COMPLETENESS PROOFS FOR THE SYSTEMS RM3 AND BN4

Ross T. Brady

§1. *Introduction*

In [1], on pp. 420-426 (written by Dunn) and on pp. 469-470 (written by Meyer), two axiomatizations of RM3 are given and both are shown to be complete with respect to the following 3-valued matrix set M_3:

$\sim$	
*t	f
*b	b
f	t

&	t	b	f
*t	t	b	f
*b	b	b	f
f	f	f	f

$\rightarrow$	t	b	f
*t	t	f	f
*b	t	b	f
f	t	t	t

There are two designated values, t and b. The method used in both of these completeness proofs relies on Dunn's Extension Theorem ([1], p. 426): Every consistent proper extension of RM has a finite characteristic matrix set. Thus the given completeness proofs for RM3 are specialized proofs which rely on RM3 being a consistent proper extension of RM.

In this paper, I present a completeness proof of an axiomatization of RM3 with respect to the matrix set M_3, and this proof does not depend on Dunn's Extension Theorem, nor on any connection RM3 has with RM. The method of this proof is fairly general and it is hoped that the proof can be modified to enable axiomatizations of other matrix sets to be obtained.

I also present a completeness proof of the same axiomatization of RM3 with respect to a 2 set-up model structure, thereby showing that M_3, and this 2 set-up model structure have the same set of valid formulae. Again, the method of proof is fairly general and it is hoped that other model structures can be axiomatized by appropriate modifications to the proof.

Further, I consider the following 4-valued matrix set M_4:

$\sim$	
*t	f
*b	b
n	n
f	t

&	t	b	n	f
*t	t	b	n	f
*b	b	b	f	f
n	n	f	n	f
f	f	f	f	f

$\rightarrow$	t	b	n	f
*t	t	f	n	f
*b	t	b	n	f
n	t	n	t	n
f	t	t	t	t

There are two designated values, t and b. The matrix set M_4 is an adaptation of the Smiley matrixes, given in [1] on pp. 161-162, which have only the one designated value, t, and have the occurrence of 'b' in the '$\rightarrow$' -matrix of M_4 replaced by 't', and the occurrences of 'n' in the '$\rightarrow$' -matrix of M_4 replaced by 'f'. The Smiley matrix set, and hence M_4, is characteristic for the system E_{fde} of tautological entailments, but M_4 is made more suitable for a system with formulae of any degree.

There is a close relationship between M_4 and M_3, i.e. whereas M_3 can be understood as an extensional logic of sentences which take one or both of the values truth and falsity, M_4 can be understood as an extensional logic of sentences which take one, both or neither of the values truth and falsity. This can be seen from the respective single set-up semantics, given in §2 and §4.

I will give an axiomatization for a system, which I will call BN4,([1]) and I will show that this axiomatization is complete with respect to the above 4-valued matrix set. As for RM3, I will also show that the axiomatization for BN4 is complete with respect to a 2 set-up model structure. Both of these completeness proofs require refinements on the methods of proof of the corresponding completeness theorems given for RM3, and these refinements are then available for use in proving completeness theorems for other matrix sets and model structures.

I will also sketch similar completeness proofs for the Lukasiewicz 3-valued logic and state simple relationships between the axiomatizations of the three systems.

§2. *Completeness of RM3 with respect to M_3*

Axiomatization of RM3. (2)

Primitives: $\sim$, $\&$, $\rightarrow$.

Definitions: $A \vee B =_{df} \sim(\sim A \,\&\, \sim B)$;

$A \leftrightarrow B =_{df} (A \rightarrow B) \,\&\, (B \rightarrow A)$.

Axioms

1. $A \rightarrow A$
2. $A \,\&\, (A \rightarrow B) \rightarrow B$
3. $A \,\&\, B \rightarrow A$
4. $A \,\&\, B \rightarrow B$
5. $(A \rightarrow B) \,\&\, (A \rightarrow C) \rightarrow .\ A \rightarrow B \,\&\, C$
6. $A \,\&\, (B \vee C) \rightarrow (A \,\&\, B) \vee (A \,\&\, C)$
7. $A \rightarrow \sim A \rightarrow \sim A$
8. $A \rightarrow \sim B \rightarrow .\ B \rightarrow \sim A$
9. $\sim\sim A \rightarrow A$
10. $\sim A \,\&\, B \rightarrow .\ A \rightarrow B$
11. $\sim A \rightarrow A \vee (A \rightarrow B)$

Rules

1. $A,\ A \rightarrow B \Rightarrow B$
2. $A,\ B \Rightarrow A \,\&\, B$
3. $A \rightarrow B,\ C \rightarrow D \Rightarrow B \rightarrow C \rightarrow .\ A \rightarrow D$

Theorems

1. $A \rightarrow A \vee B$
2. $A \rightarrow B \vee A$
3. $A \vee \sim A$
4. $A \rightarrow \sim\sim A$
5. $\sim A \rightarrow \sim(A \,\&\, B)$
6. $\sim B \rightarrow \sim(A \,\&\, B)$
7. $\sim(A \,\&\, B) \rightarrow \sim A \vee \sim B$.
8. $\sim B \,\&\, (A \rightarrow B) \rightarrow \sim A$
9. $A \vee \sim B \vee (A \rightarrow B)$. (Use Ax. 10.)
10. $B \rightarrow \sim B \vee (A \rightarrow B)$. (Use Ax. 11.)

11. $A \& \sim B \rightarrow \sim(A \rightarrow B)$.
12. $\sim A \& \sim(A \rightarrow B) \rightarrow A$. (Use Ax. 11.)
13. $B \& \sim(A \rightarrow B) \rightarrow \sim B$. (Use Ax. 11.)

Derived Rules

1. $A \rightarrow B,\ B \rightarrow C \Rightarrow A \rightarrow C$.
2. $A \rightarrow B,\ A \rightarrow C \Rightarrow A \rightarrow B \& C$.
3. $A \rightarrow C,\ B \rightarrow C \Rightarrow A \vee B \rightarrow C$.
4. $A \& B \rightarrow C,\ A \rightarrow C \vee B \Rightarrow A \rightarrow C$. (Use Ax. 6)

Single Set-up Semantics for RM3

In order to facilitate the completeness proof, the 3-valued matrix set M_3 is replaced by the following single set-up semantics with values truth (T) and falsity (F), one or both of which can be assigned to a formula under interpretation.

A *valuation V* assigns a non-empty subset of {T, F} to each sentential variable.(3) Symbolize this as follows:

$V(p) = \{T\}$, $V(q) = \{F\}$, $V(p) = \{T, F\}$, etc.

The valuations V are extended to *interpretations I* of all formulae, inductively, as follows:

(i) $I(p) = V(p)$.
(ii) $T \in I(\sim A) \Leftrightarrow F \in I(A)$.
$F \in I(\sim A) \Leftrightarrow T \in I(A)$.
(iii) $T \in I(A \& B) \Leftrightarrow T \in I(A)$ and $T \in I(B)$.
$F \in I(A \& B) \Leftrightarrow F \in I(A)$ or $F \in I(B)$.
(iv) $T \in I(A \rightarrow B) \Leftrightarrow (T \notin I(A)$ or $T \in I(B))$ and $(F \notin I(B)$ or $F \in I(A))$.
$F \in I(A \rightarrow B) \Leftrightarrow T \in I(A)$ and $F \in I(B)$.

A formula A is *valid* in this semantics iff $T \in I(A)$, for all valuations V.

