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A REFINEMENT OF THE CRAIG–LYNDON INTERPOLATION THEOREM FOR CLASSICAL FIRST-ORDER LOGIC WITH IDENTITY

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Abstract

We refine the interpolation property of classical first-order logic (without identity and without function symbols), showing that if $\Gamma \nvdash$, $\nvdash \Delta$ and $\Gamma \vdash \Delta$ then there is an interpolant χ, constructed using only non-logical vocabulary common to both members of Γ and members of Δ, such that (*i*) Γ entails χ in the first-order version of Kleene's strong three-valued logic, and (*ii*) χ entails Δ in the first-order version of Priest's Logic of Paradox. The proof proceeds *via* a careful analysis of derivations employing semantic tableaux. Lyndon's strengthening of the interpolation property falls out of an observation regarding such derivations and the steps involved in the construction of interpolants.

Through an analysis of tableaux rules for identity, the proof is then extended to classical first-order logic with identity (but without function symbols).

Keywords: Craig–Lyndon Interpolation Theorem (for classical first-order logic), Kleene's strong 3-valued logic, Priest's Logic of Paradox, Belnap's four-valued logic, block tableaux.

1. Introduction

In our (2016) we gave a constructive proof of what is there called a "non-classical refinement" of the interpolation property for classical propositional logic, *i.e.*, a constructive proof that when $\nvDash \neg\phi$, $\nvDash \psi$ and $\phi \vDash \psi$ in the $\{\wedge, \vee, \neg\}$-fragment of classical propositional logic, there is an interpolant χ, constructed using only propositional variables common to both ϕ and ψ, such that (*i*) ϕ entails χ in Kleene's strong three-valued logic (Kleene 1952, §64), here called *K*3, and (*ii*) χ entails ψ in Priest's Logic of Paradox (Priest 1979), here called *LP*. There the proof is semantic; here we employ a modification of what Raymond Smullyan (1968) calls *Hintikka's block tableaux*. We show how to extend that result to classical first-order logic (with neither identity nor function-symbols) and obtain related results for first-order *K*3 and *LP* and for Belnap's four-valued logic (Belnap 1977), here called *B*4. A refinement of Lyndon's

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strengthening falls out of an elementary observation regarding block tableau.

The results are then extended to encompass identity (but not function symbols). We obtain a refinement of Arnold Oberschelp's sharpening of the Craig–Lyndon Interpolation Theorem for classical first-order logic with identity (Oberschelp 1968). The extension to encompass identity is carried out directly, by a case-by-case examination of block tableau rules for identity.

Quite what significance should be attached, in general, to a logic's possessing an interpolation property is hard to say. As we'll see, Belnap's four-valued logic has the property that if $\Gamma \vdash_{B4} \Delta$ then there is an interpolant χ whose non-logical vocabulary occurs in formulas in Γ and in formulas in Δ and such that $\Gamma \vdash_{B4} \chi$ and $\chi \vdash_{B4} \Delta$. Kleene's three-valued logic has the property that if $\Gamma \vdash_{K3} \Delta$ and $\Gamma \nvdash_{K3} \emptyset$ then there is an interpolant χ whose non-logical vocabulary occurs in formulas in Γ and in formulas in Δ and such that $\Gamma \vdash_{K3} \chi$ and $\chi \vdash_{K3} \Delta$. Similarly, Priest's Logic of Paradox has the property that if $\Gamma \vdash_{LP} \Delta$ and $\emptyset \nvdash_{LP} \Delta$ then there is an interpolant χ whose non-logical vocabulary occurs in formulas in Γ and in formulas in Δ and such that $\Gamma \vdash_{LP} \chi$ and $\chi \vdash_{LP} \Delta$.

In between $B4$ on the one side and $K3$ and LP on the other there lies a fourth logic, the logic whose valid arguments are exactly those pronounced valid by both $K3$ and LP. We'll call this logic $K3 \sqcap LP$. (It goes by various names in the literature including *Kalman Implication* and *RM0*, the logic of R Mingle's first degree entailments.) Now, $\emptyset \nvdash_{K3} \psi, \neg\psi$ and $\phi, \neg\phi \nvdash_{LP} \emptyset$ but $\phi, \neg\phi \vdash_{K3} \psi, \neg\psi$ and $\phi, \neg\phi \vdash_{LP} \psi, \neg\psi$ and so $\phi, \neg\phi \vdash_{K3 \sqcap LP} \psi, \neg\psi$ for any ϕ and ψ. Consequently the logic $K3 \sqcap LP$ does not have an interpolation property (or at least none anything like those possessed by $B4$, $K3$, and LP).

Possession of an interpolation property is, then, neither preserved upwards to stronger logics nor downwards to weaker logics, making it unclear what, in general, is the value in possession of such a property. In the present context there are, however, specific lessons to be learned. For example, that classically valid inferences in a first order language, possibly with identity, are of one of three kinds. If $\Gamma \vdash \Delta$ then either (*i*) $\Gamma \vdash_{K3} \emptyset$, or (*ii*) $\emptyset \vdash_{LP} \Delta$, or (*iii*) $\Gamma \nvdash \emptyset$ and $\emptyset \nvdash \Delta$ and there is a formula χ which contains only non-logical vocabulary common to both Γ and Δ such that $\Gamma \vdash_{K3} \chi$ and $\chi \vdash_{LP} \Delta$. We may look on this as telling us that there's a proof procedure for classically valid first-order inferences such that all proofs can go in two separate phases, first one that uses the Principle of Non-Contradiction (no proposition is both true and false), but not the Principle of Bivalence (every proposition is true or — inclusive 'or' — false) and then one that uses the second but not the first.[1]

[1] This is reminiscent of, but importantly different from, David Makinson's observation regarding the logic we're calling $K3 \sqcap LP$: 'We can look on it as a logic that abandons either

What we present here is a strengthening of the standard Craig–Lyndon–Oberschelp Interpolation Theorem for first-order classical logic (with identity). We put upper bounds on the amount of logic needed to get from the premise(s) to the interpolant and to obtain the conclusion(s) from the interpolant. In both cases familiar three-valued logics suffice — but different logics. To the best of our knowledge, this is a refinement of the Interpolation Theorem for first-order classical logic that has not been broached previously.

The paper proceeds thus. In the second section we present block-tableaux rules which are sound and complete for first-order logic. There are four kinds of terminal nodes, nodes whose occurrence closes a branch. We distinguish *K*3-sound, *LP*-sound, and *B*4-sound tableaux derivations by which of these terminal nodes occur. In the third section we provide enough detail to establish the *K*3-soundness, *LP*-soundness, and *B*4-soundness of the various kinds of tableaux derivations. Classical completeness of the overall system of rules follows from standard results. In the fourth section we state and prove our refinement of the Craig–Lyndon Interpolation Theorem for classical first-order logic without identity. The proof proceeds by spelling out the steps for construction of an interpolant with the desired properties through careful examination of the tableaux rules. The fifth section provides semantics for the first-order extensions of Belnap's "useful four-valued logic"; with this in hand, the sixth draws out some related results from the details of the construction in the fourth section. Introducing identity raises some issues of detail. These are addressed in the seventh where tableaux rules are introduced. Again we divide these so as to yield *K*3-sound and *LP*-sound tableaux, the semantics with respect to which they are sound being given in the eighth section. The ninth and final section states and proves our refinement of the Craig–Lyndon–Oberschelp Interpolation Theorem for first-order logic with identity, this time spelling out the steps in the construction of an interpolant relating to the identity rules.

one, but not both, of the laws of contradiction and excluded third' (Makinson 1973, p. 39). To repeat what was just said in the text, in a classical inference one need at most keep the law of contradiction, but not the law of excluded third, in mind until reaching an interpolant then swap and work from the interpolant to the conclusion keeping the law of excluded third but not the law of contradiction in mind. In the case of a valid $K3 \sqcap LP$ inference, one can derive it keeping the law of contradiction, but not the law of excluded third, in mind throughout; then one can derive it again, this time keeping the law of excluded third, but not the law of contradiction, in mind throughout.

2. Block tableaux for first-order classical logic (without identity)

We begin with block tableaux for first-order classical logic (without identity and function-symbols). We have the following rules in which finite, possibly empty sets of (closed) formulas flank the colon.[2]

Conjunction

$$\dfrac{\boxed{\Gamma, \phi \wedge \psi : \Delta}}{\boxed{\Gamma, \phi, \psi : \Delta}} \qquad \dfrac{\boxed{\Gamma : \phi \wedge \psi, \Delta}}{\boxed{\Gamma : \phi, \Delta} \qquad \boxed{\Gamma : \psi, \Delta}} \qquad \dfrac{\boxed{\Gamma, \neg(\phi \wedge \psi) : \Delta}}{\boxed{\Gamma, \neg\phi : \Delta} \qquad \boxed{\Gamma, \neg\psi : \Delta}} \qquad \dfrac{\boxed{\Gamma : \neg(\phi \wedge \psi), \Delta}}{\boxed{\Gamma : \neg\phi, \neg\psi, \Delta}}$$

Disjunction

$$\dfrac{\boxed{\Gamma, \phi \vee \psi : \Delta}}{\boxed{\Gamma, \phi : \Delta} \qquad \boxed{\Gamma, \psi : \Delta}} \qquad \dfrac{\boxed{\Gamma : \phi \vee \psi, \Delta}}{\boxed{\Gamma: \phi, \psi, \Delta}} \qquad \dfrac{\boxed{\Gamma, \neg(\phi \vee \psi) : \Delta}}{\boxed{\Gamma, \neg\phi, \neg\psi : \Delta}} \qquad \dfrac{\boxed{\Gamma : \neg(\phi \vee \psi), \Delta}}{\boxed{\Gamma : \neg\phi, \Delta} \qquad \boxed{\Gamma : \neg\psi, \Delta}}$$

Implication

$$\dfrac{\boxed{\Gamma, \phi \rightarrow \psi : \Delta}}{\boxed{\Gamma, \neg\phi : \Delta} \qquad \boxed{\Gamma, \psi : \Delta}} \qquad \dfrac{\boxed{\Gamma : \phi \rightarrow \psi, \Delta}}{\boxed{\Gamma : \neg\phi, \psi, \Delta}} \qquad \dfrac{\boxed{\Gamma, \neg(\phi \rightarrow \psi) : \Delta}}{\boxed{\Gamma, \phi, \neg\psi : \Delta}} \qquad \dfrac{\boxed{\Gamma : \neg(\phi \rightarrow \psi), \Delta}}{\boxed{\Gamma : \phi, \Delta} \qquad \boxed{\Gamma : \neg\psi, \Delta}}$$

(Double) Negation

$$\dfrac{\boxed{\Gamma, \neg\neg\phi : \Delta}}{\boxed{\Gamma, \phi : \Delta}} \qquad \dfrac{\boxed{\Gamma : \neg\neg\phi, \Delta}}{\boxed{\Gamma : \phi, \Delta}}$$

Universal Quantification

$$\dfrac{\boxed{\Gamma, \forall x\phi : \Delta}}{\boxed{\Gamma, \forall x\phi, \phi\,[t/x] : \Delta}}$$

where t is an individual constant

$$\dfrac{\boxed{\Gamma : \forall x\phi, \Delta}}{\boxed{\Gamma : \phi\,[a/x], \Delta}}$$

where the individual constant a is new

$$\dfrac{\boxed{\Gamma, \neg\forall x\phi : \Delta}}{\boxed{\Gamma, \neg\phi\,[a/x] : \Delta}}$$

where the individual constant a is new

$$\dfrac{\boxed{\Gamma : \neg\forall x\phi, \Delta}}{\boxed{\Gamma : \neg\phi\,[t/x], \neg\forall x\phi, \Delta}}$$

where t is an individual constant

[2] We make no provision for free-variable formulas; all formulas are closed formulas.

Existential Quantification

$\boxed{\Gamma, \exists x\phi : \Delta}$	$\boxed{\Gamma : \exists x\phi, \Delta}$	$\boxed{\Gamma, \neg\exists x\phi : \Delta}$	$\boxed{\Gamma : \neg\exists x\phi, \Delta}$
$\boxed{\Gamma, \phi\,[a/x] : \Delta}$	$\boxed{\Gamma : \phi\,[t/x], \exists x\phi, \Delta}$	$\boxed{\Gamma, \neg\exists x\phi, \neg\phi\,[t/x] : \Delta}$	$\boxed{\Gamma : \neg\phi\,[a/x], \Delta}$
where the individual constant *a* is new	where *t* is an individual constant	where *t* is an individual constant	where the individual constant *a* is new

Branches close immediately when nodes of any of the following four forms are reached (ϕ atomic):

$$\boxed{\Gamma, \phi : \phi, \Delta} \qquad \boxed{\Gamma, \neg\phi : \neg\phi, \Delta} \qquad \boxed{\Gamma, \phi, \neg\phi : \Delta} \qquad \boxed{\Gamma: \phi, \neg\phi, \Delta}$$

A tableau is complete when no rule can be applied. We write $\Gamma \vdash \Delta$ if all branches close in some completed tableau headed by the node $\boxed{\Gamma : \Delta}$. In this case, we say that a closed tableau exists for $\Gamma : \Delta$.

All rules are applied to pairs of (possibly empty) sets of formulas. We'll call the pair to which a rule is applied *the input pair*. Rules yield one or two pairs of (possibly empty) sets of formulas. We'll call these *the output pair* or *pairs*.

Definition 1 (Parity). Add the number of negations within whose scope an occurrence of a predicate in a formula occurs to the number of conditional subformula in whose antecedent the occurrence lies. The occurrence has *even* or *odd parity* in the formula according as to whether this number is even or odd.[3]

Remark 1. As case by case examination of the tableau rules reveals, if, in the application of a rule, one or more predicates occur in a formula in an output pair, they must also occur in some formula in the input pair on the same side of the colon *with the same parity*. Consequently, if there's a closed tableau for $\Gamma : \Delta$ in which a branch closes at a node of one of the forms $\boxed{\Theta, \psi : \psi, E}$ and $\boxed{\Theta, \neg\psi : \neg\psi, E}$ all predicates occurring in ψ occur in some member of Γ and also in some member of Δ with the same parity as in ψ in the case of a $\boxed{\Theta, \psi : \psi, E}$-node and with the opposite parity in the case of a $\boxed{\Theta, \neg\psi : \neg\psi, E}$-node.

