mathscr{E}^*$, etc.

Thus, for $F(A, B)$, we get the 4 equivalent expressions:

$$F(A, B) = F'(\overline{A}, B) = F''(A, \overline{B}) = F^*(\overline{A}, \overline{B}) \tag{2.13}$$

Finally, we have arrived at $256 \times 2^2 \times 2^4 = 2^{14} = 16384$ *syllogistic expressions*! Of course, very few of them are Greek-classical, and a lot are without any practical interest.

§ 2.7.- *Examples*

The practical calculations based on our "tensor theory" are **very easy and rapid**: No matrixes calculations, no Euler-Venn graphs, no complicated rules as in the Greek logic!

Only recall that:

1.- The 2 factors must be *normal disjunctive* forms (NDF's).
2.- The *first* factor is always the *minor* premiss.
3.- The *basical star-products* are given in (2.9).
4.- The *plus* signifies always the *idempotent addition*.

2.7.1.- **Classical examples**

a) *Barbara*, an idempotent composition between two universal premisses, i.e. *between two orders*:

$$\begin{aligned}&(\mathrm{SM} + \overline{\mathrm{S}}\mathrm{M} + \overline{\mathrm{S}}\,\overline{\mathrm{M}}) \ast (\mathrm{MP} + \overline{\mathrm{M}}\mathrm{P} + \overline{\mathrm{M}}\,\overline{\mathrm{P}}) = \\ &(\mathrm{SP} + \overline{\mathrm{S}}\mathrm{P} + \overline{\mathrm{S}}\,\overline{\mathrm{P}})\end{aligned} \tag{2.14}$$

b) *Darii*: a composition between a *projector* and an *order*:

$$(\mathrm{SM}) \ast (\mathrm{MP} + \overline{\mathrm{M}}\mathrm{P} + \overline{\mathrm{M}}\,\overline{\mathrm{P}}) = (\mathrm{SP})$$

c) *Baroco-Bocardo*, a commutative composition between an *ordinary order* and a *strict order*:

$$\begin{aligned}&\left\{\begin{array}{lll}\text{Baroco, 2}^{\text{nd}}\text{ figure:} & (\mathrm{S}\overline{\mathrm{M}}) \ast (\mathrm{PM} + \overline{\mathrm{P}}\mathrm{M} + \overline{\mathrm{P}}\,\overline{\mathrm{M}}) & = (\mathrm{S}\overline{\mathrm{P}}) \\ \text{Idem in 1}^{\text{st}}\text{ figure:} & (\mathrm{S}\overline{\mathrm{M}}) \ast (\mathrm{MP} + \mathrm{M}\overline{\mathrm{P}} + \overline{\mathrm{M}}\,\overline{\mathrm{P}}) & = (\mathrm{S}\overline{\mathrm{P}})\end{array}\right. \\ &\left\{\begin{array}{lll}\text{Bocardo, 3}^{\text{rd}}\text{ figure:} & (\mathrm{MS} + \overline{\mathrm{M}}\mathrm{S} + \overline{\mathrm{M}}\,\overline{\mathrm{S}}) \ast (\mathrm{M}\overline{\mathrm{P}}) & = (\mathrm{S}\overline{\mathrm{P}}) \\ \text{Idem in 1}^{\text{st}}\text{ figure:} & (\mathrm{SM} + \mathrm{S}\overline{\mathrm{M}} + \overline{\mathrm{S}}\,\overline{\mathrm{M}}) \ast (\mathrm{M}\overline{\mathrm{P}}) & = (\mathrm{S}\overline{\mathrm{P}})\end{array}\right.\end{aligned} \tag{2.15}$$

2.7.2.- **Non classical examples**:

d) nul (or void) conclusion:

(2.16) $$(S\bar{M}) * (MP) = 0$$

e) tautologic conclusion:

(2.17) $$(SM + S\bar{M} + \bar{S}M) * (MP + M\bar{P} + \bar{M}P) = (SP + \bar{S}P + S\bar{P} + \bar{S}\,\bar{P})$$

f) an idempotent composition between two particular premisses, i.e. *between two same projectors* [2]:

(2.18) $$(SM) * (MP) = (SP)$$

2.7.3.- **Important remark**: Classically, (2.18) is said "non-concluent", and therefore rejected by the classical logic; indeed, the conclusion may be a priori, either void or not. Here, that difficulty does not exist because our predicates are always *independent*. So we shall never have $SP = 0$, and the syllogism (2.18) is perfectly correct.

2.7.4.- **Practical examples** of non-classical syllogisms: Let us take a universe which is the population of a given city. We shall have:

d) Jill is strong *and* not tall – But Jill is also tall *and* beautiful – Thus Jill does not exist!

e) Camille is strong *or* tall – But Camille is also tall *or* beautiful – Thus Camille is anybody!

f) John is strong *and* tall – But John is also tall *and* beautiful – Thus John is strong *and* beautiful.

The same, in the classical language: Some strong citizens are tall – Some tall ones are beautiful – Conclusion: Some strong citizens are beautiful (refer here to the above *important remark*).

§ 2.8.- *To sum up:*

The logician disposes thus of three algebraic methods for the syllogisms calculations:

i) The *relations method*, based on the use of relation products between matrixes (Grosjean, 1972) or between vectors (Grieder, 1983).

ii) The *relation graphs method*, very easy; it bridges method (i) and method (iii).

iii) The *tensor products method*, the most original and the most powerful of the three.

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