[Graham Priest, in [3], has also used this semantics for RM3.]

I proceed to show that M_3 and this semantics have the same valid formulae. Let V_M be the M_3-valuation corresponding to the valuation V, as follows: For all sentential variables p,

$V_M(p) = t \Leftrightarrow T \in V(p)$ and $F \notin V(p)$.
$V_M(p) = b \Leftrightarrow T \in V(p)$ and $F \in V(p)$.
$V_M(p) = f \Leftrightarrow T \notin V(p)$ and $F \in V(p)$.

Let I_M be the extension of V_M to all formulae, using the connectives $\sim$, & and $\rightarrow$, as defined by M_3. A formula A is *valid in M_3* iff $I_M(A) = t$ or b, for all valuations V_M, since the valuations V_M are exhaustive.

Lemma 1. For all formulae A, $(T \in I(A) \Leftrightarrow I_M(A) = t$ or b) and $(F \in I(A) \Leftrightarrow I_M(A) = f$ or b).

Proof. The proof is by induction on formulae. The cases for sentential variables p, and formulae $\sim A$ and A & B are clear. The case for $A \rightarrow B$ is as follows:

$T \in I(A \rightarrow B) \Leftrightarrow (T \notin I(A)$ or $T \in I(B))$
and $(F \notin I(B)$ or $F \in I(A))$.
$\Leftrightarrow (I_M(A) = f$ or $I_M(B) = t$ or b)
and $(I_M(B) = t$ or $I_M(A) = f$ or b)
$\Leftrightarrow I_M(A) = f$ or $I_M(B) = t$ or
$(I_M(A) = f$ or b and $I_M(B) = t$ or b)
$\Leftrightarrow I_M(A \rightarrow B) = t$ or b (by M_3).
$F \in I(A \rightarrow B) \Leftrightarrow T \in I(A)$ and $F \in I(B)$
$\Leftrightarrow I_M(A) = t$ or b and $I_M(B) = f$ or b
$\Leftrightarrow I_M(A \rightarrow B) = f$ or b.

Theorem 1. For all formulae A, A is valid in the single set-up semantics for RM3 $\Leftrightarrow$ A is valid in M_3.

Proof. By Lemma 1, $T \in I(A) \Leftrightarrow I_M(A) = t$ or b.
Hence, A is valid in this semantics for RM3 $\Leftrightarrow$ A is valid in M_3.

Theorem 2. (Soundness). For all formulae A,
$\vdash_{RM3} A \Leftrightarrow$ A is valid in the single set-up semantics for RM3.

Proof. I leave this to the reader.

Completeness Proof-Preliminaries

The following preliminaries are needed for the forthcoming completeness theorem, the proof of which is modelled on the Routley-Meyer completeness proof for relevant and para-consistent logics, which can be found in [5], Chapter 4 and also in [4].

An *RM3-theory a* is a set of formulae closed under provable RM3-implication and adjunction, i.e. $\vdash_{\overline{RM3}} A \to B$ and $A \in a \Rightarrow B \in a$, and $A \in a$ and $B \in a \Rightarrow A \& B \in a$.

An RM3-theory a *is prime* iff, for all formulae A and B, $A \vee B \in a \Rightarrow A \in a$ or $B \not\ni a$.

A set b of formulae is *disjunctively RM3-derivable* from a set a of formulae, written $a \succ_{RM3} b$, iff, for some $A_1, \ldots, A_m \in a (m \geq 1)$ and some $B_1, \ldots, B_n \in b (n \geq 1)$, $\vdash_{\overline{RM3}} A_1 \& \ldots \& A_m \to B_1 \vee \ldots \vee B_n$.

A set of formulae a is *RM3-maximal* iff $a \not\succ_{RM3} \bar{a}$.

Lemma 2. If the set a is RM3-maximal then a is a prime RM3-theory.

Proof. (i) Let $\vdash_{\overline{RM3}} A \to B$ and $A \in a$ and $B \notin a$. Then $a \succ_{RM3} \bar{a}$, contradicting the RM3-maximality.

(ii) Let $A \in a$ and $B \in a$ and $A \& B \notin a$. Then, since $\vdash_{\overline{RM3}} A \& B \to A \& B$, $a \succ_{RM3} \bar{a}$, contradicting the RM3-maximality.

(iii) Let $A \vee B \in a$, $A \notin a$ and $B \notin a$. Then since $\vdash_{\overline{RM3}} A \vee B \to A \vee B$, $a \succ_{RM3} \bar{a}$, contradicting the RM3 maximality.

Lemma 3. (Extension). If $a \not\succ_{RM3} b$ then there is a set a′of formulae such that $a \subseteq a\bar{a}$, $b \subseteq \bar{a}'$ and a′ is RM3-maximal.

Proof. Enumerate the formulae of RM3: $A_1, A_2, A_3, \ldots, A_n, \ldots$. Define sets of formulae a_i and b_i, for $i = 0, 1, 2, \ldots$, recursively as follows:

$_0 = a$, $b_0 = b$.

(i) If $a_i \cup \{A_{i+1}\} \succ_{RM3} b_i$ then $a_{i+1} = a_i$ and $b_{i+1} = b_i \cup \{A_{i+1}\}$.

(ii) If $a_i \cup \{A_{i+1}\} \not\succ_{RM3} b_i$ then $a_{i+1} = a_i \cup \{A_{i+1}\}$ and $b_{i+1} = b_i$.

Let $a' = \bigcup_i a_i$ and $b' = \bigcup_i b_i$. Hence $a \subseteq a'$ and $b \subseteq b'$. By construction,

$a' \cup b'$ = set of all formulae. We need to show the following:

(+) $a_i \not\succ_{RM3} b_i$, for all i.

The proof is by induction on i.

$a_0 \not\succ_{RM3} b_0$, by assumption.

Let $a_i \not\succ_{RM3} b_i$. Also, let $a_{i+1} \succ_{RM3} b_{i+1}$.

(a) Suppose $a_i \cup \{A_{i+1}\} \succ_{RM3} b_i$. Then $a_{i+1} = a_i$ and $b_{i+1} = b_i \cup \{A_{i+1}\}$. Hence $a_i \succ_{RM3} b_i \cup \{A_{i+1}\}$. Let the formulae of a_i and b_i in this derivation be respectively $B_1, \ldots, B_m$ and $C_1, \ldots, C_n$. Then $\vdash_{RM3} B_1 \& \ldots \& B_m \to C_1 \vee \ldots \vee C_n \vee A_{i+1}$, i.e. $\vdash_{RM3} B \to C \vee A_{i+1}$, where $B = B_1 \& \ldots \& B_m$ and $C = C_1 \vee \ldots \vee C_n$. Since $a_i \cup \{A_{i+1}\} \succ_{RM3} b_i$, for some conjunction B' of elements of a_i and some disjunction C' of elements of b_i, $\vdash_{RM3} B' \& A_{i+1} \to C'$. Thus, putting $B'' = B \& B'$ and $C'' = C \vee C'$, $\vdash_{RM3} B'' \& A_{i+1} \to C''$ and $\vdash_{RM3} B'' \to C'' \vee A_{i+1}$, using Derived Rules 1, 2 and 3, Axioms 3 and 4, and Theorems 1 and 2, of RM3. By Derived Rule 4, we have $\vdash_{RM3} B'' \to C''$. Hence $a_i \succ_{RM3} b_i$, contradicting our assumption.