Remark 2. Two feature of the tableaux rules are easily confirmed by inspection. Firstly, in applications of any of the rules, there is change on exactly

[3] This definition extends Henkin's neat reformulation of Lyndon's definition (Henkin 1963, p. 201, n. 7). Neither Lyndon nor Henkin consider a language containing $\rightarrow$ as primitive; Lyndon (1959, pp. 129 & 131) treats $\phi \rightarrow \psi$ as an abbreviation for $\neg\phi \vee \psi$ (which has the same effect as, but is not the letter of, what we do here). The predicate F has an occurrence with even and an occurrence with odd parity in the formula $\neg\exists x(\neg Fx \wedge (\neg Fb \rightarrow Gb))$.

one side of the colon: on the other side there is no change between the input pair and the output pair(s). (We call a rule a *left-hand rule* or a *right-hand rule* depending on which side changes.) Secondly, if in the application of a rule one side of an input pair is empty, the same side of the output pair or pairs must be empty too. Likewise, if in the application of a rule one side of an output pair is empty, the same side of the input pair (and so the same side of the other output pair, if there is one) must be empty too. Thus only left-hand rules are used in a tableau headed by a node $\boxed{\Gamma : \varnothing}$ and only right-hand rules in a tableau headed by a node $\boxed{\varnothing : \Delta}$.

LEMMA 1. If there's a closed tableau for $\Gamma : \Delta$ in which no branch reaches a node of either of the forms $\boxed{\Theta, \psi : \psi, E}$ and $\boxed{\Theta, \neg\psi : \neg\psi, E}$, it must be the case that $\Gamma \vdash \varnothing$ or $\varnothing \vdash \Delta$.

PROOF. Suppose that in a closed tableau for $\Gamma : \Delta$ no branch reaches a node of either of the forms $\boxed{\Theta, \psi : \psi, E}$ and $\boxed{\Theta, \neg\psi : \neg\psi, E}$. By REMARK 2, no rule applies to $\boxed{\varnothing : \varnothing}$, nor is it terminal, so there cannot be a closed tableau for $\varnothing : \varnothing$; consequently, at least one of Γ and Δ is non-empty. And if the other is empty, we are done, so suppose that neither Γ nor Δ is empty and consider the closed tableau for $\Gamma : \Delta$. Since left-hand rules take note of and change only what's on the left of the colon and right-hand rules only what's on the right, we can re-order the applications of rules in every branch, applying first, say, the left-hand rules; order of application, and so the branch-structure of the tableau apart, at most some "new names" may have to be changed to preserve their novelty but given the autonomy of the left and right sides, this can affect nothing essential when the new and all subsequent occurrences on that same side of the colon are uniformly substituted. Of the four types of terminal nodes, only those of the forms $\boxed{\Theta, \psi : \psi, E}$ and $\boxed{\Theta, \neg\psi : \neg\psi, E}$, the two forms which, by hypothesis, are not present in the original tableau, pay attention to what's on both sides of the colon, the other two pay attention only to one side. Having re-ordered the application of rules, either all branches close at nodes of the form $\boxed{\Theta, \psi, \neg\psi : E}$ before we get to apply the right-hand rules — in which case, by deleting the occurrences of Δ on the right-hand side of nodes, we obtain a closed tableau for $\Gamma : \varnothing$ — or at least one branch includes applications of right-hand rules *and only right-hand rules* after a certain point, working downwards from a node with Δ on the right-hand side, and *all* branches passing through that node close at nodes of the form $\boxed{\Theta : \psi, \neg\psi, E}$. In the latter case, pick one such branch, ignore everything above the node to which that first application of a right-hand rule is made, and delete what's on the left-hand side in every node from that node downwards. (It doesn't change.) We obtain a closed tableau for $\varnothing : \Delta$. □

DEFINITION 2 (*K*3-, *LP*-, AND *B*4-TABLEAUX). We write $\Gamma \vdash_{K3} \Delta$ if no branch terminates in a node of the form $\boxed{\Theta : \chi, \neg\chi, E}$ in some closed tableau for

$\Gamma : \Delta$; we call such a tableau a closed $K3$-tableau. Likewise, we write $\Gamma \vdash_{LP} \Delta$ if no branch terminates in a node of the form $\boxed{\Theta, \chi, \neg\chi : E}$ in some closed tableau for $\Gamma : \Delta$; we call such a tableau a closed LP-tableau. We write $\Gamma \vdash_{B4} \Delta$ if no branch ends in a node of either of the forms $\boxed{\Theta : \chi, \neg\chi, E}$ and $\boxed{\Theta, \chi, \neg\chi : E}$ in some closed tableau for $\Gamma : \Delta$; we call such a tableau a closed $B4$-tableau.

LEMMA 2. If $\Gamma \vdash \emptyset$ then $\Gamma \vdash_{K3} \emptyset$. Likewise, if $\emptyset \vdash \Delta$ then $\emptyset \vdash_{LP} \Delta$.

PROOF. If a closed tableau for $\Gamma : \emptyset$ exists, then, since the right-hand side of $\Gamma : \emptyset$ is empty, the right-hand side of every node in the tableau must also be empty (REMARK 2), and so no branch terminates in a node of the form $\boxed{\Theta : \chi, \neg\chi, E}$.

If $\emptyset \vdash \Delta$ then, since the left-hand side of $\emptyset : \Delta$ is empty, the left-hand side of every node in the tableau must also be empty (REMARK 2), and so no branch terminates in a node of the form $\boxed{\Theta, \chi, \neg\chi : E}$. □

DEFINITION 3 (CONTRARIES). Given a formula ϕ not of the form $\neg\psi$, its contrary is $\neg\phi$; if ϕ is of the form $\neg\psi$, ψ is its contrary. Given a set of formulas Δ, $-\Delta$ is a set containing contraries of every member of Δ (and nothing more).[4]

The Duality Principle (Syntax). It is obvious, by inspection coupled with judicious use of the rules for double negation, that a closed $K3$-tableau exists for $\boxed{\Gamma : -\Delta}$ if, and only if, a closed LP-tableau exists for $\boxed{\Delta : -\Gamma}$. Likewise, a closed $B4$-tableau exists for $\boxed{\Gamma : -\Delta}$ if, and only if, a closed $B4$-tableau exists for $\boxed{\Delta : -\Gamma}$. □

3. Semantics for first-order *K*3, *LP*, *K*3 ⊓ *LP*, and classical logic (without identity)

An interpretation $\mathfrak{A}$ for a first-order language with neither function-symbols nor identity comprises a non-empty domain D and a function $\mathcal{I}$; to each individual constant c, $\mathcal{I}$ assigns an element of D; to each n-place predicate F, $\mathcal{I}$ assigns two disjoint subsets, $\mathcal{I}^+(F)$ and $\mathcal{I}^-(F)$, of D^n.

We extend the language by adding a new constant, $\overline{d}$, for each element d in D. We extend $\mathcal{I}$ by the obvious stipulation: for all $d \in D$, $\mathcal{I}(\overline{d}) = d$. We assign values to atomic formulas of the extended language like this:

> In an interpretation $\mathfrak{A} = \langle D, \mathcal{I}\rangle$, $v_{\mathfrak{A}}(F(c_1, c_2, ..., c_n)) = 1$ if $\langle \mathcal{I}(c_1), \mathcal{I}(c_2), ..., \mathcal{I}(c_n)\rangle \in \mathcal{I}^+(F)$; $v_{\mathfrak{A}}(F(c_1, c_2, ..., c_n)) = 0$ if $\langle \mathcal{I}(c_1), \mathcal{I}(c_2), ..., \mathcal{I}(c_n)\rangle \in \mathcal{I}^-(F)$; $v_{\mathfrak{A}}(F(c_1, c_2, ..., c_n)) = ½$ otherwise.

[4] The contrary of a contrary differs, if at all, from the original formula at most in having lost a double negation from the head.

An interpretation $\mathfrak{A} = \langle D, \mathcal{I} \rangle$ is *classical* iff, for all $n \in \mathbb{N}^+$, for all n-place predicates F, $\mathcal{I}^+(F) \cup \mathcal{I}^-(F) = D^n$.

We evaluate formulas with negation, conjunction, disjunction, or (material) implication dominant in accordance with these truth-tables (Kleene 1952, Asenjo 1966, Priest 1979):

ϕ	$\neg\phi$
1	0
1/2	1/2
0	1

	$\phi \wedge \psi$	ψ: 1	1/2	0
	1	1	1/2	0
ϕ	1/2	1/2	1/2	0
	0	0	0	0

	$\phi \vee \psi$	ψ: 1	1/2	0
	1	1	1	1
ϕ	1/2	1	1/2	1/2
	0	1	1/2	0

	$\phi \rightarrow \psi$	ψ: 1	1/2	0
	1	1	1/2	0
ϕ	1/2	1	1/2	1/2
	0	1	1	1

$$v_{\mathfrak{A}}(\forall v\phi) = \min\{v_{\mathfrak{A}}(\phi[\overline{d}/v]) : d \in D\};\ v_{\mathfrak{A}}(\exists v\phi) = \max\{v_{\mathfrak{A}}(\phi[\overline{d}/v]) : d \in D\}.$$

In a classical interpretation $\mathfrak{A}$, $v_{\mathfrak{A}}$ takes only the values 0 and 1.

DEFINITION 4 ($K3$-SATISFACTION). $\mathfrak{A} \vDash_{K3} \Gamma$ iff, for all $\phi \in \Gamma$, $v_{\mathfrak{A}}(\phi) = 1$.

DEFINITION 5 (LP-SATISFACTION). $\mathfrak{A} \vDash_{LP} \Gamma$ iff, for all $\phi \in \Gamma$, $v_{\mathfrak{A}}(\phi) \neq 0$.

DEFINITION 6 ($K3$-CONSEQUENCE). $\Gamma \vDash_{K3} \Delta$ iff for no interpretation $\mathfrak{A}$ is it the case that $\mathfrak{A} \vDash_{K3} \Gamma$ and $\mathfrak{A} \vDash_{LP} -\Delta$.

DEFINITION 7 (LP-CONSEQUENCE). $\Gamma \vDash_{LP} \Delta$ iff for no interpretation $\mathfrak{A}$ is it the case that $\mathfrak{A} \vDash_{LP} \Gamma$ and $\mathfrak{A} \vDash_{K3} -\Delta$.

DEFINITION 8 ($K3 \sqcap LP$-CONSEQUENCE). $\Gamma \vDash_{K3 \sqcap LP} \Delta$ iff for no interpretation $\mathfrak{A}$ is it the case that $\mathfrak{A} \vDash_{K3} \Gamma$ and $\mathfrak{A} \vDash_{LP} -\Delta$ and for no interpretation $\mathfrak{B}$ is it the case that $\mathfrak{B} \vDash_{LP} \Gamma$ and $\mathfrak{B} \vDash_{K3} -\Delta$.

The Duality Principle (Semantics). It falls out of these definitions and the evaluation rule for negation that for any — possibly empty — sets Γ and Δ of formulas in a first-order language with neither function-symbols nor identity,

$$\Gamma \vDash_{K3} -\Delta \text{ iff } \Delta \vDash_{LP} -\Gamma$$

and

$$\Gamma \vDash_{K3 \sqcap LP} -\Delta \text{ iff } \Delta \vDash_{K3 \sqcap LP} -\Gamma.$$

DEFINITION 9 ((IN)CORRECTNESS). Say that a node $\boxed{\Gamma : \Delta}$ (where Γ and Δ are finite, possibly empty sets of formulas in a first-order language without identity or function-symbols) is $K3$-incorrect with respect to the interpretation $\mathfrak{A} = \langle D, \mathcal{I} \rangle$ iff $\mathfrak{A} \vDash_{K3} \Gamma$ and $\mathfrak{A} \vDash_{LP} -\Delta$, *i.e.*, iff, for all $\phi \in \Gamma$, $v_{\mathfrak{A}}(\phi) = 1$ and, for all $\psi \in \Delta$, $v_{\mathfrak{A}}(\psi) \neq 1$.

Say, likewise, that a node $\boxed{\Gamma : \Delta}$ is *LP*-incorrect with respect to the interpretation $\mathfrak{A} = \langle D, \mathcal{I}\rangle$ iff $\mathfrak{A} \vDash_{LP} \Gamma$ and $\mathfrak{A} \vDash_{K3} -\Delta$, *i.e.*, iff, for all $\psi \in \Delta$, $v_{\mathfrak{A}}(\psi) = 0$ and, for all $\phi \in \Gamma$, $v_{\mathfrak{A}}(\phi) \neq 0$.

Say too that a node $\boxed{\Gamma : \Delta}$ is $K3 \sqcap LP$-incorrect with respect to the interpretation $\mathfrak{A} = \langle D, \mathcal{I}\rangle$ iff $\mathfrak{A} \vDash_{K3} \Gamma$ and $\mathfrak{A} \vDash_{LP} -\Delta$ or $\mathfrak{A} \vDash_{LP} \Gamma$ and $\mathfrak{A} \vDash_{K3} -\Delta$, *i.e.*, iff $\min\{v_{\mathfrak{A}}(\phi) : \phi \in \Gamma\} > \max\{v_{\mathfrak{A}}(\psi) : \psi \in \Delta\}$.

Say that a node $\boxed{\Gamma : \Delta}$ is *classically incorrect* with respect to the interpretation $\mathfrak{A} = \langle D, \mathcal{I}\rangle$ iff it is both *K*3-incorrect and *LP*-incorrect.

