Logique & Analyse **239** (2017), 355-373

PROOF ANALYSIS OF GLOBAL CONSEQUENCE

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One can distinguish not only between different modes of assertion and denial, but also between different modes of consequence. Typically, two distinct notions of logical consequence can be defined in modal logic. One is local consequence, that is preservation of truth at a world (in any world of any model). The other is global consequence, that is preservation of truth in all worlds of a model (in any model). Although global and local consequence agree on the set of theorems (i.e. consequence claims with an empty antecedent and a single formula in the succedent), they differ in general.

Most, if not all, approaches to modal logic in the sequent calculus setting are tight to local consequence. The inference rules of modal sequent calculi are sound only if each sequent is interpreted as a local consequence claim. Not so if sequents are interpreted as global consequence claims.

Avron (1991, 2003) suggested a uniform way to recover global consequence from any "local" sequent system. Also Cobreros (2011) provides a mean to construct proof systems for global consequence out of systems for local consequence. These methods, however, suffer from different limitations like being bound to single-premise consequence claims (in the case of Cobreros) or essentially relying on the cut rule (in the case of Avron).

In this paper we propose an analytic (i.e. cut-free) account of global consequence based on two ingredients: (i) the use labelled sequent calculi in the style of Negri (2005) to internalize the semantic features of modal logic at the syntactic level; (ii) the universal modality (Goranko and Passy 1992) to simulate global consequence by means of local consequence.

* Luca Tranchini's work was supported by the Deutsche Forschungsgemeinschaft as part of the project "Logical Consequence and Paradoxical Reasoning" (Tr1112/03-1), by the Deutsche Forschungsgemeinschaft and the Agence Nationale de la Recherche as part of the French-German project "Beyond Logic" (Schr275/17-1) and by the Ministerio de Economía y Competitividad, Gobierno de España as part of the project "Non-Transitive Logics" (Grant n◦: FFI2013-46451-P).

† Pablo Cobreros thanks the project "Non-Transitive Logics" funded by the Ministerio de Economía y Competitividad, Gobierno de España (Grant n◦: FFI2013-46451-P). This work was also made possible thanks to the "Visiting Scientist Program" by the Università degli Studi di Padova.

doi: 10.2143/LEA.239.0.3237159.

The present contribution to the understanding of the different modes of consequence in modal logic is thus intended as a complementation of the investigations on the different modes of assertion and denial to which the present special issue of *Logique et Analyse* is devoted.

1. Classical Valuations and the G3c calculus

Consider the propositional language $\mathcal{L}$ built from a set *AT* of *atomic formulas* and the nullary connective $\bot$ using the binary connectives $\wedge$, $\vee$ and $\supset$. We use P and Q (possibly with sub-scripts) as meta-variables for atomic formulas, A and B (possibly with sub-scripts) for formulas, and Γ and Δ (possibly with sub-scripts) for multi-set of formulas. We define negation as follows: $\neg A =_{\text{def}} A \supset \bot$.

A classical *valuation* is a function v from *AT* to $\{1, 0\}$. The *extension* $\overline{v}$ *of a valuation* v is the function from $\mathcal{L}$ to $\{1, 0\}$ satisfying the usual conditions: $\overline{v}(P) = v(P)$, $\overline{v}(\bot) = 0$, $\overline{v}(A \wedge B) = 1$ iff $\overline{v}(A) = \overline{v}(B) = 1$ etc. *Logical consequence* for $\mathcal{L}$ is defined as follows: $\Gamma \vDash_{CL} \Delta$ iff, for all extended valuations $\overline{v}$, if $\overline{v}(A){=}1$ for all $A \in \Gamma$ then $\overline{v}(B) = 1$ for at least one $B \in \Delta$.

A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ is called the *antecedent* and Δ the *succedent* of the sequent. Rooted trees whose leaves are initial sequents and whose nodes are instances of the rules in table 1 are called ***G3c**-derivations* (rules are schemes and we call their instances rule-applications). The sequents below the inference lines are called the *consequences* of the rules (resp. rule-applications) and the sequents above the inference lines are the *premises* of the rules (resp. rule-applications). In a rule(-application), the *principal* formula is the one occurring in the consequence but not in the premises, the *active* formulas are those occurring in the premises but not in the consequence. All other formulas are said to belong to the *context*. A **G3c**-derivation of $\Gamma \Rightarrow \Delta$ is a **G3c**-derivation whose root (also called the *conclusion* of the derivation) is labelled with the sequent $\Gamma \Rightarrow \Delta$. When such a derivation exists, we say that $\Gamma \Rightarrow \Delta$ is ***G3c**-derivable*, sometimes written also $\vdash_{\mathbf{G3c}} \Gamma \Rightarrow \Delta$.

The system **G3c** constitutes a sound and complete axiomatization of the above defined notion of consequence. A sequent $\Gamma \Rightarrow \Delta$ is **G3c**-derivable if and only if (henceforth iff) $\Gamma \vDash_{\text{CL}} \Delta$. We sketch the proof in the style of Schütte (1956).

To check soundness, we can observe that the rules of **G3c** faithfully reflect the way in which extended valuations assign a truth value to logically complex sentences. This can be best appreciated by reading the rules backwards (i.e. going from the consequence to the premises) as expressing conditional claims according to the following conventions: (i) Forget about contexts; (ii) Read the sentences in the antecedents of sequents as true and those in the succedents of sequents as false (iii) Read the commas and the sequent arrow conjunctively and the presence of more than one premise disjunctively. Given these conventions, the rules $L\wedge$ and $R\wedge$ read respectively as follows: If $A \wedge B$ is true then both A and B are true. If $A \wedge B$ is false, then either A is false or B is false. These are nothing but the conditions imposed on the extended valuations.

For completeness, a counter-model for $\Gamma \vDash \Delta$ can be extracted from the failed proof-search for the sequent $\Gamma \Rightarrow \Delta$. Start constructing a tree (usually called a *reduction tree*) by applying the rules of **G3c** backwards starting from its root $\Gamma \Rightarrow \Delta$. Whenever, after an arbitrary number of backwards rule-applications, the point is reached in which one of the leaves of the reduction tree is labelled by an initial sequent, do not extend that branch any further. Whenever it is no longer possible to backward-apply any rule of **G3c** to a given leaf, and the sequent labelling the leaf is not initial, then extend the branch by copying the sequent above itself (such a backward rule is usually called *repetition*).