(b) Suppose $a_i \cup \{A_{i+1}\} \not\succ_{RM3} b_i$. Then $a_{i+1} = a_i \cup \{A_{i+1}\}$ and $b_{i+1} = b_i$. Since $a_{i+1} \vdash_{RM3} b_{i+1}$, $a_i \cup \{A_{i+1}\} \succ_{RM3} b_i$, which is a contradiction.

Hence, $a_{i+1} \not\succ_{RM3} b_{i+1}$, as required.

By (+), $a' \cap b' = \phi$, since, if $a' \cap b' \neq \phi$ then, for some A and for some i, $A \in a_i$ and $A \in b_i$, and hence $a_i \succ_{RM3} b_i$. Hence, $b' = \bar{a}'$.

In addition, $a' \not\succ_{RM3} b'$, since, if $a' \succ_{RM3} b'$ then there is an i such that $a_i \succ_{RM3} b_i$, which contradicts (+). Hence, a' is RM3-maximal, as required.

Lemma 4. (Priming). Let T be the set of theorems of RM3 and let A be a non-theorem of RM3. Then there is an RM3-theory T' such that $T \subseteq T'$, $A \notin T'$ and T' is prime.

Proof. Immediate from Lemmas 2 and 3.

Any RM3-theory T' which is prime and contains all RM3 theorems induces a valuation V, as in the semantics for RM3 given above, on the sentential variables as follows:

$T \in V(p) \Leftrightarrow p \in T'$.
$F \in V(p) \Leftrightarrow \sim p \in T'$.

We require for such a valuation that, for all sentential variables p, $T \in V(p)$ or $F \in V(p)$. Here, this is so, since $\vdash_{RM3} A \vee \sim A$ (Theorem 3), $T \subseteq T'$ and hence $p \vee \sim p \in T'$, and, by primeness of T', $p \in T'$ or $\sim p \in T'$.

Theorem 3. (Interpretation). The valuation V induced by T′ is extended to an interpretation I of all formulae, such that $(T \in I(A) \Leftrightarrow \in T')$ and $(F \in I(A) \Leftrightarrow \sim A \in T')$.

Proof. When $(T \in I(A) \Leftrightarrow A \in T')$ and $(F \in I(A) \Leftrightarrow \sim A \in T')$ are shown, the interpretation I will have the required property, $T \in I(A)$ or $F \in I(A)$, since, as above, $A \vee \sim A \in T'$ and, $A \in T'$ or $\sim A \in T'$.

The proof is by induction on formulae.

(i) $T \in I(p) \Leftrightarrow T \in V(p)$
$\Leftrightarrow p \in T'$.
$F \in I(p) \Leftrightarrow F \in V(p)$
$\Leftrightarrow \sim p \in T'$.

(ii) $T \in I(\sim A) \Leftrightarrow F \in I(A)$
$\Leftrightarrow \sim A \in T'$ (induction hypothesis)
$F \in I(\sim A) \Leftrightarrow T \in I(A)$
$\Leftrightarrow A \in T'$ (induction hypothesis)
$\Leftrightarrow \sim\sim A \in T'$, by Axiom 9 and Theorem 4 of RM3, and the fact that T′ is an RM3- theory.

(iii) $T \in I(A \& B) \Leftrightarrow T \in I(A)$ and $T \in I(B)$
$\Leftrightarrow A \in T'$ and $B \in T'$ (induction hypothesis)
$\Leftrightarrow A \& B \in T'$, by Axioms 3 and 4 of RM3, and the fact that T′ is an RM3-theory.

$F \in I(A \& B) \Leftrightarrow F \in I(A)$ or $F \in I(B)$
$\Leftrightarrow \sim A \in T'$ or $\sim B \in T'$ (induction hypothesis)
$\Leftrightarrow \sim(A \& B) \in T'$, this step being justified as follows:

By Theorems 5 and t of RM3 and the fact that T′ is an RM3-theory, if $\sim A \in T'$ or $\sim B \in T'$ then $\sim(A \& B) \in T'$.

By Theorem 7 of RM3 and the fact that T′ is a prime RM3-theory, if $\sim(A \& B) \in T'$ then $\sim A \vee \sim B \in T'$ and hence $\sim A \in T'$ or $\sim B \in T'$

(iv) $T \in I(A \to B)$ $\Leftrightarrow$ ($T \notin I(A)$ or $T \in I(B)$) and ($F \notin I(B)$ or $F \in I(A)$).
$\Leftrightarrow$ ($A \notin T'$ or $B \in T'$) and ($\sim B \notin T'$ or $\sim A \in T'$) (induction hypothesis)
$\Leftrightarrow$ $A \to B \in T'$, this step being justified as follows:

First Direction: By Axiom 2, $A \to B \in T' \Rightarrow$. $A \notin T'$ or $B \in T'$.

By Theorem 8 of RM3, $A \to B \in T' \Rightarrow$. $\sim B \notin T'$ or $\sim A \in T'$.
Second Direction: We need to prove: ($A \notin T'$ or $B \in T'$) and ($\sim B \notin T'$ or $\sim A \in T'$) $\Rightarrow A \to B \in T'$.

This is equivalent to:

($A \notin T'$ and $\sim B \notin T'$) or ($A \notin T'$ or $\sim A \in T'$) or ($B \in T'$ and $\sim B \notin T'$) or ($B \in T'$ and $\sim A \in T'$) $\Rightarrow A \to B \in T'$.

This is in turn equivalent to:

($A \notin T'$ and $\sim B \notin T' \Rightarrow A \to B \in T'$), and .....(1)
($A \notin T'$ and $\sim A \in T' \Rightarrow A \to B \in T'$), and .....(2)
($B \in T'$ and $\sim B \notin T' \Rightarrow A \to B \in T'$), and .....(3)
($B \in T'$ and $\sim A \in T' \Rightarrow A \to B \in T'$) .....(4)

By Theorem 9 of RM3, $A \in T'$ or $\sim B \in T'$ or $A \to B \in T'$, and hence $A \notin T'$ and $\sim B \notin T' \Rightarrow A \to B \in T'$, which is (1).

By Axiom 11, $\sim A \in T' \Rightarrow A \in T'$ or $A \to B \in T'$, and hence $A \notin T'$ and $\sim A \in T' \Rightarrow A \to B \in T'$, which is (2).

By Theorem 10 of RM3, $B \in T' \Rightarrow \sim B \in T'$ or $A \to B \in T'$, and hence $B \in T'$ and $\sim B \notin T' \Rightarrow A \to B \in T'$, which is (3).

By Axiom 10, $B \in T'$ and $\sim A \in T' \Rightarrow A \to B \in T$, which is (4).

$F \in I(A \to B)$ $\Leftrightarrow$ $T \in I(A)$ and $F \in I(B)$
$\Leftrightarrow$ $A \in T'$ and $\sim B \in T'$ (induction hypothesis)
$\Leftrightarrow$ $\sim(A \to B) \in T'$, this step being justified as follows:

First Direction: By Theorem 11 of RM3, $A \in T'$ and $\sim B \in T' \Rightarrow \sim(A \rightarrow B) \in T'$.

Second Direction: We need to prove: $\sim(A \rightarrow B) \in T' \Rightarrow A \in T'$ and $\sim B \in T'$.