Say, lastly, that a node $\boxed{\Gamma : \Delta}$ is *correct* in the relevant sense (*K*3, *LP*, $K3 \sqcap LP$, classical) if there is no interpretation with respect to which it is incorrect in the same sense. If $\boxed{\Gamma : \Delta}$ is correct in one or more of these senses, so too is $\boxed{\Gamma, \Gamma' : \Delta, \Delta'}$ in the same sense(s), for any sets of formulas Γ' and Δ'. (The node $\boxed{\Gamma : \Delta}$ is correct in one or more of these senses just in case Δ is a consequence of Γ in the same sense(s).)

Given the aim of the present investigation, we need soundness and completeness of the whole set of rules and, in particular, all the closure conditions for branches for classical first-order logic, but only need soundness of the rules and the designated closure conditions for what we have called closed *K*3-, *LP*-, and, when we get to them, *B*4-tableaux. Completeness proofs could be adapted from, *e.g.*, (Bloesch 1993) and (Priest 2008, Chs. 22 & 23) but we have in fact no need of them.[5]

Soundness. The tableaux rules preserve *K*3-, *LP*-, and $K3 \sqcap LP$-incorrectness with respect to any interpretation $\langle D, \mathcal{I}\rangle$ *downwards*, hence preserve classical incorrectness with respect to a (classical) interpretation downwards, in the sense that if the input pair in the application of a rule is incorrect (in the relevant sense) with respect to an interpretation, so too is the output pair, if there is only one, and so too is at least one of the output pairs, if there are two. (In the case of the rules where a new name is introduced, the function $\mathcal{I}$ must be extended to supply an appropriate interpretation of that name; it is guaranteed that there will be such when the input pair is incorrect with respect to the interpretation in play.) Nodes of the forms

[5] Should completeness fail for any of the sets of rules, the claims about failure of upwards and downwards transmission of possession of an interpolation property made in the Introduction would still stand even if one or more of the logics would then be slightly misidentified there.

Of course, the question of completeness is of independent interest. A referee remarks that Bloesch's results cover only propositional *LP* and *B*4. Syntactic and semantic duality extend his results to propositional *K*3. As for first-order, I take Bloesch at his word when he says, 'While coupled tree proof systems exist for both logics [*i.e.*, *LP* and *B*4] the tableau proof system described has several advantages over them. First, it is easier to use and second, it lends itself to first-order and modal extensions of the above logics' (p. 295). Also, one may refer to (Priest 2008, Chs. 22 & 23).

$\boxed{\Gamma, \phi : \phi, \Delta}$, $\boxed{\Gamma, \neg\phi : \neg\phi, \Delta}$, and $\boxed{\Gamma, \phi, \neg\phi : \Delta}$ are all $K3$-correct; nodes of the forms $\boxed{\Gamma, \phi : \phi, \Delta}$, $\boxed{\Gamma, \neg\phi : \neg\phi, \Delta}$, and $\boxed{\Gamma : \phi, \neg\phi, \Delta}$ are all LP-correct; all four forms are classically correct.

Correctness is preserved upwards. Thus

$$\Gamma \vdash_{K3} \Delta \text{ only if } \Gamma \vDash_{K3} \Delta$$

and

$$\Gamma \vdash_{LP} \Delta \text{ only if } \Gamma \vDash_{LP} \Delta.$$

Completeness. It is obvious, by inspection, that with judicious use of the rules for double negation a closed tableau exists for $\Gamma : \Delta$ if, and only if, a closed tableau exists for $\Gamma \cup -\Delta : \emptyset$.[6] In drawing up a closed tableau for $\Gamma \cup -\Delta : \emptyset$, only left-hand rules are used (Remark 2). But the left-hand rules are in effect just the familiar rules for tableaux employed to test classical consistency in a first-order language without identity. Borrowing from any number of classical or textbook sources, we have that if there is no closed tableau for $\Gamma \cup -\Delta : \emptyset$, equivalently, no closed standard tableau for the set $\Gamma \cup -\Delta$, there is a classical interpretation in which all members of $\Gamma \cup -\Delta$ take the value 1. Equivalently, there is a classical interpretation with respect to which $\boxed{\Gamma : \Delta}$ is classically incorrect.

$K3 \sqcap LP$ presents an interesting case. Nodes of the forms $\boxed{\Gamma, \phi, \neg\phi : \Delta}$ and $\boxed{\Gamma : \phi, \neg\phi, \Delta}$ are *not* $K3 \sqcap LP$-correct, but nodes of the form $\boxed{\Gamma, \phi, \neg\phi : \psi, \neg\psi, \Delta}$ are. Thus a block tableau presentation of this logic has all the rules for connectives and quantifiers that we have in play and exactly these three forms of terminal nodes (ϕ, ψ atomic):

$$\boxed{\Gamma, \phi : \phi, \Delta} \qquad \boxed{\Gamma, \neg\psi : \neg\psi, \Delta} \qquad \boxed{\Gamma, \phi, \neg\phi : \psi, \neg\psi, \Delta}$$

Naturally, we write $\Gamma \vdash_{K3\sqcap LP} \Delta$ if all branches terminate at nodes of these three forms in a block tableaux headed by the node $\boxed{\Gamma : \Delta}$. With that in hand we have

$$\Gamma \vdash_{K3\sqcap LP} \Delta \text{ only if } \Gamma \vDash_{K3\sqcap LP} \Delta.$$

4. A refinement of the Craig–Lyndon Interpolation Theorem

Theorem 1. If $\Gamma \vdash \Delta$ and $\Gamma \nvdash \emptyset$ and $\emptyset \nvdash \Delta$ then there is an interpolant χ, constructed using only non-logical vocabulary common to both Γ and Δ, such that

$$\Gamma \vdash_{K3} \chi \text{ and } \chi \vdash_{LP} \Delta$$

[6] That $\Gamma \cap -\Delta$ may be non-empty slightly upsets the parallelism between the tableaux in some cases but does not undermine the claim just made.

and every predicate which occurs with even parity in χ occurs with even parity in some member of Γ and also in some member of Δ and, likewise, every predicate which occurs with odd parity in χ occurs with odd parity in some member of Γ and also in some member of Δ.

PROOF. Given a closed tableau for $\Gamma : \Delta$ we associate interpolants with some, not necessarily all, nodes, working upwards from the terminal nodes. The aim is to associate interpolants with all nodes $\boxed{\Theta : E}$ such that $\Theta \nvdash \emptyset$ and $\emptyset \nvdash E$. At the same time we "reverse engineer" a closed $K3$-tableau for $\Gamma : \chi$ and a closed LP-tableau for $\chi : \Delta$ where χ is the interpolant associated with $\boxed{\Gamma : \Delta}$. By design, in the first of these no terminal node of the form $\boxed{\Theta : \psi, \neg\psi, E}$ occurs; in the second, no terminal node of the form $\boxed{\Theta, \psi, \neg\psi : E}$.

Terminal nodes. With a node of the form $\boxed{\Theta, \psi : \psi, E}$, we take ψ itself to be the interpolant. Similarly, with a node of the form $\boxed{\Theta, \neg\psi : \neg\psi, E}$, we take $\neg\psi$ to be the interpolant. Trivially, $\Theta, \psi \vdash_{K3} \psi$ and $\Theta, \neg\psi \vdash_{K3} \neg\psi$ and, likewise, $\psi \vdash_{LP} \psi, E$ and $\neg\psi \vdash_{LP} \neg\psi, E$. With terminal nodes of the other two forms we do not associate any interpolant. With nodes of the form $\boxed{\Theta, \psi, \neg\psi : E}$ we have that $\Theta, \psi, \neg\psi \vdash_{K3} \emptyset$; with nodes of the form $\boxed{\Theta : \psi, \neg\psi, E}$ we have that $\emptyset \vdash_{LP} \psi, \neg\psi, E$.

No change rules. In the case of these rules —

$$
\begin{array}{c} \boxed{\Gamma, \phi \wedge \psi : \Delta} \\ | \\ \boxed{\Gamma, \phi, \psi : \Delta} \end{array}
\qquad
\begin{array}{c} \boxed{\Gamma : \neg(\phi \wedge \psi), \Delta} \\ | \\ \boxed{\Gamma : \neg\phi, \neg\psi, \Delta} \end{array}
\qquad
\begin{array}{c} \boxed{\Gamma : \phi \vee \psi, \Delta} \\ | \\ \boxed{\Gamma : \phi, \psi, \Delta} \end{array}
\qquad
\begin{array}{c} \boxed{\Gamma, \neg(\phi \vee \psi) : \Delta} \\ | \\ \boxed{\Gamma, \neg\phi, \neg\psi : \Delta} \end{array}
$$

$$
\begin{array}{c} \boxed{\Gamma : \phi \rightarrow \psi, \Delta} \\ | \\ \boxed{\Gamma : \neg\phi, \psi, \Delta} \end{array}
\qquad
\begin{array}{c} \boxed{\Gamma, \neg(\phi \rightarrow \psi) : \Delta} \\ | \\ \boxed{\Gamma, \phi, \neg\psi : \Delta} \end{array}
\qquad
\begin{array}{c} \boxed{\Gamma, \neg\neg\phi : \Delta} \\ | \\ \boxed{\Gamma, \phi : \Delta} \end{array}
\qquad
\begin{array}{c} \boxed{\Gamma : \neg\neg\phi, \Delta} \\ | \\ \boxed{\Gamma : \phi, \Delta} \end{array}
$$

— the interpolant, if any, associated with the output pair is associated with the input pair (and no interpolant is associated with the input pair if none is associated with the output pair). In each case, where $\boxed{\Gamma : \Delta}$ is the output pair, $\boxed{\Gamma' : \Delta'}$ the input pair, and ψ the interpolant associated with the output pair, we have that one of the two steps, from $\boxed{\Gamma' : \psi}$ to $\boxed{\Gamma : \psi}$ or from $\boxed{\psi : \Delta'}$ to $\boxed{\psi : \Delta}$, consists of mere repetition and the other of an application of the very rule under examination. Thus $\Gamma' \vdash_{K3} \psi$ when $\Gamma \vdash_{K3} \psi$ and $\psi \vdash_{LP} \Delta'$ when $\psi \vdash_{LP} \Delta$.

By assumption, if no interpolant is associated with the output pair $\boxed{\Gamma : \Delta}$, then either $\Gamma \vdash_{K3} \emptyset$ or $\emptyset \vdash_{LP} \Delta$ or both. If $\Gamma' = \Gamma$ and $\Gamma \vdash_{K3} \emptyset$ we are done. Likewise if $\Delta' = \Delta$ and $\emptyset \vdash_{LP} \Delta$. In the case of all the rules, if $\Gamma' \neq \Gamma$ then

$$\begin{array}{c}\boxed{\Gamma' : \varnothing}\\ | \\ \boxed{\Gamma : \varnothing}\end{array}$$

by an application of the very rule under investigation and hence $\Gamma' \vdash_{K3} \varnothing$ when $\Gamma \vdash_{K3} \varnothing$. Similarly, in the case of all the rules, if $\Delta' \neq \Delta$ then

$$\begin{array}{c}\boxed{\varnothing : \Delta'}\\ | \\ \boxed{\varnothing : \Delta}\end{array}$$

by an application of the very rule under investigation and hence $\varnothing \vdash_{LP} \Delta'$ when $\varnothing \vdash_{LP} \Delta$.

Conjunction

$$\begin{array}{cc}\multicolumn{2}{c}{\boxed{\Gamma : \phi \wedge \psi, \Delta}}\\ \swarrow & \searrow \\ \boxed{\Gamma : \phi, \Delta} & \boxed{\Gamma : \psi, \Delta}\end{array}$$

Suppose first that the interpolant associated with $\boxed{\Gamma : \phi, \Delta}$ is χ, hence $\Gamma \vdash_{K3} \chi$ and $\chi \vdash_{LP} \phi, \Delta$, and that the interpolant associated with $\boxed{\Gamma : \psi, \Delta}$ is η, hence $\Gamma \vdash_{K3} \eta$, and $\eta \vdash_{LP} \psi, \Delta$. The steps

$$\begin{array}{cc}\multicolumn{2}{c}{\boxed{\Gamma : \chi \wedge \eta}}\\ \swarrow & \searrow \\ \boxed{\Gamma : \chi} & \boxed{\Gamma : \eta}\end{array} \quad \text{and} \quad \begin{array}{cc}\multicolumn{2}{c}{\boxed{\chi \wedge \eta : \phi \wedge \psi, \Delta}}\\ \multicolumn{2}{c}{|}\\ \multicolumn{2}{c}{\boxed{\chi, \eta : \phi \wedge \psi, \Delta}}\\ \swarrow & \searrow \\ \boxed{\chi, \eta : \phi, \Delta} & \boxed{\chi, \eta : \psi, \Delta}\end{array}$$

are in accordance with the tableaux rules. The left-hand one allows us to turn closed $K3$-tableaux for $\Gamma : \chi$ and $\Gamma : \eta$ into a closed $K3$-tableau for $\Gamma : \chi \wedge \eta$. By adding η on the left of the colon in all nodes in a closed LP-tableau for $\boxed{\chi : \phi, \Delta}$ and χ on the left of the colon in all nodes in a closed LP-tableau for $\boxed{\chi : \psi, \Delta}$ — which may require change of "new names" but nothing more — and uniting them under $\boxed{\chi, \eta : \phi \wedge \psi, \Delta}$, we obtain from the right-hand tableau fragment a closed LP-tableau for $\chi \wedge \eta : \phi \wedge \psi, \Delta$.