If the reduction-tree is finite, then all its leaves are initial sequents and the reduction tree is a **G3c**-derivation of $\Gamma \Rightarrow \Delta$. Otherwise, by König's lemma the tree has at least an infinite branch. Take the list of sequents labelling any infinite branch of the reduction tree. Construct a valuation v such that $v(P) = 1$ if P is in the antecedent of one of these sequents, $v(P) = 0$ if P is in the consequent of one such sequent. It can be easily shown that v is a valuation, and that $\overline{v}(A) = 1$ for all A in Γ and $\overline{v}(B) = 0$ for all B in Δ (for details, see Takeuti 1987, Ch. 1, § 8). Hence $\Gamma \nvDash_{CL} \Delta$.

Table 1: The rules of **G3c**

Initial Sequents:

$$P, \Gamma \Rightarrow \Delta, P$$

Propositional Rules:

$$\frac{A, B, \Gamma \Rightarrow \Delta}{(A \wedge B), \Gamma \Rightarrow \Delta}\, L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, (A \wedge B)}\, R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{(A \vee B), \Gamma \Rightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, (A \vee B)}\, R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{(A \supset B), \Gamma \Rightarrow \Delta}\, L\supset \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, (A \supset B)}\, R\supset$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\, L\bot$$

Actually, in the propositional case under consideration the space of proof-search can be effectively bounded by disallowing applications of the repetition rule in the construction of the reduction tree. In this way at each backward application of a rule, the total number of logical signs in the sequents labelling the leaves of the reduction-tree decreases by one. Thus after a finite number of steps the leaves of the reduction tree which are not initial sequents will contain only atomic formulas. If the reduction tree is not a derivation of the sequent labelling its root, then this sequent is not **G3c**-derivable.

As a corollary of completeness, the structural rules of weakening, contraction and cut are admissible in **G3c**. This means that adding them to the system does not enrich the set of derivable sequents or, equivalently, it means that if the premises of an application of one of the rules are **G3c**-derivable, so is the consequence (a syntactic proof of admissibility can be given as well, see, e.g. Negri and von Plato 2001, Ch. 3, § 2):

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\,LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\,RW$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\,LC \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\,RC$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,Cut$$

A further corollary is the sub-formula property of **G3c**-derivations: all formulas occurring in a **G3c**-derivation are subformulas of the formulas in the conclusion of the derivation.

It is well-known that extending logical systems with axioms tend to destroy the admissibility of the structural rules, and in particular of *Cut*, at least when the extension of the logic by an axiom A is obtained by adding $\Rightarrow A$ as a new initial sequent. A simple example is given by Girard (1987, p. 125): if we add to **G3c** two new initial sequents of the form $\Rightarrow P \supset Q$ and $\Rightarrow P$ (expressing respectively that $P \supset Q$ and P are taken as axioms) the sequent $\Rightarrow Q$ is derivable only if we further extended the system with the *Cut* rule:

$$\frac{\Rightarrow P \supset Q \qquad \dfrac{\Rightarrow P \qquad Q \Rightarrow Q}{P \supset Q \Rightarrow Q}\,L\supset}{\Rightarrow Q}\,Cut$$

As Negri and von Plato (1998) showed, when axioms have a particular form, it is however possible to convert them into inference rules in such a way that the addition of these rules to the base **G3c** system does not disturb the admissibility of the *Cut* rule. More precisely, axioms of the following form (called *regular*):

$$P_1 \wedge \ldots \wedge P_m \supset Q_1 \vee \ldots \vee Q_n$$

where all P_is and Q_js are atomic formulas, are transformed in rules having the following form:

$$\frac{Q_1, \Gamma \Rightarrow \Delta \qquad \ldots \qquad Q_n, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}\ Reg^*$$

where the P_is are the principal formulas and the Q_js the active formulas.

As Girard's "toy" axioms are regular, they can be converted into the following two rules:

$$\frac{P, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad\qquad \frac{Q, \Gamma \Rightarrow \Delta}{P, \Gamma \Rightarrow \Delta}$$

and the derivation of $\Rightarrow Q$ without *Cut* looks as follows:

$$\frac{\dfrac{Q \Rightarrow Q}{P \Rightarrow Q}}{\Rightarrow Q}$$

The admissibility of cut allows to (partly) recover the sub-formula property: All formulas occurring in a derivation of an extension of **G3c** using rules of the above form are either sub-formulas of some formula in the conclusion or are atomic formulas.

In order for contraction to be also admissible, the atoms P_i must in general be repeated in all premises of the rule-version of the axioms. The rule thus looks as follows:

$$\frac{Q_1, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta \qquad \ldots \qquad Q_n, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}\ Reg$$

The transformation of axioms in rules also works in a first-order setting for the universal closure of regular formulas (as well as for more general classes of axioms with which we will not be concerned here). For these theories, completeness proofs can be given in the Takeuti-style sketched above.

2. Kripke frames and local vs global consequence

A *Kripke frame* is a couple $\langle W, R\rangle$, where W is a non-empty set whose elements (indicated with w, x, y, possibly with primes) are called *worlds*, and R a binary relation on W called *accessibility*. A *Kripke model* over a frame $\langle W, R\rangle$ is a triple $\langle W, R, I\rangle$, where I is a function from pairs $\langle w, P\rangle$ ($w \in W$ and $P \in AT$) to $\{0, 1\}$.

Let $\mathcal{L}^{\Box}$ be the extension of $\mathcal{L}$ by means of a new unary connective $\Box$. The relation of *forcing* $\Vdash$ between worlds and formulas of $\mathcal{L}^{\Box}$ (in a model $\langle W, R, I\rangle$) is defined as follows:

- $w \Vdash P$ iff $I(w, P) = 1$;
- for no w, $w \Vdash \bot$;
- $w \Vdash (A \wedge B)$ iff $w \Vdash A$ and $w \Vdash B$; (analogous clauses for $\vee$ and $\supset$)
- $w \Vdash \Box A$ iff $w' \Vdash A$ for all w' such that wRw'.

A formula A is *true in a Kripke model* $\langle W, R, I\rangle$ iff $w \Vdash A$ for all $w \in W$; it is *true in a Kripke frame* $\langle W, R\rangle$ iff it is true in all models over $\langle W, R\rangle$; and it is *true in a class of frames* $\mathbb{C}$ iff it is true in all frames $\langle W, R\rangle \in \mathbb{C}$.