This is equivalent to: $(\sim(A \rightarrow B) \in T' \Rightarrow A \in T')$ .....(5)

and $(\sim(A \rightarrow B) \in T' \Rightarrow \sim B \in T')$. .....(6)

By Theorem 12 of RM3, $\sim A \in T'$ and $\sim(A \rightarrow B) \in T' \Rightarrow A \in T'$. Hence, $\sim(A \rightarrow B) \in T' \Rightarrow \sim A \notin T'$ or $A \in T'$. Since $\vdash_{RM3} A \vee \sim A$, $A \in T'$ or $\sim A \in T'$, and hence $\sim A \notin T' \Rightarrow A \in T'$. Then, $\sim(A \rightarrow B) \in T' \Rightarrow A \in T'$, which is (5).

By Theorem 13 of RM3, $B \in T'$ and $\sim(A \rightarrow B) \in T' \Rightarrow \sim B \in T'$. Hence, $\sim(A \rightarrow B) \in T' \Rightarrow B \notin T'$ or $\sim B \in T'$. Since $B \in T'$ or $\sim B \in T'$, $B \notin T' \Rightarrow \sim B \in T'$, and hence $\sim(A \rightarrow B) \in T' \Rightarrow \sim B \in T'$, which is (6).

Theorem 4. (Completeness). For all formulae A, A is valid in the single set-up semantics for RM3 $\Rightarrow \vdash_{RM3} A$.

Proof. Let A be a non-theorem of RM3. By Lemma 4, there is a prime RM3-theory T', containing all the theorems of RM3 and such that $A \notin T'$. By Theorem 3, T' induces an interpretation I on all formulae B such that $T \in I(B) \Leftrightarrow B \in T'$. Hence $T \notin I(A)$ and A is invalid in the semantics, since the induced interpretation is an extension of a valuation V of the semantics.

Corollary. For all formulae A, $\vdash_{RM3} A \Leftrightarrow A$ is valid in M_3.

§ 3. *Completeness of RM3 with respect to the 2 set-up model structure for RM3*

2 Set-up Model Structure for RM3

The *model structure for RM3* consists of $<T, K, *, R>$, where K is the set $\{T, T^*\}$, with $T \neq T^*$, $*$ is the 1-1 function on K such that $(T)^* = T^*$ and $(T^*)^* = T$, and R is the 3-place relation on K defined as follows:

$$Rabc =_{df} a \neq T \text{ or } b \neq T \text{ or } c = T.$$

This model structure has been independently determined by Meyer, and by Mortensen in [5]. Mortensen in [5] has shown that, for all formulae A, A is valid in this model structure iff A is valid in M_3. I have independently shown this by a semantic translation method, and the completeness proof here will add a further method.

The *valuations* v on sentential variables are then given at each element a, b of K such that, for all sentential variables p, RTab and $v(p, a) = T \Rightarrow v(p, b) = T$. The only case where $a \neq b$ and RTab is $a = T^*$ and $b = T$. Thus, we require only $v(p, T^*) = T \Rightarrow v(p, T) = T$, for all p.

The valuations v are extended to *interpretations I* of all formulae A in the same way as appears in [4] and in [5], Chapter 4.

A formula A is *true on valuation* v iff $I(A, T) = T$.

A formula A is *valid in this model structure* iff A is true on all valuations v.

Semantic Properties

$a \leqslant b =_{df} RTab$

The following semantic properties can easily be seen to hold, for all $a, b, c \in K$:

(i) $T^* < T$, where $a < b =_{df} a \leqslant b$ and $a \neq b$.
(ii) RTaa, i.e. $a \leqslant a$.
(iii) $a \leqslant b$ and $Rbcd \Rightarrow Racd$.
(iv) $Rabc \Rightarrow Rac^*b^*$, and hence $b \leqslant c \Rightarrow c^* \leqslant b^*$.
(v) Raaa.
(vi) Raa^*a.

By using the semantic properties (ii), (iii) and (iv), and the definition of '$\leqslant$', the following two theorems can be proved, as in [5], Chapter 4, and in [4], p. 208:

(α) For all formulae A, for all $a, b \in K$, $a \leqslant b$ and $I(A, a) = T \Rightarrow I(A, b) = T$.

(β) For all formulae A, B, $I(A \rightarrow B, T) = T \Leftrightarrow (\forall b \in K)\,(I(A, b) = T \Rightarrow I(B, b) = T)$.

Truth Conditions for $A \rightarrow B$

Using (α) and (β), the truth conditions for $A \rightarrow B$ can be simplified, as follows:

$I(A \rightarrow B, T) = T \Leftrightarrow (I(A, T) = T \Rightarrow I(B, T) = T)$
and $(I(A, T^*) = T \Rightarrow I(B, T^*) = T$, by ($\beta$) above.

$I(A \rightarrow B, T^*) = T \Leftrightarrow (\forall b, c \in K)\ (I(A, b) = T \Rightarrow I(B, c) = T)$, since RT^*bc, for all $b, c \in K$.

$\Leftrightarrow$ $(I(A, T) = T \Rightarrow I(B, T) = T)$
and $(I(A, T) = T \Rightarrow I(B, T^*) = T)$
and $(I(A, T^*) = T \Rightarrow I(B, T) = T)$
and $(I(A, T^*) = T \Rightarrow I(B, T^*) = T)$
$\Leftrightarrow$ $(I(A, T) = T \Rightarrow I(B, T^*) = T)$
and $(I(A, T^*) = T \Rightarrow I(B, T^*) = T)$
by property (i) and (α) above.
$\Leftrightarrow$ $(I(A, T) = T \Rightarrow I(B, T^*) = T)$,
by property (i) and (α) above.

Theorem 5. (Soundness). For all formulae A, $\vdash_{RM3} A \Rightarrow A$ is valid in the 2 set-up model structure for RM3.

Proof. The proof follows similar lines to that in [5], Chapter 4.

Completeness Proof-Preliminaries

This completeness proof is also modelled on the Routley-Meyer completeness proof. The definitions of RM3-theory, prime RM3-theory, disjunctively RM3-derivable and RM3-maximal, given in § 2, are also required here. Lemmas 2, 3 and 4 are also required.

Consider a prime RM3-theory T_L containing all the theorems of RM3 and such that $A \notin T_L$, for some non-theorem A of RM3. Such a theory T_L is established by the Priming Lemma.

Define T_L^* as follows:

$$T_L^* =_{df} \{A \mid \sim A \notin T_L\}.$$

Lemma 6. T_L^* has the following properties:

(i) $A \in T_L \Leftrightarrow \sim A \notin T_L^*$.

(ii) T_L^* is an RM3-theory.
(iii) $T_L^* \subseteq T_L$

Proof. I leave this to the reader.

For the model structure for RM3, let the valuation v be determined as follows:

For all sentential variables p, $v(p, T) = T \Leftrightarrow p \in T_L$, and $v(p, T^*) = T \Leftrightarrow p \in T_L^*$.

To satisfy the valuation condition on v, we only require, for all p, $v(p, T^*) = T \Rightarrow v(p, T) = T$. This is so because $T^* \subseteq T$, by Lemma 6.

Theorem 6. (Interpretation). The valuation v is extended to an interpretation I of all formulae, such that $I(A, T) = T \Leftrightarrow A \in T_L$, and $I(A, T^*) = T \Leftrightarrow A \in T_L^*$.

Proof. The proof is by induction on formulae. The proof is clear as it follows along the lines of a similar proof in [5], Chapter 4, and the case for $A \to B$ can be dealt with as in Theorem 3.

Theorem 7. (Completeness). For all formulae A, A is valid in the model structure for RM3 $\Rightarrow \vdash_{RM3}$ A.