If no interpolant is associated with $\boxed{\Gamma : \phi, \Delta}$ but η is associated with $\boxed{\Gamma : \psi, \Delta}$ we proceed as follows. By assumption, $\Gamma \vdash_{K3} \eta$ and $\eta \vdash_{K3} \psi, \Delta$; by assumption, too, either $\Gamma \vdash_{K3} \varnothing$ or $\varnothing \vdash_{LP} \phi, \Delta$. *If* $\Gamma \vdash_{K3} \varnothing$, no interpolant is associated with $\boxed{\Gamma : \phi \wedge \psi, \Delta}$. *If*, on the other hand, $\varnothing \vdash_{LP} \phi, \Delta$ then η is associated with $\boxed{\Gamma : \phi \wedge \psi, \Delta}$, for we can add η on the left in all nodes of a closed LP-tableau for $\varnothing : \phi, \Delta$ to obtain a closed LP-tableau for $\eta : \phi, \Delta$ and, by an application of the rule in question —

$$\begin{array}{cc}\multicolumn{2}{c}{\boxed{\eta : \phi \wedge \psi, \Delta}}\\ \swarrow & \searrow \\ \boxed{\eta : \phi, \Delta} & \boxed{\eta : \psi, \Delta}\end{array}$$

— we unite the closed *LP*-tableaux for $\eta : \phi, \Delta$ and $\eta : \psi, \Delta$ in a closed *LP*-tableau for $\eta : \phi \wedge \psi, \Delta$.

Likewise, *mutatis mutandis*, if no interpolant is associated with $\boxed{\Gamma : \psi, \Delta}$ but there is an interpolant associated with $\boxed{\Gamma : \phi, \Delta}$

If no interpolant is associated with $\boxed{\Gamma : \phi, \Delta}$ and none is associated with $\boxed{\Gamma : \psi, \Delta}$, none is associated with $\boxed{\Gamma : \phi \wedge \psi, \Delta}$. By assumption, either $\Gamma \vdash_{K3} \emptyset$ or both $\emptyset \vdash_{LP} \phi, \Delta$ and $\emptyset \vdash_{LP} \psi, \Delta$. If $\Gamma \vdash_{K3} \emptyset$, nothing changes travelling north, and if $\emptyset \vdash_{LP} \phi, \Delta$ and $\emptyset \vdash_{LP} \psi, \Delta$, we can, using the rule under investigation, entirely properly unite closed *LP*-tableaux for $\emptyset : \phi, \Delta$ and $\emptyset : \psi, \Delta$ under the node $\boxed{\emptyset : \phi \wedge \psi, \Delta}$ to obtain a closed *LP*-tableau for $\emptyset \vdash_{LP} \phi \wedge \psi, \Delta$.

$$\begin{array}{c} \boxed{\Gamma, \neg(\phi \wedge \psi) : \Delta} \\ \boxed{\Gamma, \neg\phi : \Delta} \quad \boxed{\Gamma, \neg\psi : \Delta} \end{array}$$

Suppose first that the interpolant associated with $\boxed{\Gamma, \neg\phi : \Delta}$ is χ, hence $\Gamma, \phi \vdash_{K3} \chi$ and $\chi \vdash_{LP} \Delta$, and that the interpolant associated with $\boxed{\Gamma, \neg\psi : \Delta}$ is η, hence $\Gamma, \neg\psi \vdash_{K3} \eta$ and $\eta \vdash_{LP} \Delta$. The steps

$$\begin{array}{c} \boxed{\Gamma, \neg(\phi \wedge \psi) : \chi \vee \eta} \\ \boxed{\Gamma, \neg(\phi \wedge \psi) : \chi, \eta} \\ \boxed{\Gamma, \neg\phi : \chi, \eta} \quad \boxed{\Gamma, \neg\psi : \chi, \eta} \end{array} \qquad \text{and} \qquad \begin{array}{c} \boxed{\chi \vee \eta : \Delta} \\ \boxed{\chi : \Delta} \quad \boxed{\eta : \Delta} \end{array}$$

are in accordance with the tableaux rules. By adding η on the right of the colon in all nodes in a closed *K*3-tableau for $\Gamma, \neg\phi : \chi$ and χ on the right of the colon in all nodes in a closed *K*3-tableau for $\Gamma, \neg\psi : \eta$ — which may require change of "new names" but nothing more — and uniting them under $\boxed{\Gamma, \neg(\phi \wedge \psi) : \chi, \eta}$, we obtain from the left-hand tableau fragment a closed *K*3-tableau for $\Gamma, \neg(\phi \wedge \psi) : \chi \vee \eta$. The right-hand fragment allows us to turn closed *LP*-tableaux for $\chi : \Delta$ and $\eta : \Delta$ into a closed *LP*-tableau for $\chi \vee \eta : \Delta$.

If no interpolant is associated with $\boxed{\Gamma, \neg\phi : \Delta}$ but η is associated with $\boxed{\Gamma, \neg\psi : \Delta}$ we proceed as follows. By assumption, $\Gamma, \neg\psi \vdash_{K3} \eta$ and $\eta \vdash_{LP} \Delta$; by assumption, too, either $\Gamma, \neg\phi \vdash_{K3} \emptyset$ or $\emptyset \vdash_{LP} \Delta$. *If* $\emptyset \vdash_{LP} \Delta$, no interpolant is associated with $\boxed{\Gamma, \neg(\phi \wedge \psi) : \Delta}$. *If*, on the other hand, $\Gamma, \phi \vdash_{K3} \emptyset$ then η is associated with $\boxed{\Gamma, \neg(\phi \wedge \psi) : \Delta}$, for we can add η on the right in all nodes of a closed *K*3-tableau for $\Gamma, \neg\phi : \emptyset$ to obtain a closed *K*3-tableau for $\Gamma, \neg\phi : \eta$ and, by an application of the rule in question —

$$\begin{array}{c} \boxed{\Gamma, \neg(\phi \wedge \psi) : \eta} \\ \boxed{\Gamma, \neg\phi : \eta} \quad \boxed{\Gamma, \neg\psi : \eta} \end{array}$$

— we unite the closed $K3$-tableaux for $\Gamma, \neg\phi : \eta$ and $\Gamma, \neg\psi : \eta$ in a closed $K3$-tableau for $\Gamma, \neg(\phi \wedge \psi) : \eta$.

Likewise, *mutatis mutandis*, if none is associated with $\boxed{\Gamma, \neg\psi : \Delta}$ but there is an interpolant associated with $\boxed{\Gamma, \neg\phi : \Delta}$.

If no interpolant is associated with $\boxed{\Gamma, \neg\phi : \Delta}$ and none is associated with $\boxed{\Gamma, \neg\psi : \Delta}$, none is associated with $\boxed{\Gamma, \neg(\phi \wedge \psi) : \eta}$. By assumption, either $\emptyset \vdash_{LP} \Delta$ or both $\Gamma, \neg\phi \vdash_{K3} \emptyset$ and $\Gamma, \neg\psi \vdash_{K3} \emptyset$. If $\emptyset \vdash_{LP} \Delta$, again nothing changes travelling north, and if $\Gamma, \neg\phi \vdash_{K3} \emptyset$ and $\Gamma, \neg\psi \vdash_{K3} \emptyset$, we can, using the rule under investigation, entirely properly unite closed tableaux for $\Gamma, \neg\phi : \emptyset$ and $\Gamma, \neg\psi : \emptyset$ under the node $\boxed{\Gamma, \neg(\phi \wedge \psi) : \emptyset}$ to to obtain a closed $K3$-tableau for $\Gamma, \neg(\phi \wedge \psi) \vdash_{K3} \emptyset$.

Disjunction and Implication. The rules governing disjunction and implication are treated similarly.

Universal quantification

$$\begin{array}{c} \boxed{\Gamma, \forall x\phi : \Delta} \\ | \\ \boxed{\Gamma, \forall x\phi,\ \phi\,[t/x]\ : \Delta} \end{array}$$

Suppose first that the interpolant associated with $\boxed{\Gamma, \forall x\phi,\ \phi\,[t/x] : \Delta}$ is χ, so that $\Gamma, \forall x\phi,\ \phi\,[t/x] \vdash_{K3} \chi$ and $\chi \vdash_{LP} \Delta$. By an application of the rule in question, we have that $\Gamma, \forall x\phi \vdash_{K3} \chi$. If the individual constant t occurs in Γ or does not occur in χ, only non-logical vocabulary common to $\Gamma \cup \{\forall x\phi\}$ and Δ occurs in χ and so we retain χ as interpolant. If, on the other hand, t doesn't occur in Γ but does occur in χ then

$$\begin{array}{c} \boxed{\Gamma, \forall x\phi : \forall v\chi\,[v/t]} \\ | \\ \boxed{\Gamma, \forall x\phi : \chi} \\ | \\ \boxed{\Gamma, \forall x\phi,\ \phi\,[t/x] : \chi} \end{array} \qquad \text{and} \qquad \begin{array}{c} \boxed{\forall v\chi\,[v/t] : \Delta} \\ | \\ \boxed{\forall v\chi\,[v/t], \chi : \Delta} \end{array}$$

are produced by correct applications of the tableaux rules, v being some variable foreign to χ and t being new in the context in which it is introduced in place of v in $\chi[v/t]$ on the right-hand side in the left-hand tableau fragment. Appending a closed $K3$-tableaux for $\Gamma, \forall x\phi,\ \phi\,[t/x] : \chi$, we obtain a closed $K3$-tableau for $\Gamma, \forall x\phi : \forall v\chi\,[v/t]$. Appending a closed LP-tableau for $\chi : \Delta$ and inserting $\forall v\chi\,[v/t]$ on the left of the colon in all its nodes, we obtain a closed LP-tableau for $\forall v\chi\,[v/t]: \Delta$. Thus $\forall v\chi\,[v/t]$ serves as interpolant, containing only non-logical vocabulary common to $\Gamma \cup \{\forall x\phi\}$ and Δ.

If no interpolant is associated with $\boxed{\Gamma, \forall x\phi,\ \phi\,[t/x]\ : \Delta}$, none is associated with $\boxed{\Gamma, \forall x\phi : \Delta}$, for if $\emptyset \vdash_{LP} \Delta$, nothing changes and if $\Gamma, \forall x\phi,\ \phi\,[t/x] \vdash_{K3}$

Ø, a closed $K3$-tableau for $\Gamma, \forall x\phi, \phi\,[t/x] : \emptyset$ can be expanded to one for $\Gamma, \forall x\phi : \emptyset$ (and starts with an application of the very rule examined here).

$$\begin{array}{c} \boxed{\Gamma : \forall x\phi, \Delta} \\ | \\ \boxed{\Gamma : \phi\,[a/x], \Delta} \end{array}$$

where the individual
constant a is new

Suppose first that the interpolant associated with $\boxed{\Gamma : \phi\,[a/x], \Delta}$ is χ, so that $\Gamma \vdash_{K3} \chi$ and $\chi \vdash_{LP} \phi[a/x], \Delta$, and that the name a does not occur in any member of $\Gamma \cup \Delta \cup \{\forall x\phi\}$. Since only non-logical vocabulary common to both Γ and $\Delta \cup \{\phi[a/x]\}$ and Δ occurs in χ, a does not occur in χ; hence the step with input pair $\chi : \forall x\phi, \Delta$ and output pair $\chi : \phi\,[a/x], \Delta$ proceeds in accordance with the rules and allows us to turn a closed LP-tableau for $\chi : \phi\,[a/x], \Delta$ into a closed LP-tableau for $\chi : \forall x\phi, \Delta$.

If no interpolant is associated with $\boxed{\Gamma : \phi\,[a/x], \Delta}$, none is associated with $\boxed{\Gamma : \forall x\phi, \Delta}$, for if $\Gamma \vdash_{K3} \emptyset$, nothing changes, and a closed LP-tableau for $\emptyset : \phi\,[a/x], \Delta$, a new, extends to one for $\emptyset : \forall x\phi, \Delta$ by an application of the very rule in question here.

$$\begin{array}{c} \boxed{\Gamma, \neg\forall x\phi : \Delta} \\ | \\ \boxed{\Gamma, \neg\phi[a/x] : \Delta} \end{array}$$

where the individual
constant a is new

Suppose first that the interpolant associated with $\boxed{\Gamma, \neg\phi[a/x] : \Delta}$ is χ, so that $\Gamma, \neg\phi[a/x] \vdash_{K3} \chi$ and $\chi \vdash_{LP} \Delta$, and the name a does not occur in any member of $\Gamma \cup \Delta \cup \{\neg\forall x\phi\}$. Since only non-logical vocabulary common to both $\Gamma \cup \{\neg\phi[a/x]\}$ and Δ occurs in χ, a does not occur in χ; hence the step with input pair $\Gamma, \neg\forall x\phi : \chi$ and output pair $\Gamma, \neg\phi[a/x] : \chi$ proceeds in accordance with the rules and allows us to turn a closed $K3$-tableau for $\Gamma, \neg\phi\,[a/x] : \chi$ into a closed $K3$-tableau for $\Gamma, \neg\forall x\phi : \chi$.