There are at least two ways of generalizing to the modal setting the notion of logical consequence of the classical propositional setting. With the terminology of Blackburn et al. (2001, Ch. 1, §5) we speak of *local consequence* $\vDash_L$ and of *global consequence* $\vDash_G$ (see also Fitting and Mendelsohn 1998, Ch. 1, §9):

- $\Gamma \vDash_L \Delta$ relative to $\mathbb{C}$ iff for any model $\langle W, R, I\rangle$ over a frame $\langle W, R\rangle \in \mathbb{C}$, for all $w \in W$, if $w \Vdash A$ for all $A \in \Gamma$ then $w \Vdash B$ for some $B \in \Delta$;
- $\Gamma \vDash_G \Delta$ relative to $\mathbb{C}$ iff for any model $\langle W, R, I\rangle$ over a frame $\langle W, R\rangle \in \mathbb{C}$, if for all $w \in W$ $w \Vdash A$ for all $A \in \Gamma$, then for some $B \in \Delta$ for all $w \in W$ $w \Vdash B$.

It easy to check that both notions of consequence satisfy Tarski's conditions, that is both $\vDash_L$ and $\vDash_G$ are reflexive, transitive[1] and monotone relationships between multi-sets of formulas. The notion of truth in a model is sometimes called validity (in a model) and the notion of forcing (of A at w) is sometimes called truth of A in w. For this reason global and local consequence are also called consequence as *validity-preservation* and consequence as *truth-preservation* respectively.[2]

[1] By 'transitivity' we mean that if $\Gamma \vDash \Delta, A$ and $A, \Sigma \vDash \Theta$ then $\Gamma, \Sigma \vDash \Delta, \Theta$ (and *not* that if $\Gamma \vDash \Delta$ and $\Delta \vDash \Sigma$ then $\Gamma \vDash \Sigma$).

[2] It is worth stressing the importance of the respective positions of the quantification over worlds and formulas in the definition of global consequence. Consider these two:

(a) If $\forall w\ \forall A \in \Gamma \quad w \Vdash A$ then $\forall w\ \exists B \in \Delta \quad w \Vdash B$
(b) If $\forall A \in \Gamma\ \forall w \quad w \Vdash A$ then $\exists B \in \Delta\ \forall w \quad w \Vdash B$

Consider now their contrapositives:

(a*) If $\exists w\ \forall B \in \Delta \quad w \nVdash B$ then $\exists w\ \exists A \in \Gamma \quad w \nVdash A$
(b*) If $\forall B \in \Delta\ \exists w \quad w \nVdash B$ then $\exists A \in \Gamma\ \exists w \quad w \nVdash A$

Observe that (a*) says that if there is a world at which all Bs are false, then there is a world at which some A's are false. But this is not global consequence, since in order for all conclu-

When $\Gamma = \emptyset$ and $\Delta = A$, both notions coincide with the notion of truth in the class of frames $\mathbb{C}$. However, when Γ is non-empty or Δ contains more than one formula, they differ in a significant manner.

Relative to the class of all frames, which we will henceforth call $\mathbb{K}$, we have for instance that $A \nvDash_L \Box A$ and $A \vDash_G \Box A$, although both $\nvDash_L A \supset \Box A$ and $\nvDash_G A \supset \Box A$. As exhaustively discussed in Hakli and Negri (2012) this mismatch is the reason of the disagreement about whether the (semantic counterpart) of the deduction theorem holds in modal logic: Whereas $\Gamma, A \vDash_L B, \Delta$ iff $\Gamma \vDash_L A \supset B, \Delta$, the analogous bi-conditional does not hold for global consequence (i.e. it is *not* the case that $\Gamma, A \vDash_G B, \Delta$ iff $\Gamma \vDash_G A \supset B, \Delta$).

This situation suggests that we have to take a certain care if we want to construct sequent calculi for modal logic. Were a sequent $\Gamma \Rightarrow \Delta$ expected to express a global consequence claim, at least some of the rules for the calculus **G3c** should be rejected. In particular, the rule $R\supset$ would permit to pass over from correct global consequence claims to incorrect ones. To be able to obtain sequent calculi for modal logics as extensions of **G3c**, it seems therefore necessary to interpret sequents as expressing local consequence claims.

The mismatch between local and global consequence is even more striking in a multiple conclusion setting. For any atomic formula P, we have that $\vDash_L P, \neg P$ and $\nvDash_G P, \neg P$, although both $\vDash_L P \vee \neg P$ and $\vDash_G P \vee \neg P$. Since global consequence is transitive, $P \vee \neg P \nvDash_G P, \neg P$. However, by reflexivity and monotonicity of $\vDash_G$, both $P \vDash_G P, \neg P$ and $\neg P \vDash_G P, \neg P$. Thus not only $R\supset$, but also $L\vee$ would be unsound if sequents were interpreted as global consequence claims.[3]

In spite of this, Avron (e.g. 1991) has suggested a uniform way to obtain a proof-theoretic account of global consequence from one of local consequence (he refers to the two notions of consequence as *external* and *internal* respectively). Suppose a sequent calculus characterizing the notion of local consequence relative to a certain class of frames $\mathbb{C}$ is given (by this we mean that the sequent $\Gamma \Rightarrow \Delta$ is derivable in the calculus iff $\Gamma \vDash_L \Delta$ relative to $\mathbb{C}$). Avron first introduces the notion of *derivation from assumptions*, by allowing the leaves of a derivation to be arbitrary sequents and not just initial sequents. He then observes that $\Gamma \vDash_G B$ relative to the class of frames in questions iff the sequent $\Rightarrow B$ is derivable *from* all sequents A for all $\Rightarrow A \in \Gamma$ (see also Avron 1988, Troelstra 1992, §7.9, for a discussion of internal and external consequence in the context of linear logic).

sions to be not true in a model, they have all to be false at some world but not necessarily at the same. Clearly, it's (b*) and thus (b) what captures global consequence.

[3] For this reason, global consequence is sometimes said to be non-truth-functional, see Cobreros and Tranchini (2014) for a discussion of this claim in the context of supervaluationism.

Although Avron considers global consequence claims with only one sentence in the succedent, his account can be straightforwardly generalized to the multiple-succedent case: $\Gamma \vDash_G \Delta$ relative to $\mathbb{C}$ iff for some $B \in \Delta$ the sequent $\Rightarrow B$ is derivable from all sequents $\Rightarrow A$ (for $A \in \Gamma$) in a calculus which is sound and complete with respect to $\vDash_L$ relative to $\mathbb{C}$.