Proof. Let A be a non-theorem of RM3. By Lemma 4, there is a prime RM3-theory T_L, containing all the theorems of RM3 and such that $A \notin T_L$. By Theorem 6, T_L, together with T_L^* as defined above, determine a valuation v for the RM3 model structure, which is extended to an interpretation I such that, for all formulae B, $I(B, T) = T \Leftrightarrow B \in T_L$. Hence, $I(A, T) = F$, for this model, and A is invalid in the RM3 model structure.

§ 4. *Completeness of BN4 with respect to M_4*

Axiomatization of BN4

Primitives: $\sim$, &, $\to$.
Definitions: $A \vee B =_{df} \sim(\sim A \mathbin{\&} \sim B)$; $A \leftrightarrow B =_{df} (A \to B) \mathbin{\&} (B \to A)$.

Axioms.

1. $A \to A$.
2. $A \& B \to A$.
3. $A \& B \to B$.
4. $(A \to B) \& (A \to C) \to . \ A \to B \& C$.
5. $A \& (B \vee C) \to (A \& B) \vee (A \& C)$.
6. $A \to {\sim}B \to . \ B \to {\sim}A$.
7. ${\sim}{\sim}A \to A$.
8. ${\sim}A \& B \to . \ A \to B$.
9. ${\sim}A \to A \vee (A \to B)$.
10. $A \vee {\sim}B \vee (A \to B)$.
11. $A \to . \ A \to {\sim}A \to {\sim}A$.
12. $A \vee ({\sim}A \to . \ A \to B)$.

Rules.

1. $A, B \Rightarrow A \& B$.
2. $A, A \to B \Rightarrow B$.
3. $A \to B, C \to D \Rightarrow B \to C \to . \ A \to D$.
4. $C \vee A, C \vee (A \to B) \Rightarrow C \vee B$.

Theorems.

1. $A \to A \vee B$.
2. $(A \& B) \& C \to A \& (B \& C)$.
3. $A \to B \vee A$.
4. $A \vee (B \vee C) \to (A \vee B) \vee C$.
5. $(A \vee B) \& (A \vee C) \to A \vee (B \& C)$.
6. $A \vee A \to A$.
7. $(A \vee B) \vee C \to A \vee (B \vee C)$.
8. $A \to {\sim}{\sim}A$.
9. ${\sim}A \to {\sim}(A \& B)$.
10. ${\sim}B \to {\sim}(A \& B)$.
11. ${\sim}(A \& B) \to {\sim}A \vee {\sim}B$.
12. $A \to B \to . \ {\sim}B \to {\sim}A$.
13. $B \to {\sim}B \vee (A \to B)$. (Use Ax. 9).
14. $A \& {\sim}B \to . \ A \& {\sim}B \to {\sim}(A \to B)$. (Use Ax. 11).
15. $A \vee ({\sim}(A \to B) \to A)$. (Use Ax. 12).
16. ${\sim}(A \to B) \to {\sim}({\sim}B \to {\sim}A)$.

Derived Rule.

1. $A \to B \Rightarrow C \vee A \to C \vee B$.

Single Set-up Semantics for BN4

As for RM3, a single set-up semantics is employed to facilitate the completeness proof. It should be noted that the only difference between this semantics and the corresponding one for RM3 is that this semantics allows neither, as well as both, of the value truth and falsity to be assigned to formulae. This semantics for BN4 is an adaptation of Dunn's semantics for first-degree entailments in [2], the essential differences being in the truth and falsity conditon (iv) for $A \to B$ and the definition of validity.

A *valuation V* assigns a subset of $\{T, F\}$ to each sentential variable. Symbolize this as follows: $V(p) = \{T\}$, $V(p) = \{F\}$, $V(q) = \{T, F\}$, $V(q) = \phi$, etc.

The valuations V are extended to *interpretations I* of all formulae, inductively as follows:

(i) $I(p) = V(p)$.

(ii) $T \in I(\sim A) \Leftrightarrow F \in I(A)$.
$F \in I(\sim A) \Leftrightarrow T \in I(A)$.

(iii) $T \in I(A \& B) \Leftrightarrow T \in I(A)$ and $T \in I(B)$.
$F \in I(A \& B) \Leftrightarrow F \in I(A)$ or $F \in I(B)$.

(iv) $T \in I(A \to B) \Leftrightarrow (T \notin I(A)$ or $T \in I(B))$ and $(F \notin I(B)$ or $F \in I(A))$.
$F \in I(A \to B) \Leftrightarrow T \in I(A)$ and $F \in I(B)$.

A formula A is *valid* in this semantics iff $T \in I(A)$, for all valuations V.

In order to show that M_4 and this semantics have the same valid formulae, let V_M be the M_4-valuation corresponding to the valuation V, as follows:

For all sentential variables p,

$V_M(p) = t \Leftrightarrow T \in V(p)$ and $F \notin V(p)$.
$V_M(p) = b \Leftrightarrow T \in V(p)$ and $F \in V(p)$.
$V_M(p) = n \Leftrightarrow T \notin V(p)$ and $F \notin V(p)$.
$V_M(p) = f \Leftrightarrow T \notin V(p)$ and $F \in V(p)$.

Let I_M be the extension of V_M to all formulae, using the connectives $\sim$, & and $\rightarrow$, as defined by M_4.

A formula A is *valid in* M_4 iff $I_M(A) = t$ or b, for all valuations V_M, since the valuations V_M are exhaustive.

Lemma 7. For all formulae A, $(I(A) = T \Leftrightarrow I_M(A) = t$ or b) and $(I(A) = F \Leftrightarrow I_M(A) = f$ or b).

Proof. The proof is by induction on formulae. I leave this for the reader.

Theorem 8. For all formulae A, A is valid in the single set-up semantics for BN4 $\Leftrightarrow$ A is valid in M_4.

Proof. By Lemma 7, $T \in I(A) \Leftrightarrow I_M(A) = t$ or b.
Hence, A is valid in this semantics for BN4 $\Leftrightarrow$ A is valid in M_4.

Theorem 9. (Soundness). For all formulae A, $\vdash_{BN4} A \Rightarrow A$ is valid in the single set-up semantics for BN4.

Proof. I leave this for the reader.

Completeness Proof-Preliminaries

Although the preliminaries are similar to those required for the completeness proof for RM3, refinements are made to the notions of BN4-theory and BN4-derivability.

A *BN4-theory a* is a set of formulae closed under the following closure conditions:

(ii) $A \in a$ and $B \in a \Rightarrow A \& B \in a$.
(ii) $\vdash_{BN4} A \rightarrow B$ and $A \in a \Rightarrow B \in a$.
(iii) $A \rightarrow B \in a$ and $A \in a \Rightarrow B \in a$.
(iv) $C \vee (A \rightarrow B) \in a$ and $C \vee A \in a \Rightarrow C \vee B \in a$.

A BN4-theory a is *prime* iff, for all formulae A and B, $A \vee B \in a \Rightarrow A \in a$ or $B \in a$.

A formula B is BN4-derivable from a formula A, written $A \vdash_{BN4} B$, iff B is derivable from A by successive applications of the following rules:

(i) $A, B \Rightarrow A \& B$.
(ii) $A \Rightarrow B$, where $\vdash_{BN4} A \rightarrow B$.
(iii) $A, A \rightarrow B \Rightarrow B$.
(iv) $C \vee A, C \vee (A \rightarrow B) \Rightarrow C \vee B$.