If no interpolant is associated with $\boxed{\Gamma, \neg\phi[a/x] : \Delta}$, none is associated with $\boxed{\Gamma, \neg\forall x\phi : \Delta}$ for if $\emptyset \vdash_{LP} \Delta$, nothing changes, and a closed $K3$-tableau for $\Gamma, \neg\phi[a/x] : \emptyset$, a new, extends to one for $\Gamma, \neg\forall x\phi : \emptyset$ by an application of the very rule in question here.

$$\begin{array}{c} \boxed{\Gamma : \neg\forall x\phi, \Delta} \\ | \\ \boxed{\Gamma : \neg\phi[t/x], \neg\forall x\phi, \Delta} \end{array}$$

Suppose first that the interpolant associated with $\boxed{\Gamma : \neg\phi[t/x], \neg\forall x\phi, \Delta}$ is χ, so that $\Gamma \vdash_{K3} \chi$ and $\chi \vdash_{LP} \neg\phi\,[t/x], \neg\forall x\phi, \Delta$. By an application of the rule in question, we have that $\chi \vdash_{LP} \neg\forall x\phi, \Delta$. If t occurs in Δ or does not occur in χ then only non-logical vocabulary common to Γ and $\Delta \cup \{\neg\forall x\phi\}$ occurs in χ and we retain χ as interpolant. If, on the other hand, t doesn't occur in Δ but does occur in χ then

$$\begin{array}{c} \boxed{\Gamma : \exists v\chi[v/t]} \\ | \\ \boxed{\Gamma : \chi, \exists v\chi[v/t]} \end{array} \quad \text{and} \quad \begin{array}{c} \boxed{\exists v\chi\,[v/t] : \neg\forall x\phi, \Delta} \\ | \\ \boxed{\chi : \neg\forall x\phi, \Delta} \\ | \\ \boxed{\chi : \neg\phi[t/x], \neg\forall x\phi, \Delta} \end{array}$$

are produced by correct applications of the tableaux rules, v being some variable foreign to χ and t being new in the context in which it is introduced in place of v in $\chi\,[v/t]$ on the left-hand side. Appending a closed LP-tableaux for $\chi : \neg\phi\,[t/x], \neg\forall x\phi, \Delta$, we obtain a closed LP-tableau for $\exists v\chi\,[v/t] : \neg\forall x\phi, \Delta$. Appending a closed $K3$-tableau for $\Gamma : \chi$ and inserting $\exists v\chi\,[v/t]$ on the right of the colon in all its nodes, we obtain a closed $K3$-tableau for $\Gamma : \exists v\chi\,[v/t]$. Thus $\exists v\chi\,[v/t]$ serves as interpolant, containing only non-logical vocabulary common to Γ and $\Delta \cup \{\neg\forall x\phi\}$.

If no interpolant is associated with $\boxed{\Gamma : \neg\phi\,[t/x], \Delta}$, none is associated with $\boxed{\chi : \neg\forall x\phi, \Delta}$, for if $\Gamma \vdash_{K3} \emptyset$, nothing changes, and if $\emptyset \vdash_{LP} \neg\phi\,[t/x], \neg\forall x\phi, \Delta$ a closed LP-tableau for $\emptyset : \neg\phi[t/x], \neg\forall x\phi, \Delta$ can be expanded to one for $\emptyset : \neg\forall x\phi, \Delta$ (and starts with an application of the very rule examined here).

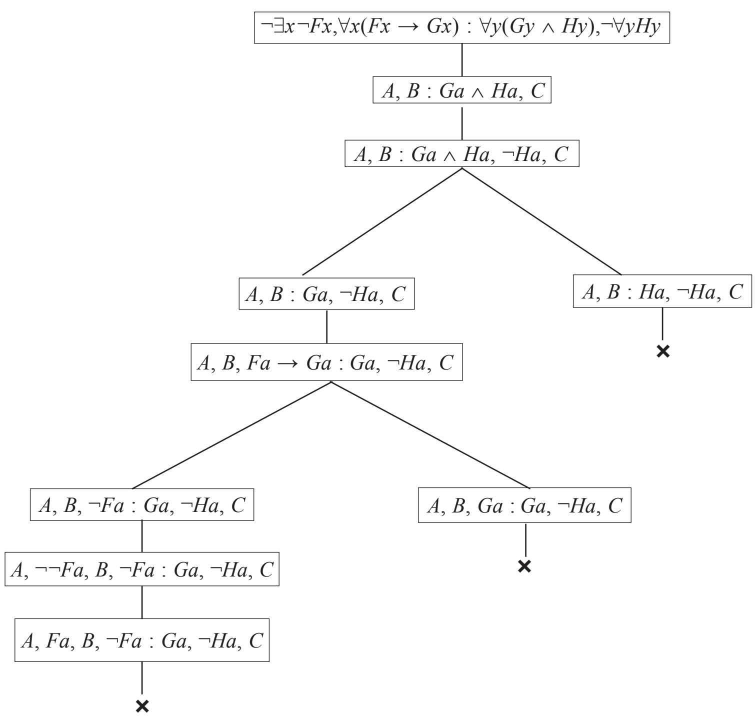
Existential quantification. The rules governing the existential quantifier are dealt with similarly.

Interpolants percolate upwards from terminal nodes of the forms $\boxed{\Theta, \psi : \psi, E}$ and $\boxed{\Theta, \neg\psi : \neg\psi, E}$, ψ atomic, possibly undergoing change, possibly being eliminated. By LEMMA 1, when $\Gamma \nvdash \emptyset$ and $\emptyset \nvdash \Delta$, there must be at least one such node. Moreover, as no rule increases the stock of predicates in play on either side of the colon, every predicate occurring in each such node must occur in both Γ and Δ. REMARK 1 shows that such predicates must occur with the same parity in the node at the head of the tableau. As inspection of the instructions for the formation of interpolants, if any, associated with input pairs from interpolants associated with output pairs confirms, predicates occurring in the interpolant, if any, associated with an input pair occur with the same parity as they occur in the interpolant associated with the output pair or pairs, if there's just one such interpolant, or as in one or other of the interpolants associated with the output pairs, when there are two such interpolants. Consequently, every predicate which occurs in the interpolant associated with $\Gamma : \Delta$ occurs with the same parity in some member of Γ and also in some member of Δ. Lastly, the parity of the occurrence of a predicate in an interpolant is singularly easy to determine for, from the instructions for the formation of interpolants, interpolants contain no occurrence of the conditional and negations occur only in subformulas of the form $\neg\phi$, ϕ atomic.

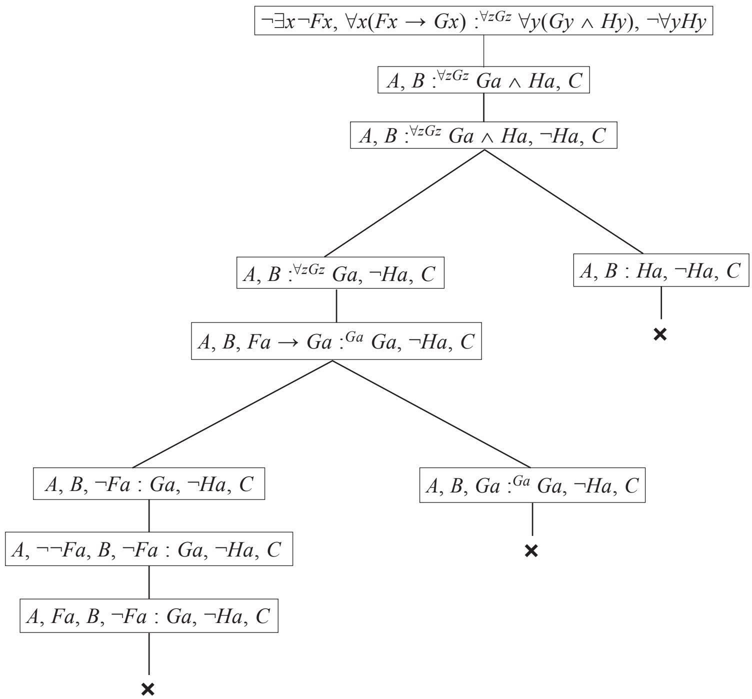
If we have a closed tableau for $\Gamma : \Delta$ with which *no* interpolant is associated, then, in virtue of our procedure, we have that either $\Gamma \vdash_{K3} \emptyset$, so $\Gamma \vdash \emptyset$, or $\emptyset \vdash_{LP} \Delta$, so $\emptyset \vdash \Delta$, or both. □

REMARK 3. Let us say that a formula is responsible for terminating a branch if it plays the role of ϕ in any of the four forms of terminal node given on p. 393. In our "reverse engineering" of the $K3$-and LP-tableaux involving the interpolant in the proof of THEOREM 1, when an interpolant exists, the terminal nodes may not be identical to terminal nodes in the original closed tableau but the construction employed ensures that this much does carry over: the formulas responsible for terminating branches are responsible in the closed tableau on which the construction is based and they terminate them in terminal nodes of the same form as in in the original tableau.

EXAMPLE 1. The classically valid $\neg\exists x\neg Fx, \forall x(Fx \rightarrow Gx) \vDash \forall y(Gy \wedge Hy), \neg\forall yHy$ is neither $K3$-correct nor LP-correct. Here's a tableau derivation, where A abbreviates $\neg\exists x\neg Fx$, B abbreviates $\forall x(Fx \rightarrow Gx)$ and C abbreviates $\neg\forall yHy$:

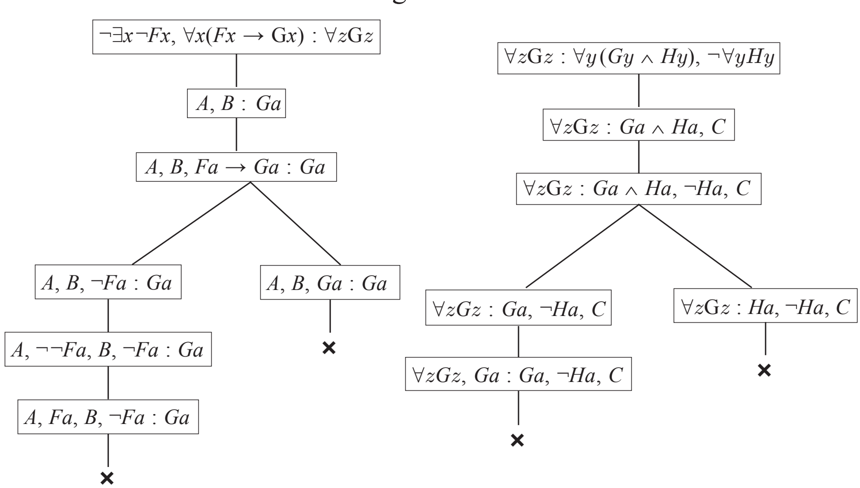


Here's the derivation decorated with interpolants:



where we can use any variable in place of z.

The instructions accompanying the proof of THEOREM 1 give us the left-hand closed $K3$-tableau and the right-hand closed LP-tableau:



The predicate G occurs with even parity in the interpolant $\forall zGz$ and in $\forall x(Fx \rightarrow Gx)$, to the left of the colon in $\boxed{\neg\exists x\neg Fx, \forall x(Fx \rightarrow Gx) : \forall y(Gy \wedge Hy), \neg\forall yHy}$, and in $\forall y\,(Gy \wedge Hy)$, to the right.

EXAMPLE 2

$$\boxed{(\exists xFx \wedge \neg\exists xFx) \vee \forall xGx : (\exists xFx \vee \neg\exists xFx) \vee \exists xHx}$$
$$|$$
$$\boxed{(\exists xFx \wedge \neg\exists xFx) \vee \forall xGx : \exists xFx \vee \neg\exists xFx, \exists xHx}$$
$$|$$
$$\boxed{(\exists xFx \wedge \neg\exists xFx) \vee \forall xGx : \exists xFx, \neg\exists xFx, \exists xHx}$$
$$|$$
$$\boxed{(\exists xFx \wedge \neg\exists xFx) \vee \forall xGx : \exists xFx, \neg Fa, \exists xHx}$$
$$|$$
$$\boxed{(\exists xFx \wedge \neg\exists xFx) \vee \forall xGx : Fa, \exists xFx, \neg Fa, \exists xHx}$$
$$|$$
$$\times$$

is a derivation *fully decorated with interpolants* — there are none. But as deleting everything on the left-hand side shows, $\varnothing \vdash (\exists xFx \vee \neg\exists xFx) \vee \exists xHx$.

REMARK 4. What this example shows is that we cannot weaken the antecedent of THEOREM 1 merely to the requirement that Γ and Δ share non-logical vocabulary and keep the same method of construction for interpolants. But the method is appropriate to the task at hand, for as an interpretation $\langle D, \mathcal{I}\rangle$ in which $\mathcal{I}^+(F) = \mathcal{I}^-(F) = \varnothing$ and $\mathcal{I}^+(G) = D$ makes clear, no formula containing F as sole predicate is a $K3$-consequence of $(\exists xFx \wedge \neg\exists xFx) \vee \forall xGx$.

5. Semantics for the first-order extension of Belnap's four-valued logic

A $B4$-interpretation $\mathfrak{A}$ for a first-order language with neither function-symbols nor identity comprises a non-empty domain D and a function $\mathcal{I}$; to each individual constant c, $\mathcal{I}$ assigns an element of D; to each n-place predicate F, $\mathcal{I}$ assigns a function $\mathcal{I}(F)$ mapping elements of D^n into the set $\{0, n, b, 1\}$.