In this way, from any account of local consequence we can recover one of global consequence. The notion of derivability from assumptions suffers however of the same problem of the notion of derivability in axiomatic extensions of purely logical calculi mentioned in the section 1: The failure of cut-admissibility. In particular, a complete account of global consequence can be attained only if the base calculus is equipped with the *Cut* rule.

It is true that, in particular cases, one can show that only restricted forms of cut are needed.[4] However, it is far from obvious that significant constraints on the application of the *Cut* rule can always be found.

In a different context, Cobreros (2011) shows how to obtain a proof system for global consequence out of a proof system for local consequence. His strategy consists in adding a rule reflecting the inference $A \vDash_G \Box A$ that, as we mentioned above, is globally but not locally valid. Cobreros shows that the addition of such a rule to a proof system for local validity renders a system complete for global validity. Now the problem is, as previously remarked, that some rules like $R\supset$ are not sound when sequents are interpreted as global consequence statement. Cobreros overcomes this by restricting those rules. But his strategy, once again, makes essential use of the cut rule. In addition, Cobreros' strategy is formulated for single-conclusion arguments and it has no straightforward generalization to the multiple-conclusion case. In order to obtain a cut-free analysis of global consequence we will introduce in the next section labelled sequent calculi in the style of Negri (2005) which will be then used to provide an account of global consequence overcoming the weakness of Avron's and Cobreros' approaches.

3. Labelled sequent calculi and the universal modality

In section 1, we observed how the sequent calculus **G3c** fully internalizes the semantics of the classical propositional language $\mathcal{L}$. In order to be able to fully internalize the more complex semantics of the language $\mathcal{L}^\Box$, we need to add some structure to the sequents. One way of doing this is by

[4] For instance, Avron (see 2003, § 7.3) uses Ohnishi and Matsumoto's (1957) sequent calculus for (the local notion of consequence of) S5 to obtain, using the notion of derivation from assumptions along the lines suggested above, a complete account of global consequence using only analytic cuts, i.e. applications of the *Cut* rule in which the cut-formula occurs either in the conclusions or in one of the assumptions of the derivation.

replacing formulas with labelled formulas. A *labelled formula* is an expression of the form $x : A$, where x belong to a set of variables and A is a formula of $\mathcal{L}^{\Box}$. Sequents will be now taken to be expressions of the form $\Gamma \Rightarrow \Delta$ with Γ and Δ multi-sets of labelled formulas and of *relational atoms*, which are additional atomic formulas of the form xRy.

Henceforth, we will use Γ and Δ in antecedents and succedents of sequents for multi-sets of labelled formulas and relational atoms (and keep use them in local and global consequence claims for multi-sets of formulas). Sometimes we use $x : \Gamma$ for $\{x : A \mid A \in \Gamma\}$ and $\Box\Gamma$ for $\{\Box\, A \mid A \in \Gamma\}$ (in such cases Γ is assumed to be a multi-set of formulas).

A sequent calculus which is sound and complete with respect to local consequence relative to the class of frames $\mathbb{K}$ is obtained by first decorating the rules of **G3c** by labelling principal and active formulas with the same label. The bottom-up reading of the rules follows the same conventions described in the section 1 with the following difference: in **G3c** a formula A on the left (right) of the sequent arrow reads 'A is true (false)'; now a labelled formula x: A on the left (right) reads 'A is (not) forced at x'. Again the rules faithfully reflect the semantic clauses.

This holds true also for the following rules for $\Box$, which encode the corresponding clause of the forcing relation:

$$\frac{y : A, xRy, \Gamma \Rightarrow \Delta}{x : \Box A, xRy, \Gamma \Rightarrow \Delta}\, L\Box^{*} \qquad\qquad \frac{xRy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\, R\Box$$

The left rule says that if $\Box A$ is forced at x and y is accessible from x, then A is forced at y; the right rule says that if $\Box A$ is not forced at x, then y is accessible from x and A is not forced at y. To fully grasp the semantic clause, the rules should capture the idea that the condition expressed by the left rule holds for all y, while the condition expressed by right rule is to be understood as satisfied if there is at least one relevant y. This is done by requiring in the right rule that y does not occur in the conclusion of the rule. In order to show the admissibility of contraction, $L\Box^{*}$ must be modified by repeating the formula $x : \Box A$ in the premise.

Finally, additional initial sequents for relational atoms are added, though these do not modify the set of derivable sequents containing only labelled formulas.

We call the resulting calculus **G3K** (its rules are given in table 2). The sequent calculus **G3K** is sound and complete for local consequence relative the class of all frames $\mathbb{K}$ in the following sense: $\Gamma \vDash_L \Delta$ relative to $\mathbb{K}$ iff the sequent $x : \Gamma\; x : \Delta$ is **G3K**-derivable. As in the case of **G3c**, completeness is established by extracting counter-models from the reduction tree generated by backwards applying the rules of the calculus (and the repetition rule). All structural rules are admissible (see, for details, Negri 2005).

Particular classes of frames are characterized by different properties of the accessibility relation R. In the cases of the classes of reflexive frames $\mathbb{T}$, of symmetric frames $\mathbb{B}$, or of transitive frames $\mathbb{4}$, these properties can be expressed as first-order formulas which are the universal closure of regular formulas: $\forall x(xRx)$; $\forall xy(xRy \supset yRx)$; $\forall xyz(xRy \wedge yRz \supset xRz)$.