A set b of formulae is BN4-derivable from a set a of formulae, written $a \vdash_{BN4} b$, iff, for some $A_1, \ldots, A_m \in a$ ($m \geq 1$) and for some $B_1, \ldots, B_n \in b$ ($n \geq 1$), $A_1 \& \ldots \& A_m \vdash_{BN4} B_1 \vee \ldots \vee B_n$.

A set of formulae a is *BN4-maximal* iff $a \nvdash_{BN4} \bar{a}$.

Lemma 8. If the set a is BN4-maximal then a is a prime BN4-theory.

Proof:

(i) Let $A \in a$, $B \in a$ and $A \& B \notin a$. Then, since $\vdash_{BN4} A \& B \rightarrow A \& B$ and $A \& B \vdash_{BN4} A \& B$, $a \vdash_{BN4} \bar{a}$, contradicting the BN4-maximality.

(ii) Let $\vdash_{BN4} A \rightarrow B$, $A \in a$ and $B \notin a$. Then, since $A \vdash_{BN4} B$, $a \vdash_{BN4} \bar{a}$, contradicting the BN4-maximality.

(iii) Let $A \rightarrow B \in a$, $A \in a$ and $B \notin a$. Since $\vdash_{BN4} A \& B \rightarrow A$ and $\vdash_{BN4} A \& B \rightarrow B$, $(A \rightarrow B) \& A \vdash_{BN4} A$ and $(A \rightarrow B) \& A \vdash_{BN4} A \rightarrow B$, and hence $(A \rightarrow B) \& A \vdash_{BN4} B$, by rule (iii). Then $a \vdash_{BN4} \bar{a}$, contradicting the BN4-maximality.

(iv) Let $C \vee (A \rightarrow B) \in a$, $C \vee A \in a$ and $C \vee B \notin a$. Since $\vdash_{BN4} A \& B \rightarrow A$ and $\vdash_{BN4} A \& B \rightarrow B$, $(C \vee (A \rightarrow B)) \& (C \vee A) \vdash_{BN4} C \vee (A \rightarrow B)$ and $(C \vee (A \rightarrow B)) \& (C \vee A) \vdash_{BN4} C \vee A$, and hence $(C \vee (A \rightarrow B)) \& (C \vee A) \vdash_{BN4} C \vee B$, by rule (iv). Then $a \vdash_{BN4} \bar{a}$, contradicting the BN4-maximality.

(v) Let $A \vee B \in a$, $A \notin a$ and $B \notin a$. Since $\vdash_{BN4} A \vee B \rightarrow A \vee B$, $A \vee B \vdash_{BN4} A \vee B$ and hence $a \vdash_{BN4} \bar{a}$, contradicting the BN4-maximality.

Lemma 9. (Extension). If $a \nvdash_{BN4} b$ then there is a set a' of formulae such that $a \subseteq a'$, $b \subseteq \bar{a}'$ and a' is BN4-maximal.

Proof. Enumerate the formulae of BN4: $A_1, A_2, A_3, \ldots, A_n, \ldots$. Define sets of formulae a_i and b_i, for $i = 0, 1, 2, \ldots$, recursively as follows:

$$a_0 = a,\ b_0 = b.$$

(i) If $a_i \cup \{A_{i+1}\} \vdash_{\overline{\mathrm{BN4}}} b_i$ then $a_{i+1} = a_i$ and $b_{i+1} = b_i \cup \{A_{i+1}\}$.

(ii) If $a_i \cup \{A_{i+1}\} \nvdash_{\overline{\mathrm{BN4}}} b_i$ then $a_{i+1} = a_i \cup \{A_{i+1}\}$ and $b_{i+1} = b_i$.

Let $a' = \bigcup_i a_i$ and $b' = \bigcup_i b_i$. Hence $a \subseteq a'$ and $b \subseteq b'$.

By construction, $a' \cup b'$ = set of all formulae. We need to show:

(ξ) $a_i \nvdash_{\overline{\mathrm{BN4}}} b_i$, for all i.

The proof is by induction on i.

$a_0 \nvdash_{\overline{\mathrm{BN4}}} b_0$, by assumption.

Let $a_i \nvdash_{\overline{\mathrm{BN4}}} b_i$ and let $a_{i+1} \vdash_{\overline{\mathrm{BN4}}} b_{i+1}$.

(a) Suppose $a_i \cup \{A_{i+1}\} \vdash_{\overline{\mathrm{BN4}}} b_i$. Then $a_{i+1} = a_i$ and $b_{i+1} = b_i \cup \{A_{i+1}\}$. Hence $a_i \vdash_{\overline{\mathrm{BN4}}} b_i \cup \{A_{i+1}\}$. Let formulae of a_i and b_i in this derivation be respectively $B_1, \ldots, B_m$ and $C_1, \ldots, C_n$. Then $B_1 \& \ldots \& B_m \vdash_{\overline{\mathrm{BN4}}} C_1 \vee \ldots \vee C_n \vee A_{i+1}$, i.e. $B \vdash_{\overline{\mathrm{BN4}}} C \vee A_{i+1}$, where $B = B_1 \& \ldots \& B_m$ and $C = C_1 \vee \ldots \vee C_n$. Since $a_i \cup \{A_{i+1}\} \vdash_{\overline{\mathrm{BN4}}} b_i$, for some conjunction B' of elements of a_i and some disjunction C' of elements of b_i, $B' \& A_{i+1} \vdash_{\overline{\mathrm{BN4}}} C'$. Putting $B'' = B \& B'$ and $C'' = C' \vee C$, $B'' \& A_{i+1} \vdash_{\overline{\mathrm{BN4}}} C''$, $B'' \vdash_{\overline{\mathrm{BN4}}} C'' \vee A_{i+1}$, and hence $B'' \vdash_{\overline{\mathrm{BN4}}} C''$ can be obtained as follows:

(α) $B'' \& A_{i+1} \vdash_{\overline{\mathrm{BN4}}} C''$

By Theorem 1 of BN4, $C' \vdash_{\overline{\mathrm{BN4}}} C' \vee C$ and, since $B' \& A_{i+1} \vdash_{\overline{\mathrm{BN4}}} C'$, $B' \& A_{i+1} \vdash_{\overline{\mathrm{BN4}}} C''$. By Axiom 3, $B \& (B' \& A_{i+1}) \vdash_{\overline{\mathrm{BN4}}} B' \& A_{i+1}$, and hence $B \& (B' \& A_{i+1}) \vdash_{\overline{\mathrm{BN4}}} C''$. By Theorem 2 of BN4, $(B \& B') \& A_{i+1} \vdash_{\overline{\mathrm{BN4}}} B \& (B' \& A_{i+1})$ and hence $B'' \& A_{i+1} \vdash_{\overline{\mathrm{BN4}}} C''$.

(β) $B'' \vdash_{\overline{\mathrm{BN4}}} C'' \vee A_{i+1}$

By Axiom 2, $B \& B' \vdash_{\overline{\mathrm{BN4}}} B$ and, since $B \vdash_{\overline{\mathrm{BN4}}} C \vee A_{i+1}$, $B'' \vdash_{\overline{\mathrm{BN4}}} C \vee A_{i+1}$. By Theorem 3 of BN4, $C \vee A_{i+1} \vdash_{\overline{\mathrm{BN4}}} C' \vee (C \vee A_{i+1})$, and hence $B'' \vdash_{\overline{\mathrm{BN4}}} C' \vee (C \vee A_{i+1})$. By Theorem 4 of BN4, $C' \vee (C \vee A_{i+1}) \vdash_{\overline{\mathrm{BN4}}} (C' \vee C) \vee A_{i+1}$, and hence $B'' \vdash_{\overline{\mathrm{BN4}}} C'' \vee A_{i+1}$.