As before we extend the language by adding a new constant, $\overline{d}$, for each element d in D. Again we extend $\mathcal{I}$ by the stipulation: for all $d \in D$, $\mathcal{I}(\overline{d}) = d$. Relative to the interpretation $\mathfrak{A} = \langle D, \mathcal{I}\rangle$, we assign values to atomic formulas of the extended language by the constraint:

$$v_{\mathfrak{A}}(F(c_1, c_2, \ldots, c_n)) = \mathcal{I}(F)\,(\mathcal{I}(c_1), \mathcal{I}(c_2), \ldots, \mathcal{I}(c_n)).$$

We evaluate formulas with negation, conjunction, disjunction, or (material) implication dominant in accordance with these truth-tables:

ϕ	$\neg\phi$
1	0
b	b
n	n
0	1

	$\phi \wedge \psi$	ψ: 1	b	n	0
ϕ	1	1	b	n	0
	b	b	b	0	0
	n	n	0	n	0
	0	0	0	0	0

	$\phi \vee \psi$	ψ: 1	b	n	0
ϕ	1	1	1	1	1
	b	1	b	1	b
	n	1	1	n	n
	0	1	b	n	0

	$\phi \to \psi$	ψ: 1	b	n	0
ϕ	1	1	b	n	0
	b	1	b	1	b
	n	1	1	n	n
	0	1	1	1	1

- $v_{\mathfrak{A}}(\forall v\phi) = 1$ if, for all $d \in D$, $v_{\mathfrak{A}}(\phi[\overline{d}/v]) = 1$;
- $v_{\mathfrak{A}}(\forall v\phi) = b$ if, for all $d \in D$, $v_{\mathfrak{A}}(\phi[\overline{d}/v]) \in \{1,b\}$ and, for at least one $d \in D$, $v_{\mathfrak{A}}(\phi[\overline{d}/v]) = b$;
- $v_{\mathfrak{A}}(\forall v\phi) = n$ if, for all $d \in D$, $v_{\mathfrak{A}}(\phi[\overline{d}/v]) \in \{1,n\}$ and, for at least one $d \in D$, $v_{\mathfrak{A}}(\phi[\overline{d}/v]) = n$;
- $v_{\mathfrak{A}}(\forall v\phi) = 0$ if, for some $d \in D$, $v_{\mathfrak{A}}(\phi[\overline{d}/v]) = 0$ or if, for some pair d_1, d_2 in D, $v_{\mathfrak{A}}(\phi[\overline{d_1}/v]) = b$ and $v_{\mathfrak{A}}(\phi[\overline{d_2}/v]) = n$.
- $v_{\mathfrak{A}}(\exists v\phi) = 1$ if, for some $d \in D$, $v_{\mathfrak{A}}(\phi[\overline{d}/v]) = 1$ or if, for some pair d_1, d_2 in D, $v_{\mathfrak{A}}(\phi[\overline{d_1}/v]) = b$ and $v_{\mathfrak{A}}(\phi[\overline{d_2}/v]) = n$;
- $v_{\mathfrak{A}}(\exists v\phi) = b$ if, for all $d \in D$, $v_{\mathfrak{A}}(\phi[\overline{d}/v]) \in \{b,0\}$ and, for at least one $d \in D$, $v_{\mathfrak{A}}(\phi[\overline{d}/v]) = b$;
- $v_{\mathfrak{A}}(\exists v\phi) = n$ if, for all $d \in D$, $v_{\mathfrak{A}}(\phi[\overline{d}/v]) \in \{n,0\}$ and, for at least one $d \in D$, $v_{\mathfrak{A}}(\phi[\overline{d}/v]) = n$;
- $v_{\mathfrak{A}}(\exists v\phi) = 0$ if, for all $d \in D$, $v_{\mathfrak{A}}(\phi[\overline{d}/v]) = 0$.

DEFINITION 10 ($B4$ (JOINT) SATISFACTION). $\mathfrak{A} \vDash_{B4} \Gamma$ iff, for all $\phi \in \Gamma$, $v_{\mathfrak{A}}(\phi) \in \{1, b\}$.

DEFINITION 11 ($B4$ (JOINT) REFUTATION). $\mathfrak{A} \dashv_{B4} \Gamma$ iff, for all $\phi \in \Gamma$, $v_{\mathfrak{A}}(\phi) \in \{n, 0\}$.

DEFINITION 12 ($B4$-CONSEQUENCE). $\Gamma \vDash_{B4} \Delta$ iff for no interpretation $\mathfrak{A}$ is it the case that $\mathfrak{A} \vDash_{B4} \Gamma$ and $\mathfrak{A} \dashv_{B4} \Delta$.

DEFINITION 13 ($B4$-(IN)CORRECTNESS). Say that a node $\boxed{\Gamma : \Delta}$ (where Γ and Δ are finite, possibly empty sets of of formulas in a first-order language without identity or function-symbols) is $B4$-incorrect with respect to the $B4$-interpretation $\mathfrak{A} = \langle D, \mathcal{I}\rangle$ iff $\mathfrak{A} \vDash_{B4} \Gamma$ and $\mathfrak{A} \dashv_{B4} \Delta$.

Say that a node $\boxed{\Gamma : \Delta}$ is $B4$-correct if there is no $B4$-interpretation with respect to which it is incorrect. If $\boxed{\Gamma : \Delta}$ is $B4$-correct, so too is $\boxed{\Gamma, \Gamma' : \Delta, \Delta'}$. (The node $\boxed{\Gamma : \Delta}$ is $B4$-correct just in case Δ is a $B4$-consequence of Γ.)

Soundness. The tableaux rules preserve $B4$-incorrectness with respect to any $B4$-interpretation $\langle D, \mathcal{I}\rangle$ *downwards* in the sense that if the input pair in the application of a rule is $B4$-incorrect with respect to a $B4$-interpretation, so too is the output pair, if there is only one, and so too is at least one of the output pairs, if there are two. (In the case of the rules where a new name is introduced, the function $\mathcal{I}$ must be extended to supply an appropriate interpretation of that name; it is guaranteed that there will be such when the input pair is incorrect with respect to the interpretation in play.) Nodes of the forms $\boxed{\Gamma, \phi : \phi, \Delta}$ and $\boxed{\Gamma, \neg\phi : \neg\phi, \Delta}$ are $B4$-correct.

6. Further lessons from the proof technique

COROLLARY 1.

(a) If $\Gamma \vdash_{K3} \Delta$ and $\Gamma \nvdash_{K3} \varnothing$ then there is an interpolant χ, constructed using only non-logical vocabulary common to both Γ and Δ, such that

$$\Gamma \vdash_{K3} \chi \text{ and } \chi \vdash_{B4} \Delta$$

and every predicate which occurs with even parity in χ occurs with even parity in some member of Γ and also in some member of Δ and, likewise, every predicate which occurs with odd parity in χ occurs with odd parity in some member of Γ and also in some member of Δ.

(b) If $\Gamma \vdash_{LP} \Delta$ and $\varnothing \nvdash_{LP} \Delta$ then there is an interpolant χ, constructed using only non-logical vocabulary common to both Γ and Δ, such that

$$\Gamma \vdash_{B4} \chi \text{ and } \chi \vdash_{LP} \Delta$$

and every predicate which occurs with even parity in χ occurs with even parity in some member of Γ and also in some member of Δ and, likewise, every predicate which occurs with odd parity in χ occurs with odd parity in some member of Γ and also in some member of Δ.

(c) If $\Gamma \vdash_{B4} \Delta$ then there is an interpolant χ, constructed using only non-logical vocabulary common to both Γ and Δ, such that

$$\Gamma \vdash_{B4} \chi \text{ and } \chi \vdash_{B4} \Delta$$

and every predicate which occurs with even parity in χ occurs with even parity in some member of Γ and also in some member of Δ and, likewise, every predicate which occurs with odd parity in χ occurs with odd parity in some member of Γ and also in some member of Δ.

PROOF. A closed $K3$-tableau for $\Gamma : \Delta$ contains no terminal nodes of the form $\Theta : \psi, \neg\psi, E$, so when we carry out the construction of the interpolant χ, exactly as instructed above, we obtain a closed $K3$-tableaux for $\Gamma : \chi$ in which no terminal nodes of the form $\boxed{\Theta : \psi, \neg\psi, E}$ occurs and a closed

LP-tableaux for $\chi : \Delta$ in which no terminal nodes of either of the forms $\boxed{\Theta : \psi, \neg\psi, E}$ and $\boxed{\Theta, \psi, \neg\psi : E}$ occurs (REMARK 3).

Parts (b) and (c) are demonstrated analogously. □

In light of the Duality Principle and rather obvious facts about use of the double negation rules, (a) and (b) entail each other.

REMARK 5. The presence of terminal nodes of the form $\boxed{\Gamma, \phi, \neg\phi : \psi, \neg\psi, \Delta}$ in $K3 \sqcap LP$-tableaux renders the technique inapplicable to that logic, nor is it obvious how it could be adapted.

7. Block tableaux for first-order classical logic with identity

As $a = b \vDash Fa \rightarrow Fb$ and $Fa \wedge \neg Fb \vDash \neg(a = b)$ but none of the formulas is logically true or logically false, $a = b$, which contains no non-logical predicate, must be the interpolant in the first case, $\neg(a = b)$ in the second, if we are to broaden the scope of the Craig–Lyndon Interpolation Theorem to encompass first-order logic with identity (*cf.* Oberschelp 1968, p. 271). With that aim in mind, we must add rules for identity. What we call the *LP* rules follow the treatment of identity in (Priest 2006, §§5.3 & 6.7).[7] We add these rules for identity where $\phi(s/t)$ results from ϕ by substitution of the individual constant s for one or more occurrences of the individual constant t in ϕ:[8]

*K*3 **rules**

$$\begin{array}{cccc}
\boxed{\Gamma, \phi, t = s : \Delta} & \boxed{\Gamma, \phi, s = t : \Delta} & \boxed{\Gamma, \phi : \neg(t = s), \Delta} & \boxed{\Gamma, \phi : \neg(s = t), \Delta} \\
| & | & | & | \\
\boxed{\Gamma, \phi, \phi(s/t), t = s : \Delta} & \boxed{\Gamma, \phi, \phi(s/t), s = t : \Delta} & \boxed{\Gamma, \phi, \phi(s/t) : \neg(t = s), \Delta} & \boxed{\Gamma, \phi, \phi(s/t) : \neg(s = t), \Delta}
\end{array}$$

LP **rules**

$$\begin{array}{cccc}
\boxed{\Gamma, t = s : \phi, \Delta} & \boxed{\Gamma, s = t : \phi, \Delta} & \boxed{\Gamma : \neg(t = s), \phi, \Delta} & \boxed{\Gamma : \neg(s = t), \phi, \Delta} \\
| & | & | & | \\
\boxed{\Gamma, t = s : \phi, \phi(s/t), \Delta} & \boxed{\Gamma, s = t : \phi, \phi(s/t), \Delta} & \boxed{\Gamma : \neg(t = s), \phi, \phi(s/t), \Delta} & \boxed{\Gamma : \neg(s = t), \phi, \phi(s/t), \Delta}
\end{array}$$

[7] As a referee points out, the treatment of identity in (Priest 2010) is rather different.

[8] Should t not occur in ϕ, attempted application of any of these rules yields mere repetition which is redundant.

and add these terminal nodes:

$$\boxed{\Gamma, \neg(t = t) : \Delta} \qquad\qquad \boxed{\Gamma : t = t, \Delta}$$

Again, a tableau is complete when no rule can be applied. We write $\Gamma \vdash_{=} \Delta$ if all branches close in some completed tableau employing identity rules and/or involving identity terminal nodes and headed by the node $\boxed{\Gamma : \Delta}$. Again we then say that a closed tableau exists for $\Gamma : \Delta$.

We must expand upon the definitions of $K3$-and LP-tableaux.

DEFINITION 14 ($K3_{=}$-AND $LP_{=}$-TABLEAUX WITH IDENTITY). We now write $\Gamma \vdash_{K3_{=}} \Delta$ if no branch terminates in a node of the forms $\boxed{\Gamma : \chi, \neg\chi, \Delta}$ or $\boxed{\Gamma : t = t, \Delta}$ and there is at most application of $K3$ rules for identity in some closed tableau for $\Gamma : \Delta$; we call such a tableau a closed $K3_{=}$-tableau. Likewise, we now write $\Gamma \vdash_{LP_{=}} \Delta$ if no branch ends in a node of the forms $\boxed{\Gamma, \chi, \neg\chi : \Delta}$ or $\boxed{\Gamma, \neg(t = t) : \Delta}$ and there is at most application of LP rules for identity in some closed tableau for $\Gamma : \Delta$; we call such a tableau a closed $LP_{=}$-tableau.

The division of rules into $K3$ and LP rules may require some explanation. It is undertaken with four aims in view: (*i*) the overall system must be (sound and) complete for classical first-order logic with identity; (*ii*) we respect the fact that *classically* $\emptyset \vdash t = t$ and $\neg(t = t) \vdash \emptyset$ are really just two ways of saying the same thing, that $t = t$ is always true, never false; (*iii*) $K3$ should suffice for demonstrating classical inconsistency but need do little more, likewise LP should suffice for demonstrating classical logical truth but need do little more, so we may retain $K3$'s lack of theorems and LP's lack of "antitheorems" (formulas which entail everything); (*iv*) we wish to maintain the syntactic duality between $K3$ and LP when identity is added. However unnatural they may appear from other perspectives, the semantics given below reflects these aims.

REMARKS 1 and 2 still apply in this extended context and the Duality Principle (Syntax) still holds. But although REMARK 2 holds in the letter, it fails in spirit, for an occurrence of an identity on the left or of a negated identity on the right can provide the basis for changes on the other side of the colon. And LEMMA 1 now fails. *E.g.*,

$$\begin{array}{c} \boxed{a = b : b = a} \\ | \\ \boxed{a = b : a = a} \\ | \\ \times \end{array} \qquad \text{and} \qquad \begin{array}{c} \boxed{Fa, \neg Fb : \neg(a = b)} \\ | \\ \boxed{Fa, \neg Fa : \neg(a = b)} \\ | \\ \times \end{array}$$

Nevertheless we can extend our refinement of the Craig–Lyndon Interpolation Theorem to first-order logic with identity in a fairly straightforward way. In place of LEMMA 1 we have

LEMMA 3. If there's a closed tableau for $\Gamma : \Delta$ in which no branch reaches a node of either of the forms $\boxed{\Theta, \psi : \psi, E}$ and $\boxed{\Theta, \neg\psi : \neg\psi, E}$, and none of the four rules

$$\begin{array}{cccc}
\boxed{\Gamma, t = s : \phi, \Delta} & \boxed{\Gamma, \phi : \neg(t = s), \Delta} & \boxed{\Gamma, s = t : \phi, \Delta} & \boxed{\Gamma, \phi : \neg(s = t), \Delta} \\
| & | & | & | \\
\boxed{\Gamma, t = s : \phi, \phi(s/t), \Delta} & \boxed{\Gamma, \phi, \phi(s/t) : \neg(t = s), \Delta} & \boxed{\Gamma, s = t : \phi, \phi(s/t), \Delta} & \boxed{\Gamma, \phi, \phi(s/t) : \neg(s = t), \Delta}
\end{array}$$

is used, we have that $\Gamma \vdash_{=} \varnothing$ or $\varnothing \vdash_{=} \Delta$.

PROOF mimics the proof of LEMMA 1.