Table 2: Negri's (2005) rules of **G3K**

Initial Sequents:

$$x : P, \Gamma \Rightarrow \Delta, x : P \qquad\qquad xRy, \Gamma \Rightarrow \Delta, xRy$$

Propositional Rules:

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : (A \wedge B), \Gamma \Rightarrow \Delta}\, L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : (A \wedge B)}\, R\wedge$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \qquad x : B, \Gamma \Rightarrow \Delta}{x : (A \vee B), \Gamma \Rightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : (A \vee B)}\, R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \qquad x : B, \Gamma \Rightarrow \Delta}{x : (A \supset B), \Gamma \Rightarrow \Delta}\, L\supset \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : (A \supset B)}\, R\supset$$

$$\frac{}{x : \bot, \Gamma \Rightarrow \Delta}\, L\bot$$

Modal Rules:

$$\frac{y : A,\, x : \Box A,\, xRy,\, \Gamma \Rightarrow \Delta}{x : \Box A,\, xRy,\, \Gamma \Rightarrow \Delta}\, L\Box \qquad \frac{xRy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\, R\Box$$

[where P is an atomic formula, and y does not occur in the conclusion of $R\Box$]

Using the transformation of axioms in rules mentioned in section 1, they can be converted into rules whose addition to the calculus **G3K** does not disturb the admissibility of structural rules:

$$\frac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, \mathit{Ref} \qquad \frac{yRx, xRy, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\, \mathit{Sym} \qquad \frac{xRz, xRy, yRz, \Gamma \Rightarrow \Delta}{xRy, yRz, \Gamma \Rightarrow \Delta}\, \mathit{Trans}$$

With the help of these rules, different extensions of **G3K** are obtained which are sound and complete with respect to the notion of (local) consequence relative to the different classes of frames. For instance, the system **G3S5** for the class of frames $\mathbb{T}\cap\mathbb{B}\cap\mathbb{4}$ in which R is an equivalence relation is obtained by adding all of *Ref*, *Sym* and *Trans* to **G3K**; the system **G3S4** by adding to **G3K** only *Ref* and *Trans*.

To give an example of how the rules governing the accessibility relation interact with the modal rules, we show how in **G3S4** it is possible to derive the axiom *4*. Observe that, when constructing the derivation from its root, the point is reached where, without the rule *Trans* expressing the transitivity of R, it would not be possible to apply $L\Box$ so as to reach an initial sequent:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{xRz, xRy, yRz, z : A \Rightarrow z : A}{xRz, xRy, yRz, x : \Box A \Rightarrow z : A}\,L\Box}{xRy, yRz, x : \Box A \Rightarrow z : A}\,Trans}{xRy, x : \Box A \Rightarrow y : \Box A}\,R\Box}{x : \Box A \Rightarrow x : \Box\Box A}\,R\Box}{\Rightarrow x : \Box A \supset \Box\Box A}\,R\supset$$

Following the same pattern, we introduce the extension of **G3K** capturing the notion of local consequence relative to the class of frames $\mathbb{U}$, whose accessibility relation is the universal relation (Goranko and Passy 1992). The axiom characterizing the universal accessibility relation is $\forall xy(xRy)$. This can be converted into the following rule:

$$\dfrac{xRy, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,Univ$$

We call the resulting system **G3U**. When R is the universal relation, we will sometimes use U in place of R.

What is typical of the class of universal frames $\mathbb{U}$ is that the Kripke models defined over these frames can be given in a simplified fashion without mentioning the accessibility relation at all.

More precisely we introduce the language $\mathcal{L}^{\boxed{u}}$ obtained by extending $\mathcal{L}$ with a sign for a universal modality $\boxed{u}$. We call $()^u$ the translation of $\mathcal{L}^{\Box}$ into $\mathcal{L}^{\boxed{u}}$ which is homophonic up to the clause stating that $(\Box A)^u = \boxed{u}A^u$. We use Γ^u for $\{A^u | A \in \Gamma\}$.

If $\langle W, R, I\rangle$ is a Kripke model for $\mathcal{L}^{\Box}$ we call $\langle W, I\rangle$ a u-model for $\mathcal{L}^{\boxed{u}}$. The notion of truth in a u-model and of local$_u$ and global$_u$ consequence (notation $\vDash^u_L$, $\vDash^u_G$) are defined in the same way as truth in a Kripke model, local and global consequence respectively, using the notion of u-forcing in place of forcing. The notion of u-forcing in a u-model, $\Vdash^u$, is obtained by replacing in the definition of forcing all occurrences of $\Vdash$ with occurrences of $\Vdash^u$ and by replacing the clause for $\Box$ with the following:

- $w \Vdash^u \boxed{u}A$ iff $w' \Vdash^u A$ for all $w' \in W$.

Any Kripke model based on a universal Kripke frame $\langle W, U, I\rangle$ is equivalent to the u-model $\langle W, I\rangle$ in the following sense: $w \Vdash A$ in $\langle W, U, I\rangle$ iff $w \Vdash^u A^u$ in $\langle W, I\rangle$. Thus, $\Gamma \vDash_L \Delta$ relative to $\mathbb{U}$ iff $\Gamma^u \vDash^u_L \Delta^u$ and $\Gamma \vDash_G \Delta$ relative to $\mathbb{U}$ iff $\Gamma^u \vDash^u_G \Delta^u$.

At the syntactic level, the sequent calculus **G3U** can be simplified as well, since the rule *Univ* makes redundant the presence of relational atoms in the modal rules. As the derivation of the axiom 4 given above shows, when we search (i.e. construct from the bottom) a derivation for a sequent, the presence of relational atoms in the rules for $\Box$ may restrict the possibility of applying the rules. The rule *Univ* permits to generate any relational atom at wish. Therefore it makes pointless the presence of relational atoms in the rules for the modality $\Box$. A sequent calculus equivalent to **G3U**, to be called **G3u**, can be obtained by replacing in **G3K** the rules for $\Box$ with the following rules for $\boxed{u}$:

$$\frac{y : A,\, x : \boxed{u}A,\, \Gamma \Rightarrow \Delta}{x : \boxed{u}A,\, \Gamma \Rightarrow \Delta}\; L\boxed{u} \qquad\qquad \frac{\Gamma \Rightarrow \Delta,\, y : A}{\Gamma \Rightarrow \Delta,\, x : \boxed{u}A}\; R\boxed{u}$$

[In $R\boxed{u}$, y does not occur in the conclusion of the rule]

and by dropping the initial sequents for the accessibility relation. Completeness and decidability in the style of Takeuti can be given for **G3u** by slightly modifying the one for **G3c**. For decidability, the only complication is due to the fact that the formula $x : \boxed{u}A$ occurs both in the premise and in the conclusion of $L\boxed{u}$. Thus backwards applications of this rule do not reduce the number of logical constants in the sequents labelling the leaves of the reduction tree. However, no more than one application of the rule $L\boxed{u}$ with the same principal formula is needed in the proof-search for a given sequent (this can be established along the lines of Negri 2005, corollary 6.5). Thus by opportunely modifying the closing condition in the construction on the reduction tree, proof-search can be shown to terminate always.