(γ) $B'' \vdash_{\overline{\mathrm{BN4}}} C''$

By Theorem 3 of BN4, $B'' \vdash_{\overline{BN4}} C'' \vee B''$ and hence, by (β), $B'' \vdash_{\overline{BN4}} C'' \vee B''$, $C'' \vee A_{i+1}$. Then $B'' \vdash_{\overline{BN4}} (C'' \vee B'') \& (C'' \vee A_{i+1})$ and, by Theorem 5 of BN4, $(B \& A_{i+1})$ and then $B'' \vdash_{\overline{BN4}} C'' \vee (B'' \& A_{i+1})$. We now need to show that $(A \vdash_{\overline{BN4}} B) \Rightarrow (C \vee A \vdash_{\overline{BN4}} C \vee B)$, since, using it, we obtain, by (α), $C'' \vee (B'' \& A_{i+1}) \vdash_{\overline{BN4}} C'' \vee C''$ and hence $B'' \vdash_{\overline{BN4}} C'' \vee C''$. By Theorem 6 of BN4, $C'' \vee C'' \vdash_{\overline{BN4}} C''$ and then $B'' \vdash_{\overline{BN4}} C''$, as required.

(δ) $(A \vdash_{\overline{BN4}} B) \Rightarrow (C \vee A \vdash_{\overline{BN4}} C \vee B)$.

The proof is by induction on the derivation of B from A by using rules (i)-(iv). We show that '$C \vee$' can be placed in front of each step of this derivation.

(i) The rule $D, E \Rightarrow D \& E$ is applied. $C \vee D$, $C \vee E \Rightarrow (C \vee D) \& (C \vee E)$ and, by Theorem 5 of BN4, $(C \vee D) \& (C \vee E) \Rightarrow C \vee (D \& E)$ and hence $C \vee D, C \vee E \vdash_{\overline{BN4}} C \vee (D \& E)$.

(ii) The rule $D \Rightarrow E$ is applied, where $\vdash_{\overline{BN4}} D \rightarrow E$. By Derived Rule 1 of BN4, $\vdash_{\overline{BN4}} C \vee D \rightarrow C \vee E$ and hence $C \vee D \vdash_{\overline{BN4}} C \vee E$.

(iii) The rule $D, D \rightarrow E \Rightarrow E$ is applied. By rule (iv), $C \vee D$, $C \vee (D \rightarrow E) \vdash_{\overline{BN4}} C \vee E$.

(iv) The rule $F \vee D, F \vee (D \rightarrow E) \Rightarrow F \vee E$ is applied.

By Theorem 5 of BN4, $C \vee (F \vee (D \rightarrow E)) \Rightarrow (C \vee F) \vee (D \rightarrow E)$ and $C \vee (F \vee D) \Rightarrow (C \vee F) \vee D$.

Hence, $C \vee (F \vee D)$, $C \vee (F \vee (D \rightarrow E)) \vdash_{\overline{BN4}} (C \vee F) \vee D$, $(C \vee F) \vee (D \rightarrow E)$ and hence, by rule (iv), $C \vee (F \vee D)$. By Theorem 7 of BN4, $(C \vee F) \vee E \vdash_{\overline{BN4}} C \vee (F \vee E)$ and hence $C \vee (F \vee D)$, $C \vee (F \vee (D \vee E)) \vdash_{\overline{BN4}} C \vee (F \vee E)$.

Proceeding with the proof of Lemma 8, since $B'' \vdash_{\overline{BN4}} C''$, $a_i \vdash_{\overline{BN4}} b_i$, contradicting our assumption.

(*b*) Suppose $a_i \cup \{A_{i+1}\} \nvdash_{BN4} b_i$. Then $a_{i+1} = a_i \cup \{A_{i+1}\}$ and $b_{i+1} = b_i$. Since $a_{i+1} \vdash_{\overline{BN4}} b_{i+1}$, $a_i \cup \{A_{i+1}\} \vdash_{\overline{BN4}} b_i$, which is a contradiction.

Hence, $a_{i+1} \nvdash_{BN4} b_{i+1}$, as required.

By (ξ), $a' \cap b' = \emptyset$, since, if $a' \cap b' \neq \emptyset$ then, for some A, for some i, $A \in a_i$ and $A \in b_i$ and hence $a_i \vdash_{\overline{BN4}} b_i$. Hence, $b' = \overline{}a'$.

In addition, $a' \nvdash_{BN4} b'$, since, if $a' \vdash_{\overline{BN4}} b'$ then there is an i such that $a_i \vdash_{\overline{BN4}} b_i$, which contradicts ($\xi$).

Hence, a' is BN4-maximal, as required.

Lemma 10. (Priming). Let T be the set of theorems of BN4 and let A be a non-theorem of BN4. Then there is a BN4-theory T' such that $T \subseteq T'$, $A \notin T'$ and T' is prime.

Proof. To prove $T \nvdash_{BN4} \{A\}$, let $T \vdash_{BN4} \{A\}$. Then, for some $A_1, \ldots, A_m \in T$, $A_1 \& .. \& A_m \vdash_{BN4} A$. Since T is a BN4 theory, $A_1 \& \ldots \& A_m \in T$ and, further, $A \in T$, because T is closed under each of the four rules (i)-(iv) that can be used in establishing BN4-derivability. However, $A \notin T$ and hence $T \nvdash_{BN4} \{A\}$. By Lemma 9, there is a set T' such that $T \subseteq T'$, $A \notin T'$ and T' is BN4-maximal. Hence, by Lemma 8, T' is a prime BN4-theory.

Any BN4-theory T' which is prime and contains all BN4 theorems induces a valuation V, as in the semantics for BN4 above, on the sentential variables, as follows:

$$T \in V(p) \Leftrightarrow p \in T'.$$
$$F \in V(p) \Leftrightarrow \sim p \in T'.$$

Theorem 10. (Interpretation). The valuation V induced by T' is extended to an interpretation I of all formulae, such that $(T \in I(A) \Leftrightarrow A \in T')$ and $(F \in I(A) \Leftrightarrow \sim A \in T')$.

Proof. The proof is by induction on formulae, and follows similar lines to that of Theorem 3. Use is made of Axioms 2, 3, 7, 8, 9 and 10 and Theorems 8-16 of BN4.

Theorem 11. (Completeness). For all formulae A, A is valid in the single set-up semantics for $BN4 \Rightarrow \vdash_{BN4} A$.

Proof. Let A be a non-theorem of BN4. By Lemma 10, there is a prime BN4-theory T', containing all the theorems of BN4 such that $A \notin T'$. By Theorem 10, T' induces an interpretation I on all formulae B such that $T \in I(B) \Leftrightarrow B \in T'$. Hence $T \notin I(A)$ and A is invalid in the semantics, since the induced interpretation is an extension of a valuation V of the semantics.

Corollary. For all formulae A, $\vdash_{BN4} A \Leftrightarrow A$ is valid in M_4.