Likewise, proof of the next lemma mimics proof of LEMMA 2, noting that *LP* rules for identity, being right-hand rules, have no purchase when the right side of a node is empty and, similarly, *K*3 rules for identity, being left-hand rules, have no purchase when the left side is empty.

LEMMA 4. If $\Gamma \vdash_{=} \varnothing$ then $\Gamma \vdash_{K3_{=}} \varnothing$. Likewise, if $\varnothing \vdash_{=} \Delta$ then $\varnothing \vdash_{LP_{=}} \Delta$.

8. Semantics for first-order *K*3, *LP*, and classical logic with identity

We add these semantic evaluation clauses for identity: in an interpretation $\mathfrak{A} = \langle D, \mathcal{I} \rangle$ $\mathcal{I}^{+}(=) \subseteq \{\langle d, d \rangle : d \in D\}$ and $\mathcal{I}^{-}(=) = D^2 - \{\langle d, d \rangle : d \in D\}$. Consequently, $v_{\mathfrak{A}}(\neg(t = s)) < 1$ if, and only if, $v_{\mathfrak{A}}(t = s) > 0$ if, and only if $\mathcal{I}(t) = \mathcal{I}(s)$.

In a classical interpretation, $\mathcal{I}^{+}(=) = \{\langle d, d \rangle : d \in D\}$.

The notions of *K*3- and *LP*-satisfaction, consequence, incorrectness, and correctness are expanded to accommodate identity.

Soundness. The *K*3 identity rules preserve *K*3-incorrectness with respect to an interpretation $\mathfrak{A} = \langle D, \mathcal{I} \rangle$ downwards. Nodes of the form $\boxed{\Gamma, \neg(t = t) : \Delta}$ are *K*3-correct but not *LP*-correct.

The *LP* identity rules preserve *LP*-incorrectness with respect to an interpretation $\mathfrak{A} = \langle D, \mathcal{I} \rangle$ downwards. Nodes of the form $\boxed{\Gamma : t = t, \Delta}$ are *LP*-correct but not *K*3-correct.

Classical incorrectness with respect to a classical interpretation is preserved downwards. Nodes of the forms $\boxed{\Gamma, \neg(t = t) : \Delta}$ and $\boxed{\Gamma : t = t, \Delta}$ are classically correct.

(In point of fact, the *K*3 and the *LP* identity rules both preserve both *K*3- *and* *LP*-incorrectness. It's of interest to note, then, that in the definitions of *K*3- and *LP*-tableaux, and so in our proof of our interpolation theorem below, we make use only of the *K*3 rules' preservation of *K*3-incorrectness and the *LP* rules' preservation of *LP*-incorrectness. Recalling the remark on p. 397 regarding the aim of the present investigation, we require only soundness. Moreover, this is a step in the direction of delineating how much classical logic is needed to demonstrate classical inconsistency, how much to demonstrate classical logic truth.)

Completeness. Again, with judicious use of the rules for double negation a closed tableau exists for $\Gamma : \Delta$ if, and only if, a closed tableau exists for $\Gamma \cup -\Delta : \varnothing$. In a closed tableau for $\Gamma \cup -\Delta : \varnothing$, only left-hand rules are used (REMARK 2). But the left-hand rules are in effect just the familiar rules for tableaux employed to test classical consistency in a first-order language with identity. Borrowing again from any number of classical or textbook sources, we have that if there is no closed tableau for $\Gamma \cup -\Delta : \varnothing$, equivalently, no closed standard tableau for the set $\Gamma \cup -\Delta$, there is a classical interpretation in which all members of $\Gamma \cup -\Delta$, take the value 1. Equivalently, there is a classical interpretation with respect to which $\boxed{\Gamma : \Delta}$ is classically incorrect.

9. A refinement of the Craig–Lyndon Interpolation Theorem with identity

THEOREM 2. If $\Gamma \vdash_{=} \Delta$ and $\Gamma \nvdash_{=} \varnothing$ and $\varnothing \nvdash_{=} \Delta$ then there is an interpolant χ, constructed using only non-logical vocabulary common to both Γ and Δ and, if present in $\Gamma \cup \Delta$, possibly the identity-predicate such that

$$\Gamma \vdash_{K3_{=}} \chi \text{ and } \chi \vdash_{LP_{=}} \Delta$$

and every non-logical predicate which occurs with even parity in χ occurs with even parity in some member of Γ and also in some member of Δ and, likewise, every predicate which occurs with odd parity in χ occurs with odd parity in some member of Γ and also in some member of Δ.

PROOF. The steps we need, supplementary to those in the proof of Theorem 1, are these:

Terminal nodes. No interpolants are associated with terminal nodes of the forms $\boxed{\Gamma, \neg(t = t) : \Delta}$ and $\boxed{\Gamma : t = t, \Delta}$. For any Γ, Δ and t, we have that $\Gamma, \neg(t = t) \vdash_{K3_{=}} \varnothing$ and $\varnothing \vdash_{LP_{=}} t = t, \Delta$.

No change rules. In the case of these rules —

$$\begin{array}{cccc}
\boxed{\Gamma, \phi, t = s : \Delta} & \boxed{\Gamma, \phi, s = t : \Delta} & \boxed{\Gamma : \neg(t = s), \phi, \Delta} & \boxed{\Gamma : \neg(s = t), \phi, \Delta} \\
| & | & | & | \\
\boxed{\Gamma, \phi, \phi(s/t), t = s : \Delta} & \boxed{\Gamma, \phi, \phi(s/t), s = t : \Delta} & \boxed{\Gamma : \neg(t = s), \phi, \phi(s/t), \Delta} & \boxed{\Gamma : \neg(s = t), \phi, \phi(s/t), \Delta}
\end{array}$$

— the interpolant, if any, associated with the output pair is associated with the input pair (and no interpolant is associated with the input pair if none is associated with the output pair). In each case, where $\boxed{\Gamma : \Delta}$ is the output pair, $\boxed{\Gamma' : \Delta'}$ the input pair, and ψ the interpolant, we have that one of the two steps, from $\boxed{\Gamma' : \psi}$ to $\boxed{\Gamma : \psi}$ or from $\boxed{\psi : \Delta'}$ to $\boxed{\psi : \Delta}$, consists of redundant repetition and the other of an application of the very rule under examination. So $\Gamma' \vdash_{K3} \psi$ when $\Gamma \vdash_{K3} \psi$ and $\psi \vdash_{LP} \Delta'$ when $\psi \vdash_{LP} \Delta$.

By assumption, if no interpolant is associated with the output pair $\boxed{\Gamma : \Delta}$, then either $\Gamma \vdash_{K3} \varnothing$ or $\varnothing \vdash_{LP} \Delta$ or both. If $\Gamma' = \Gamma$ and $\Gamma \vdash_{K3} \varnothing$ we are done. Likewise if $\Delta' = \Delta$ and $\varnothing \vdash_{LP} \Delta$. In the case of all the rules, if $\Gamma' \neq \Gamma$ then

$$\begin{array}{c} \boxed{\Gamma' : \varnothing} \\ | \\ \boxed{\Gamma : \varnothing} \end{array}$$

by an application of the very rule under investigation and hence $\Gamma' \vdash_{K3} \varnothing$. Similarly, in the case of all the rules, if $\Delta' \neq \Delta$ then

$$\begin{array}{c} \boxed{\varnothing : \Delta'} \\ | \\ \boxed{\varnothing : \Delta} \end{array}$$

by an application of the very rule under investigation and hence $\varnothing \vdash_{LP} \Delta'$.

The other identity rules

$$\begin{array}{c} \boxed{\Gamma, t = s : \phi, \Delta} \\ | \\ \boxed{\Gamma, t = s : \phi, \phi(s/t), \Delta} \end{array}$$

Suppose first that the interpolant associated with $\boxed{\Gamma, t = s : \phi, \phi(s/t), \Delta}$ is χ, so that $\Gamma, t = s \vdash_{K3_=} \chi$ and $\chi \vdash_{LP_=} \phi, \phi(s/t), \Delta$, and only non-logical vocabulary common to $\Gamma \cup \{t = s\}$ and $\Delta \cup \{\phi, \phi(s/t)\}$ occurs in χ.

$$\begin{array}{cc} \multicolumn{2}{c}{\boxed{\Gamma, t = s : \chi \wedge (t = s)}} \\ \diagup & \diagdown \\ \boxed{\Gamma, t = s : \chi} & \boxed{\Gamma, t = s : t = s} \end{array} \quad \text{and} \quad \begin{array}{c} \boxed{\chi \wedge (t = s) : \phi, \Delta} \\ | \\ \boxed{\chi, t = s : \phi, \Delta} \\ | \\ \boxed{\chi, t = s : \phi, \phi(s/t), \Delta} \end{array}$$

proceed in accordance with the rules. Appending a closed $K3_=$-tableau for $\Gamma, t = s : \chi$ at $\Gamma, t = s : \chi$ in the left-hand tableau, we obtain a closed $K3_=$-tableau for $\Gamma, t = s : \chi \wedge (t = s)$. Adding $t = s$ on the left at all nodes in a closed $LP_=$-tableau for $\chi : \phi, \phi(s/t), \Delta$, the right-hand tableau fragment lets us construct a closed $LP_=$-tableau for $\chi \wedge (t = s) : \phi, \Delta$. If s occurs in $\Delta \cup \{\phi\}$ we are done. If not, let v be a variable foreign to $\chi \wedge (t = s)$. We may extend the tableau for $\Gamma, t = s : \chi \wedge (t = s)$ and $\chi, t = s : \phi, \phi(s/t), \Delta$ upwards by adding at the tops, respectively, these steps which comply with the rules, the latter exactly because s does not occur in $\Delta \cup \{\phi\}$, thereby obtaining a closed $K3_=$-tableau for $\Gamma, t = s : \exists v(\chi[v/s] \wedge (t = v))$ and a closed $LP_=$-tableau for $\exists v(\chi[v/s] \wedge (t = v)) : \phi, \Delta$.

$$\begin{array}{c} \boxed{\Gamma, t = s : \exists v(\chi[v/s] \wedge (t = v))} \\ | \\ \boxed{\Gamma, t = s : \chi \wedge (t = s)} \end{array} \quad \text{and} \quad \begin{array}{c} \boxed{\exists v(\chi[v/s] \wedge (t = v)) : \phi, \Delta} \\ | \\ \boxed{\chi \wedge (t = s) : \phi, \Delta} \end{array}$$

Only non-logical vocabulary common to $\Gamma \cup \{t = s\}$ and $\Delta \cup \{\phi\}$ occurs in $\exists v(\chi[v/s] \wedge (t = v))$ so the latter serves as interpolant for $\Gamma, t = s : \phi, \Delta$.

If no interpolant is associated with $\boxed{\Gamma, t = s : \phi, \phi(s/t), \Delta}$, then, by assumption, $\Gamma, t = s \vdash_{K3_=} \emptyset$ or $\emptyset \vdash_{LP_=} \phi, \phi(s/t), \Delta$.

- If $\Gamma, t = s \vdash_{K3_=} \emptyset$ then we associate no interpolant with $\boxed{\Gamma, t = s : \phi, \Delta}$.
- If $\emptyset \vdash_{LP_=} \phi, \phi(s/t), \Delta$ and s does *not* occur in $\Delta \cup \{\phi\}$ we again associate no interpolant with $\boxed{\Gamma, t = s : \phi, \Delta}$, for, s being new to $\Delta \cup \{\phi\}$, deleting $\phi(s/t)$ and its progeny from all branches in a closed $LP_=$-tableau for $\emptyset : \phi, \phi(s/t), \Delta$ leaves us with a closed $LP_=$-tableau for $\emptyset \vdash_{LP_=} \phi, \Delta$. (There is never choice about which rule to apply to a formula, at most there is choice regarding individual constants that substitute for variables; ϕ's progeny can differ from $\phi(s/t)$'s at most by having t where $\phi(s/t)$'s have s, something that cannot put off the closing of branches.)[9]
- If $\emptyset \vdash_{LP_=} \phi, \phi(s/t), \Delta$ and s occurs in $\Delta \cup \{\phi\}$ then, by an application of the rule under examination, we obtain

$$\begin{array}{c} \boxed{t = s : \phi, \Delta} \\ | \\ \boxed{t = s : \phi, \phi(s/t), \Delta} \end{array}$$

and can append a closed LP-tableau for $\boxed{\emptyset : \phi, \phi(s/t), \Delta}$ below $\boxed{t = s : \phi, \phi(s/t), \Delta}$ then add $t = s$ on the left-hand side there and at every node downwards from there. As $\Gamma, t = s \vdash_{K3_=} t = s$, the formula $t = s$ has the properties required of an interpolant; in particular, it contains only non-logical vocabulary common to both $\Gamma \cup \{t = s\}$ and $\Delta \cup \{\phi\}$.