Whereas in **G3U** it is possible to derive sequents containing relational atoms of the form xRy, this is obviously not possible in **G3u**. However, if we use $\widetilde{\Gamma}$ for $\{x : A \mid x : A \in \Gamma\}$, we can state the equivalence between **G3U** and **G3u** as follows:

$$\vdash_{\mathbf{G3U}} \Gamma \Rightarrow \Delta \text{ iff } \vdash_{\mathbf{G3u}} \widetilde{\Gamma}^{u} \Rightarrow \Delta^{u}$$

provided Δ contains no relational atoms. The proof is by straightforward induction on the number of inference rules applied in a **G3U**-derivation of $\Gamma \Rightarrow \Delta$.

We conclude the section with a final remark. In a mono-modal setting such as the one so far described, the universal modality is equivalent to an S5-modality. That is, the systems **G3U**, **G3u** and **G3S5** are equivalent (with a *caveat* on the derivability of sequents containing occurrences of relational atoms in the different systems). In fact, the system **G3u** is, up to minor differences, the same as Mint's (1992, p. 45) tableaux-like calculus *GS*5.

From a semantic perspective, although all $\mathbb{U}$ frames are also $\mathbb{T}\cap\mathbb{B}\cap\mathbb{4}$ frames (and therefore all $\mathbb{U}$ models are $\mathbb{T}\cap\mathbb{B}\cap\mathbb{4}$ models), the converse does not hold, as a $\mathbb{T}\cap\mathbb{B}\cap\mathbb{4}$ frame may be constituted by disconnected equivalence classes of worlds. However, by an application of the generated submodel theorem (see, e.g. Blackburn et al. 2001, Proposition 2.6), one can always obtain a countermodel for a consequence claim relative to U from a countermodel relative to $\mathbb{T}\cap\mathbb{B}\cap\mathbb{4}$ by opportunely "chopping" all disconnected equivalence classes of worlds but one from the S5 countermodel (see e.g. Priest 2008, Theorem 3.7.5).

On the other hand, in a bi-modal setting (and hence, *a fortiori*, in a multi-modal setting) a universal modality and an S5 modality differ.

A bi-modal language $\mathcal{L}^{\Box\Box'}$, is an extension of $\mathcal{L}^{\Box}$ with a further modal connectives $\Box'$. A bi-modal Kripke frame is an triple $\langle W, R, R'\rangle$ where W is a non-empty set and R and R' are relations on W. A bi-modal Kripke model over a frame $\langle W, R, R'\rangle$ is a quadruple $\langle W, R, R', I\rangle$, where I is a function from from pairs $\langle w, P\rangle$ ($w \in W$ and $P \in AT$) to $\{0, 1\}$. The definition of forcing between world and formulas (in a bi-modal Kripke model) is obtained from the definition of forcing in standard Kripke models by duplicating the clause for $\Box$ and replacing in one of the two copies of it occurrences of $\Box$ and R with occurrences of $\Box'$ and R'. The definition of local and global consequence are obtained from the previous ones by replacing Kripke models and frames with bi-modal Kripke models and frames.

Clearly, relative to the class of all bi-modal frames in which $R' = U$ we have that both $\neg p \vDash_L \Box\neg \boxed{u}p$ and $\neg p \vDash_G \Box\neg \boxed{u}p$ (and hence both $\vDash_L \neg p \supset \Box\neg \boxed{u}p$ and $\vDash_G \neg p \supset \Box\neg \boxed{u}p$). In the following bi-modal frame R' is an equivalence relation and thus a modality $\Box'$ associated to R' would be an S5 modality:

$$\begin{tikzcd} w_0 \arrow[loop above, "R'"] \arrow[r, "R"] & w_1 \arrow[loop above, "R'"] \end{tikzcd}$$

However, $R' \neq U$. Let's consider a model on this frame such that $w_0 \nVdash p$ and $w_1 \Vdash p$. In this model we have that $w_0 \Vdash \neg p$ but $w_0 \nvDash \Box\neg\Box' p$. Thus, relative to the class of all bi-modal frames in which R' is an equivalence relation $\neg p \nvDash_L \Box\neg\Box' p$ (and hence $\nvDash_L \neg p \supset \Box\neg\Box' p$ and $\nvDash_G \neg p \supset \Box\neg\Box' p$).

In the next section we will consider the universal modality in a bi-modal setting. Although this kind of examples won't show up (since we will never embed the universal modality in the scope of another modal operator), we wished to stress the universal (rather than the S5) character of the modality $\boxed{u}$ for the sake of conceptual clarity.

4. Proof analysis of global consequence

In this section, we develop a proof-theoretic analysis of global consequence. In section 4.1, we reduce global consequence claims relative to the class of frames $\mathbb{C}$ in the language $\mathcal{L}^{\Box}$ to local consequence claims relative to $\mathbb{C}$ in the bi-modal language $\mathcal{L}^{\Box\boxed{u}}$ obtained by extending $\mathcal{L}^{\Box}$ with the universal modality. In section 4.2 we show that the analysis does not actually hinges on the availability in the object language of a sign for a universal modality and we modify the previous analysis so to avoid to resort to the extended language.

4.1. *Global consequence in a bi-modal setting*

Labelled sequent calculi for classes of bi-modal frames can be obtained by adding to **G3K**:

- a copy of the **G3K** rules for $\Box$ in which occurrences of $\Box$ and R are replaced (respectively) by occurrences of $\Box'$ and R';
- initial sequents of the form $xR'y, \Gamma \Rightarrow \Delta, xR'y$;
- rules governing R and R'.

As a first attempt to characterize global consequence we consider the class $\mathbb{KU}$ of bi-modal frames in which R is an arbitrary relation and R' is the universal relation U, thus writing $\boxed{u}$ for $\Box'$. The sequent calculus **G3KU** for this class of bi-modal frames is obtained by adding to **G3K** the following: (i) initial sequents for U; (ii) rules for $\boxed{u}$, which are just copies of the rules for $\Box$, with $\boxed{u}$ in place of $\Box$ and U in place of R; (iii) the rule *Univ* to express the universality of the relation U.

As in the mono-modal case, the semantic and the syntactic presentations of a bi-modal setting in which one of the two modalities is universal can be simplified. Semantically, we consider just mono-modal Kripke frames (i.e. frames with only one accessibiltiy relation R) and encode the relation U in the clause of forcing for $\boxed{u}$ as in the previous section. Syntactically, we can simply add to **G3K** the rules for $\boxed{u}$ of **G3u**. We call the resulting system **G3Ku**.