§ 5. *Completeness of BN4 with respect to the 2 set-up model structure for BN4*

The *model structure for BN4* ([4]) consists of $< T, K, *, R >$, where K is the set $\{T, T^*\}$, with $T \neq T^*$, $*$ is the 1-1 function on K such that $(T)^* = T^*$ and $(T^*)^* = T$, and R is the 3-place relation on K defined as follows:

$$Rabc =_{df} (a \neq T \text{ or } b = c) \text{ and } (a \neq T^* \text{ or } (b = T \text{ and } c = T^*)).$$

The *valuations* v on sentential variables are then given at each element a, b of K such that, for all sentential variables p, RTab and $v(p, a) = T \Rightarrow v(p, b) = T$. Since $RTab \Leftrightarrow a = b$, this condition always applies.

The valuations v are extended to *interpretations I* of all formulae A in the same way as appears in [4] and in [5], Chapter 4.

A formula A is *true on valuation* v iff $I(A, T) = T$.

A formula A is *valid in this model structure* iff A is true on all valuations v.

Semantic Properties

$$a \leqslant b =_{df} RTab$$

The following semantic properties can easily be seen to hold, for all a, b, $c \in K$:

(i) $a \leqslant b \Leftrightarrow a = b$.
(ii) $Rabc \Rightarrow Rac^*b^*$.

Truth Conditions for $A \rightarrow B$

$I(A \rightarrow B, T) = T \Leftrightarrow (I(A, T) = T \Rightarrow I(B, T) = T)$ and $(I(A, T^*) = T \Rightarrow I(B, T^*) = T)$, since $RTab \Leftrightarrow a = b$.

$I(A \rightarrow B, T^*) = T \Leftrightarrow I(A, T) = T \Rightarrow I(B, T^*) = T$, since $RT^*ab \Leftrightarrow a = T$ and $b = T^*$.

Theorem 12. (Soundness). For all formulae A, $\vdash_{BN4} A \Rightarrow A$ is valid in the 2 set-up model structure for BN4.

Proof. The proof follows similar lines to that in [5], Chapter 4.

Completeness Proof-Preliminaries

The definitions of BN4-theory, prime BN4-theory, BN4-derivable and BN4-maximal, given in § 4, are also required here. Lemmas 8, 9 and 10 are also required.

Consider a prime BN4-theory T_L containing all the theorems of BN4 and such that $A \notin T_L$, for some non-theorem A of BN4. Such a theory T_L is established by Lemma 10.

Define T_L^* as follows:

$$T_L^* =_{df} \{A \mid {\sim}A \notin T_L\}.$$

Lemma 11. T_L^* has the following properties:

(i) $A \in T_L \Leftrightarrow {\sim}A \notin T_L^*$

(ii) T_L^* is closed under Adjunction and provable BN4-implication, i.e. T_L^* is closed under the first two closure conditions of a BN4-theory.

Proof. I leave this to the reader.

For the model structure for BN4, let the valuation v be determined as follows:

For all sentential variables p, $v(p, T) = T \Leftrightarrow p \in T_L$, and $v(p, T^*) = T \Leftrightarrow p \in T_L^*$. The valuation condition on v applies.

Theorem 13. (Interpretation). The valuation v is extended to an interpretation I of all formulae, such that $I(A, T) = T \Leftrightarrow A \in T_L$, and $I(A, T^*) = T \Leftrightarrow A \in T_L^*$.

Proof. The proof is by induction on formulae. The proof is clear as it follows along similar lines to a proof in [5], Chapter 4. The case for $A \rightarrow B$ can be dealt with as in Theorem 3.

Theorem 14. (Completeness). For all formulae A, A is valid in the model structure for BN4 $\Rightarrow \vdash_{BN4}$ A.

Proof. Let A be a non-theorem of BN4. By Lemma 10, there is a prime BN4-theory T_L, containing all the theorems of BN4 and such that $A \notin T_L$. By Theorem 13, T_L, together with T_L^* as defined above, determine a valuation v for the BN4 model structure which is extended to an interpretation I such that, for all formulae B, $I(B, T) = T \Leftrightarrow B \in T_L$.

Hence, $I(A, T) = F$ for this model, and A is invalid in the BN4 model structure. (5)

§ 6. *Concluding Remark*

A similar treatment can be given for Lukasiewicz's 3-valued logic as was given for RM3 and BN4 above. This logic is represented by the following matrix set L_3:

~	
*t	f
n	n
f	t

&	t	n	f
*t	t	n	f
n	n	n	f
f	f	f	f

→	t	n	f
*t	t	n	f
n	t	t	n
f	t	t	t

It can be shown that any formula A is valid in L_3 iff A is valid in the following single set-up semantics. The single set-up semantics for L_3 is the same as that for RM3, except that valuations V assign a *proper subset* of {T, F} to each sentential variable. The valuations V are extended to interpretations I using the same T- and F- conditions as for RM3. The valuations correspond as follows:

$V_M(p) = t \Leftrightarrow T \in V(p)$ and $F \notin V(p)$.
$V_M(p) = n \Leftrightarrow T \notin V(p)$ and $F \notin V(p)$.
$V_M(p) = f \Leftrightarrow T \notin V(p)$ and $F \in V(p)$.

Using a completeness proof similar to that for BN4, L_3 can be axiomatized as L3: BN4 + $A \,\&\, {\sim}A \rightarrow . A \,\&\, {\sim}A \rightarrow B$, where $A \,\&\, {\sim}A \rightarrow . A \,\&\, {\sim}A \rightarrow B$ ensures that the prime L3- theory T' cannot contain both A and ~A, for any formula A, and thus the valuation V induced by T' is always a proper subset of {T, F}.

There is also a 2 set-up model structure for L3, which is the same as the one for BN4 except that Rabc is defined as ($a \neq T$ or $b \neq T^*$ or $c = T^*$) and ($a \neq T^*$ or ($b = T$ and $c = T^*$)). The valuations v must satisfy the condition: For all sentential variables p, $v(p, T) = T \Rightarrow v(p, T^*) = T$. The axiomatization L3 above can be shown to be complete with respect to this model structure, in a similar manner to that for BN4.

It is also worth noting that a completeness proof for RM3 can be carried out in the same manner as for BN4, resulting in the axiomatization of RM3 as BN4 + A $\vee$ ~A.

La Trobe University
and
Australian National University

Ross T. BRADY

NOTES

(1) This name is chosen because the system contains the basic system B of [5], Chapter 4, and has a characteristic 4-valued matrix set, one of the values being 'n', representing neither truth nor falsity.

(2) I acknowledge help from R. K. MEYER in getting the axiomatization into this form.

(3) The idea of allowing both values T and F to be assigned to formulae comes from J.M. DUNN. In [2], Dunn also allows neither T nor F to be assigned to formulae.

(4) I obtained this model structure for BN4 by using the Mortensen method of constructing model structures from matrix sets, as described in [5].

(5) I acknowledge help from a referee of Logique et Analyse in putting this paper in a suitable form for publication.

REFERENCES

[1] A.R. ANDERSON and N.D. BELNAP, Jr., «Entailment, the Logic of Relevance and Necessity,» Vol. I, Princeton, 1975.

[2] J.M. DUNN, «Intuitive Semantics for First-Degree Entailments and 'Coupled Trees',» Philosophical Studies, Vol. 29 (1976), pp. 149-168.

[3] G. PRIEST, «Son of Logic of Paradox», presented at A.N.U. Logic seminar, 1978.

[4] R. ROUTLEY and R.K. MEYER, «The Semantics of Entailment», in H. Leblanc (ed.), «Truth, Syntax and Modality», pp. 199-243.

[5] R. ROUTLEY and R.K. MEYER *et al.*, «Relevant Logics and their Rivals,» Nijhoff (forthcoming in 1982).