[9] Semantically, were there an interpretation $\mathfrak{A} = \langle D, \mathscr{I} \rangle$ under which $\boxed{\emptyset : \phi, \Delta}$ is LP-incorrect, then, given that s does not occur in $\Delta \cup \{\phi\}$, the interpretation which differs from $\mathfrak{A}$ at most in assigning to s what $\mathscr{I}$ assigns to t would render $\boxed{\emptyset : \phi, \phi(s/t), \Delta}$ LP-incorrect, contrary to hypothesis.

$$\begin{array}{c} \boxed{\Gamma,\ \phi : \neg(t = s),\ \Delta} \\ | \\ \boxed{\Gamma,\ \phi,\ \phi(s/t) : \neg(t = s),\ \Delta} \end{array}$$

Suppose first that the interpolant associated with $\boxed{\Gamma,\ \phi,\ \phi(s/t) : \neg(t = s),\ \Delta}$ is χ so that $\Gamma, \phi, \phi(s/t) \vdash_{K3_=} \chi$ and $\chi \vdash_{LP_=} \neg(t = s), \Delta$, and only non-logical vocabulary common to $\Gamma \cup \{\phi, \phi(s/t)\}$ and $\Delta \cup \{\neg(t = s)\}$ occurs in χ.

$$\begin{array}{c} \boxed{\Gamma,\ \phi : \chi \vee \neg(t = s)} \\ | \\ \boxed{\Gamma,\ \phi : \chi,\ \neg(t = s)} \\ | \\ \boxed{\Gamma,\ \phi,\ \phi(s/t) : \chi,\ \neg(t = s)} \end{array} \quad \text{and} \quad \begin{array}{c} \boxed{\chi \vee \neg(t = s) : \neg(t = s),\ \Delta} \\ \diagup \qquad \diagdown \\ \boxed{\chi : \neg(t = s),\ \Delta} \quad \boxed{\neg(t = s) : \neg(t = s),\ \Delta} \end{array}$$

proceed in accordance with the rules. Appending a closed $LP_=$-tableau for $\chi : \neg(t = s), \Delta$ at $\chi : \neg(t = s), \Delta$ in the right-hand tableau, we obtain a closed $LP_=$-tableau for $\chi \vee \neg(t = s) : \neg(t = s), \Delta$. Adding $\neg(t = s)$ on the right at all nodes in a closed $K3_=$-tableau for $\Gamma, \phi, \phi(s/t) : \chi$, the left-hand tableau fragment lets us construct a closed $K3_=$-tableau for $\Gamma, \phi : \chi \vee \neg(t = s)$. If s occurs in $\Gamma \cup \{\phi\}$ we are done. If not, let v be a variable foreign to $\chi \vee \neg(t = s)$. We may extend the tableau for $\Gamma, \phi : \chi \vee \neg(t = s)$ and $\chi \vee \neg(t = s) : \neg(t = s), \Delta$ upwards by adding at the tops, respectively, these steps which comply with the rules, the first exactly because s does not occur in $\Gamma \cup \{\phi\}$, thereby obtaining a closed $K3_=$-tableau for $\Gamma, \phi : \forall v(\chi[v/s] \vee \neg(t = v))$ and a closed $LP_=$-tableau for $\forall v(\chi\,[v/s] \vee \neg(t = v)) : \neg(t = s), \Delta$.

$$\begin{array}{c} \boxed{\Gamma,\ \phi : \forall v(\chi[v/s] \vee \neg(t = v))} \\ | \\ \boxed{\Gamma,\ \phi : \chi \vee \neg(t = s)} \end{array} \quad \text{and} \quad \begin{array}{c} \boxed{\forall v(\chi[v/s] \vee \neg(t = v)) : \neg(t = s),\ \Delta} \\ | \\ \boxed{\chi \vee \neg(t = s) : \neg(t = s),\ \Delta} \end{array}$$

Only non-logical vocabulary common to $\Gamma \cup \{\phi\}$ and $\Delta \cup \{\neg(t = s)\}$ occurs in $\forall v(\chi\,[v/s] \vee \neg(t = v))$ so the latter serves as interpolant for $\Gamma, \phi : \neg(t = s), \Delta$.

If no interpolant is associated with $\boxed{\Gamma,\ \phi,\ \phi(s/t) : \neg(t = s),\ \Delta}$ then, by assumption, $\Gamma, \phi, \phi(s/t) \vdash_{K3_=} \emptyset$ or $\emptyset \vdash_{LP_=} \neg(t = s), \Delta$.

- If $\emptyset \vdash_{LP_=} \neg(t = s), \Delta$ then we associate no interpolant with $\boxed{\Gamma,\ \phi : \neg(t = s),\ \Delta}$
- If $\Gamma, \phi, \phi(s/t) \vdash_{K3_=} \emptyset$ and s does *not* occur in $\Gamma \cup \{\phi\}$, we again associate no interpolant with $\boxed{\Gamma,\ \phi : \neg(t = s),\ \Delta}$, for, s being new to $\Gamma \cup \{\phi\}$, deleting $\phi(s/t)$ and its progeny from all branches in a closed $K3_=$-tableau for $\Gamma, \phi, \phi(s/t) : \emptyset$ leaves us with a closed $LP_=$-tableau for $\Gamma, \phi : \emptyset$ (for the reasons adduced above).

◦ If $\Gamma, \phi, \phi(s/t) \vdash_{K3_=} \varnothing$ and s occurs in $\Gamma \cup \{\phi\}$ then, by an application of the rule under examination, we obtain

$$\begin{array}{c}\boxed{\Gamma, \phi : \neg(t = s)}\\ | \\ \boxed{\Gamma, \phi, \phi(s/t) : \neg(t = s)}\end{array}$$

and can append a closed $K3$-tableau for $\boxed{\Gamma, \phi, \phi(s/t) : \varnothing}$ below $\boxed{\Gamma, \phi, \phi(s/t) : \varnothing}$ then add $\neg(t = s)$ on the right-hand side there and at every node downwards from there. As $\neg(t = s) \vdash_{LP_=} \neg(t = s), \Delta$, the formula $\neg(t = s)$ has the properties required of an interpolant; in particular, it contains only non-logical vocabulary common to both $\Gamma \cup \{\phi\}$ and $\Delta \cup \{\neg(t = s)\}$.

The remaining two rules for identity are treated as the preceding two.

Again interpolants percolate upwards but this time it may be from terminal nodes of the forms $\boxed{\Theta, \psi : \psi, E}$ and $\boxed{\Theta, \neg\psi : \neg\psi, E}$, ψ atomic, or it may be from the input pair at applications of the rules

$$\begin{array}{cccc}\boxed{\Gamma, t = s : \phi, \Delta} & \boxed{\Gamma, s = t : \phi, \Delta} & \boxed{\Gamma, \phi : \neg(t = s), \Delta} & \boxed{\Gamma, \phi : \neg(s = t), \Delta}\\ | & | & | & | \\ \boxed{\Gamma, t = s : \phi, \phi(s/t), \Delta} & \boxed{\Gamma, s = t : \phi, \phi(s/t), \Delta} & \boxed{\Gamma, \phi, \phi(s/t) : \neg(t = s), \Delta} & \boxed{\Gamma, \phi, \phi(s/t) : \neg(s = t), \Delta}\end{array}$$

Again possibly they undergo change, possibly they are eliminated. By LEMMA 3, when $\Gamma \nvdash \varnothing$ and $\varnothing \nvdash \Delta$, there must be at least one such node or one such application of a rule — and if there is no such terminal node, the construction is such that one such application of a rule must result in the introduction of an interpolant that is an identity or the negation of an identity. Moreover, as no rule increases the stock of predicates in play on either side of the colon, every predicate occurring in each such terminal node must occur in both Γ and Δ. REMARK 1 shows that such predicates must occur with the same parity in the node at the head of the tableau. Inspection of the instructions for the formation of interpolants shows that *non-logical* predicates enter interpolants *only* from terminal nodes. And as inspection of the instructions for the formation of interpolants, if any, associated with input pairs from interpolants associated with output pairs confirms, non-logical predicates occurring in the interpolant, if any, associated with an input pair occur with the same parity as they occur in the interpolant associated with the output pair or pairs, if there's just one such interpolant, or as in one or other of the interpolants associated with the output pairs, when there are two such interpolants. Consequently, every non-logical predicate which occurs in the interpolant associated with the node at the head of the tableau occurs with the same parity in some member of Γ and also in some member of Δ. Lastly, it remains the case that interpolants contain no occurrence of the conditional and negations occur only in subformulas of the form $\neg\phi$, ϕ atomic.

If we have a closed tableau for $\Gamma : \Delta$ with which *no* interpolant is associated, then, in virtue of our procedure, we have that either $\Gamma \vdash \emptyset$ or $\emptyset \vdash \Delta$ or both. $\square$

Examination of the instructions for constructing the interpolant χ from a closed tableau for $\Gamma : \Delta$ show

THEOREM 3 (Oberschelp 1968, Theorem 2). If '=' occurs with even parity in χ, then it occurs with even parity in some formula in Γ. And if '=' occurs with odd parity in χ, then it occurs with odd parity in some formula in Δ.

From this it follows immediately,

COROLLARY 2 (Oberschelp 1968, Corollary). If $\Gamma \vdash \Delta$ and '=' occurs only with odd polarity in Γ and only with even polarity in Δ then any interpolant contains no occurrence of '='.

EXAMPLE 3. An example we have encountered already, namely,

$$\boxed{Fa, \neg Fb : \neg(a = b)}$$
$$\boxed{Fa, \neg Fb, \neg Fa : \neg(a = b)}$$
$$\times$$

is a case in which we associate no interpolant with the terminal node but the node above it acquires $\neg(a = b)$ as interpolant.

EXAMPLE 4. Here are two different closed tableaux with the same initial node.

$$\boxed{a = b, Fb : \neg(a = c), Fc}$$
$$\boxed{a = b, Fb : \neg(a = c), Fc, Fa}$$
$$\boxed{a = b, Fb, Fa : \neg(a = c), Fc, Fa}$$
$$\times$$

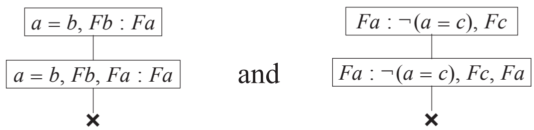
$$\boxed{a = b, Fb : \neg(a = c), Fc}$$
$$\boxed{a = b, Fb, Fa : \neg(a = c), Fc}$$
$$\boxed{a = b, Fb, Fa, Fc : \neg(a = c), Fc}$$
$$\times$$

Decorating with interpolants, we obtain

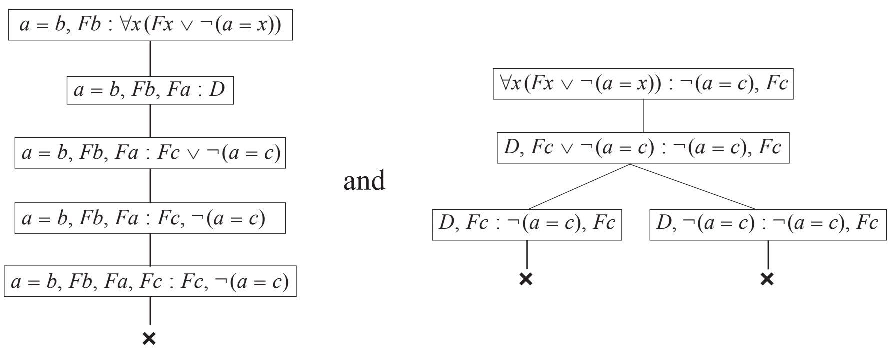
$$\boxed{a = b, Fb :{}^{Fa} \neg(a = c), Fc}$$
$$\boxed{a = b, Fb :{}^{Fa} \neg(a = c), Fc, Fa}$$
$$\boxed{a = b, Fb, Fa :{}^{Fa} \neg(a = c), Fc, Fa}$$
$$\times$$

$$\boxed{a = b, Fb :{}^{\forall x(Fx \vee \neg(a = x))} \neg(a = c), Fc}$$
$$\boxed{a = b, Fb, Fa :{}^{\forall x(Fx \vee \neg(a = x))} \neg(a = c), Fc}$$
$$\boxed{a = b, Fb, Fa, Fc :{}^{Fc} \neg(a = c), Fc}$$
$$\times$$

From these we extract these pairs of derivations, $K3$-derivations on the left, LP-derivations on the right. For the left-hand derivation:



For the right-hand derivation, where D abbreviates $\forall x(Fx \vee \neg(a = x))$:



Coupled with LEMMA 4, the syntactic Principle of Duality gives us these two principles of "classical recapture":

K3 $\Gamma \vdash_{=} \Delta$ only if (i) $\Gamma \vdash_{K3=} \varnothing$, or ($ii$) $-\Delta \vdash_{K3=} \varnothing$, or ($iii$) there is a formula χ which contains only non-logical vocabulary common to both Γ and Δ such that $\Gamma \vdash_{K3=} \chi$ and $-\Delta \vdash_{K3=} -\chi$.

LP $\Gamma \vdash_{=} \Delta$ only if (i) $\varnothing \vdash_{LP=} -\Gamma$, or ($ii$) $\varnothing \vdash_{LP=} \Delta$, or ($iii$) there is a formula χ which contains only non-logical vocabulary common to both Γ and Δ such that $-\chi \vdash_{LP=} -\Gamma$ and $\chi \vdash_{LP=} \Delta$.

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Corrections added in proof

Nodes of the form $\boxed{\Gamma, \phi, \neg\phi : \phi, \neg\phi, \Delta}$, ϕ atomic, as arise in the closed tableau for $\phi \wedge \neg\phi : \phi \vee \neg\phi$, are four ways classically, triply $K3$- and LP-, and doubly $B4$-terminal. Other nodes such as $\boxed{\Gamma, \phi, \neg\phi : \phi, \Delta}$ and $\boxed{\Gamma, \neg\phi : \phi, \neg\phi, \Delta}$ are equally open to more than one “reading”. In such cases, the rubric on p. 399 for starting the construction of interpolants at terminal nodes yields conflicting instructions. To avoid conflict, simply choose one of the available readings; nothing goes amiss in obtaining Theorems 1 & 2. The definitions of $K3$-, LP- and $B4$-tableaux in Definition 2 and the statements and proofs of Lemmas 1 & 3 need to be recast to speak of absence of appeal to readings rather than absence of nodes *tout court*. Again, nothing goes amiss.

On pp. 414-5, $\exists v(\chi[v/s] \wedge (t = v))$ should be added on the right in $\boxed{\Gamma, t=s : \chi \wedge (t=s)}$ and in all nodes appended below when obtaining a closed $K3_=$-tableau for $\Gamma, t = s : \exists v(\chi[v/s] \wedge (t = v))$. Likewise, *mutatis mutandis*, on p. 416 when obtaining a closed $LP_=$-tableau for $\forall v(\chi[v/s] \vee \neg(t = v)) : \neg(t = s), \Delta$.