In this bi-modal setting we can account for global consequence claims relative to the class of all frames $\mathbb{K}$ formulated in the language $\mathcal{L}^{\Box}$. From the definitions of global and local consequence, one clearly has that (Goranko and Passy 1992, Proposition 2.1, see also Cobreros 2008, p. 298):

$$\Gamma \vDash_G \Delta \text{ relative to } \mathbb{K} \text{ iff } \boxed{u}\Gamma \vDash_L \boxed{u}\Delta \text{ relative to } \mathbb{KU}$$

Thus, to check whether $\Gamma \vDash_G \Delta$ relative to $\mathbb{K}$, one has to check whether the sequent $\vdash_{\mathbf{G3KU}} x : \boxed{u}\Gamma \Rightarrow x : \boxed{u}\Delta$.

To give concrete examples, we can reconsider the two claims $A \vDash_G \Box A$ and $\nvDash_G P, \neg P$. That the first holds means that we can derive in **G3Ku** the sequent $x : \boxed{u}A \Rightarrow x : \boxed{u}\Box A$:

$$\dfrac{\dfrac{\dfrac{yRz, z : A, x : \boxed{u}A \Rightarrow z : A}{yRz, x : \boxed{u}A \Rightarrow z : A}\,L\boxed{u}}{x : \boxed{u}A \Rightarrow y : \Box A}\,R\Box}{x : \boxed{u}A \Rightarrow x : \boxed{u}\Box A}\,R\boxed{u}$$

The second claim does not hold, which means that we cannot derive in **G3Ku** the sequent $\Rightarrow x\colon \boxed{u}P,\ x\colon \boxed{u}\neg P$. In fact, in the bottom-up search for a derivation of this sequent we arrive at the point in which the condition on the labels in the $R\boxed{u}$ rule makes it impossible to reach an initial sequent, due to a mismatch between the labels:

$$\dfrac{\dfrac{\dfrac{y : P \Rightarrow z : P,\ y : \bot}{y : P \Rightarrow x : \boxed{u}\,P,\ y : \bot}R\boxed{u}}{\Rightarrow x : \boxed{u}\,P,\ y : \neg P}R\supset}{\Rightarrow x : \boxed{u}\,P,\ x : \boxed{u}\neg P}R\boxed{u}$$

This way of proceedings works not only for global consequence claims which are correct relative to the class of all frames $\mathbb{K}$, but also for the global consequence claims which are correct relative to any class of frames for which labelled calculi in the style of Negri can be developed.

For example, we could apply the same analysis to the class of global consequence claims which are correct relative to the class of reflexive and transitive frames $\mathbb{T} \cap \mathbb{4}$. To do this, we should consider the class of bi-modal frames with a reflexive and transitive relation R and the universal relation U. The sequent calculus **G3S4u** for this class of frames is obtained by adding the rules for $\boxed{u}$ to **G3S4**. A global consequence claim $\Gamma \vDash_G \Delta$ is correct relative to $\mathbb{T} \cap \mathbb{4}$ iff the corresponding sequent $x : \boxed{u}\Gamma \Rightarrow x : \boxed{u}\Delta$ is derivable in **G3S4u**.

The same holds true of the global consequence claims which are correct relative to the class of universal frames $\mathbb{U}$, though in this case there is no need of extending the language with an extra universal modality. To test whether a global consequence claim $\Gamma \vDash_G \Delta$ holds relative to $\mathbb{U}$, just add a (universal) box in front of all Γs and Δs and check whether the sequent $x : \boxed{u}\Gamma \Rightarrow x : \boxed{u}\Delta$ is derivable in **G3u**.

4.2. *Suppressing the universal modality*

The adoption of the universal modality allows to give a straightforward analysis of global consequence claims. However, it induces a mismatch between the language whose global consequence claims we want to analyse i.e. $\mathcal{L}^{\Box}$, and the language in which the analysis is carried out, i.e. $\mathcal{L}^{\Box\boxed{u}}$ (apart in the case in which the modality of the language under investigation is itself universal).

This mismatch is however not necessary. The universal modality makes only explicit the quantification on all worlds which is used in the definition of global consequence. Therefore, it does not seem essential to the analysis of global consequence which was given in the previous subsection.

Let's sum up the conclusions of the previous section.

The local consequence claim $\Gamma \vDash_L \Delta$ holds relative to the class of all frames $\mathbb{K}$ iff in all models over all frames if all the Γs are forced at a world w so is at least one of the Δs. The correctness of a local consequence claim $\Gamma \vDash_L \Delta$ over the set of all frames corresponds to the **G3K** derivability of $x : \Gamma \Rightarrow x : \Delta$. Hence, the label x in the sequents points at the fact that we are checking whether all sentences in Γ and Δ are forced at the very same point (the point is arbitrary and this gives the desired result).

A global consequence claim $\Gamma \vDash_G \Delta$ holds iff in all models over all frames if all the Γs are true in the model so is at least one of the Δs. The correctness of a global consequence claim $\Gamma \vDash_G \Delta$ corresponds to the derivability of the sequent $x : \boxed{u}\Gamma \Rightarrow x : \boxed{u}\Delta$ in **G3Ku**.

The sequent to be derived contains, as it were, redundant information. Each formula A composing the global consequent claim enters the sequent dressed up with a modality and a label. However, the label and the modality cancel each other out. The label says "Look for whether $\boxed{u}A$ is forced at the world x", whereas the modality in front of A says "Check whether A is forced not only at x, but at all worlds in the model".

How to overcome this roundabout way of using labelled calculi to check global consequence claims? One possible solution is to recast the rules for $\boxed{u}$ — which tell how to add the universal modality in front of a sentence in the context of a label — as telling how to *erase* labels from sentences. Contrary to a labelled formula $x : A$, which expresses the fact that A is forced at x, a de-labelled formula A, like a labelled $\boxed{u}$-formula $x : \boxed{u}A$ in **G3Ku**, expresses the truth of A in the model.

In place of adding the universal modality to **G3K**, we extend **G3K** by allowing not only labelled formulas and relational atoms to appear in sequents, but also de-labelled formulas (Γ and Δ when occurring in sequents are thus now understood as multi-set of relational atoms, labelled and delabelled formulas) and we replace the rules for $\boxed{u}$ with the following rules:

$$\frac{y : A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ L\textsf{label} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, A}\ R\textsf{label}$$

[In Rlabel, y does not occur in the conclusion of the rule]

We call the resulting system **G3K***.

The adoption of **G3K*** allows to overcome the mismatch between languages resulting by the adoption of **G3Ku** which was observed at the beginning of this subsection: to check whether the global consequence $\Gamma \vDash_G \Delta$ claim holds relative to $\mathbb{K}$ we have to check whether the sequent $\Gamma \Rightarrow \Delta$ (where Γ and Δ just contain de-labelled formulas) is derivable in **G3K***.

In the above example we showed how to check in **G3Ku** that the global claim $A \vDash_G \Box A$ is correct. In **G3K*** we check the correctness of the claim in the following way:

$$\dfrac{\dfrac{\dfrac{z : A, A \Rightarrow z : A}{yRz, A \Rightarrow z : A}\,L\text{label}}{A \Rightarrow y : \Box A}\,L\Box}{A \Rightarrow \Box A}\,R\text{label}$$

5. Concluding remarks

As tentative conclusions we can say the following: using labelled calculi we can provide a straightforward analysis not only of local consequence but also of global consequence for a wide variety of modal logics.

Although the mismatch between the language in which the global consequence claims are formulated and that of the calculus in which their proof-theoretic analysis is given is resolved, there may be still some room for dissatisfaction. Namely, in the proof-system **G3K*** the rules governing the logical constants are purely local, and the global steps are used only to "lift" formulas up to the global level. Hence, while the semantic notion of global consequence is defined independently of local consequence, the proof-theoretic account of global consequence is heavily dependent on the one of local consequence.

This is certainly true. However, we do not think that it is a reason for denying the goodness of the analysis. The rules for the connectives just reflect the inductive definition of the notion of forcing. In a local consequence claim $\Gamma \vDash_L \Delta$, what is "transmitted" from the Γ to Δ is exactly the notion of being forced at a world. Therefore, to give a proof-theoretic account of local consequence there is no need for extra syntactic devices beyond the labelling of sentences. It is sufficient to observe that the correctness of $\Gamma \vDash_L \Delta$ relative to a class of frames $\mathbb{C}$ corresponds to the derivability of $x : \Gamma \Rightarrow x : \Delta$ in the extension of **G3K** containing the rules encodings the properties of the frames in $\mathbb{C}$.

In global consequence claims $\Gamma \vDash_G \Delta$, what is "transmitted" from Γ to Δ is the notion of being forced at all worlds. Since the rules for the connectives reflect what goes on at a single world, to give a proof-theoretic account of this notion we need some way to lift the focus from a single world to the totality of worlds. This can be either done by explicitly introducing the universal modality $\boxed{u}$ or by the structural device of de-labelling formulas.

The semantic notion of global consequence is defined by quantifying over the worlds at which the formulas composing the consequence claim are forced. This interaction between the global notion of consequence and the local notion of forcing is reflected at the proof-theoretic level, in that the rules for de-labelling sentences just record the meta-linguistic quantification over all worlds used in the semantic definition.

One cannot therefore accuse the proof-theoretic analysis of making global consequence depend on local consequence, since this dependency is already there at the semantic level, where the notion of global consequence is defined in the first place.

References

[1] Avron, A. (1988), The semantics and proof theory of linear logic, *Theoretical Computer Science* **57**(2), 161–184.

[2] Avron, A. (1991), Simple consequence relations, *Information and Computation* **92**(1), 105–139.

[3] Avron, A. (2003), Classical Gentzen-type methods in propositional many-valued logics, *in* M. Fitting and E. Orlowska (eds), *Beyond Two: Theory and Applications of Multiple-Valued Logic*, Vol. 114 of *Studies in Fuzziness and Soft Computing*, Physica-Verlag HD, pp. 117–155.

[4] Blackburn, P., de Rijke, M. and Venema, Y. (2001), *Modal Logic*, number 53 in *Cambridge Tracts in Theoretical Computer Science*, Cambridge University Press.

[5] Cobreros, P. (2008), Supervaluationism and logical consequence: A third way, *Studia Logica* **90**(3), 291–312.

[6] Cobreros, P. (2011), Varzi on supervaluationism and logical consequence, *Mind* **120**(479), 833–43.

[7] Cobreros, P. and Tranchini, L. (2014), Supervaluationism: Truth, value and degree functionality, *Thought: A Journal of Philosophy* **3**(2), 132–144.

[8] Fitting, M. and Mendelsohn, R. L. (1998), *First-order modal logic*, number 277 in *Synthese Library*, Kluwer.

[9] Girard, J.-Y. (1987), *Proof Theory and Logical Complexity*, Vol. 1, Bibliopolis, Napoli.

[10] Goranko, V. and Passy, S. (1992), Using the universal modality: Gains and questions, *Journal of Logic and Computation* **2**(1), 5–30.

[11] Hakli, R. and Negri, S. (2012), Does the deduction theorem fail for modal logic?, *Synthese* **187**(3), 849–867.

[12] Mints, G. E. (1992), *A Short Introduction to Modal Logic*, CSLI Lecture Notes, Stanford.

[13] Negri, S. (2005), Proof analysis in modal logic, *Journal of Philosophical Logic* **34**(5–6), 507–544.

[14] Negri, S. and von Plato, J. (1998), Cut elimination in the presence of axioms, *The Bulletin of Symbolic Logic* **4**(4), 418–435.

[15] Negri, S. and von Plato, J. (2001), *Structural Proof Theory*, Cambridge Univeristy Press.

[16] Ohnishi, M. and Matsumoto, K. (1957), Gentzen methods in modal calculi, *Osaka Mathematical Journal* **9**, 113–130.

[17] Priest, G. (2008), *An introduction to non-classical logic*, Cambridge Univesity Press.

[18] SCHÜTTE, K. (1956), Ein system des verknüpfenden schliessens, *Archiv für mathematische Logik und Grundlagenforschung* **2**(2–4), 55–67.
[19] TAKEUTI, G. (1987), *Proof theory*, number 81 in *Studies in Logic and Foundations of Mathematics*, 2 edn, Elsevier.
[20] TROELSTRA, A. S. (1992), *Lectures on Linear Logic*, CSLI Lecture Notes, Stanford.

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