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PERSPECTIVAL LOGIC OF ACCEPTANCE AND REJECTION

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ABSTRACT

This paper aims at developing a logical theory of perspectival epistemic attitudes. After presenting a standard framework for modeling acceptance, where the epistemic space of an agent coincides with a unique epistemic cell, more complex systems are introduced, which are characterized by the existence of many connected epistemic cells, and different possible attitudes towards a proposition, both positive and negative, are discussed. In doing that, we also propose some interesting ways in which the systems can be interpreted on well known epistemological standpoints.

1. Introduction

A preliminary view on the connections between the epistemic state of acceptance, the epistemic act of assenting, and the speech act of asserting is that, in asserting, we express assent and, in assenting, we initiate a state of acceptance. A correspondent view on the connections between the epistemic state of rejection, the epistemic act of dissenting, and the speech act of denial is that, in denying, we express dissent and, in dissenting, we initiate a state of rejection. Thus, if we endorse an equivalence thesis like

ET1: denying = asserting a negation

the possibility is open to define dissenting in terms of assenting, since expressing a dissent comes to coincide with expressing an assent to a negation, and rejection in terms of acceptance, since initiating a state of rejection comes to coincide with initiating a state of acceptance of a negation. Hence, we can conclude that

ET2: rejecting = accepting a negation

ET2 can be put into question from different points of view (see [9] for a general introduction). In particular, we can object either (i) that rejection is a primitive epistemic attitude, and so it is not to be identified with acceptance of a negation, or (ii) that epistemic attitudes are perspectival, so that different notions of acceptance and rejection are definable, whose connections are not completely captured by a principle like **ET2**.

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The main aim of this paper is to explore different ways of negating the equivalence thesis and to develop a logical framework for capturing the idea that epistemic attitudes are perspectival. The paper is structured as follows. In the rest of this section, two ways of negating the equivalence thesis are proposed and the perspectival approach to epistemic attitudes is outlined. In section 2, non-perspectival systems are introduced and discussed. In section 3, perspectival systems are introduced and appropriately axiomatized.

1.1. *The negations of the equivalence thesis*

Let us say that a position is *epistemically paraconsistent* if it admits of both $\mathbf{A}\varphi$ and $\neg\mathbf{R}\neg\varphi$ and *epistemically paracomplete* if it admits of both $\mathbf{R}\varphi$ and $\neg\mathbf{A}\neg\varphi$. We will focus on two general ways of negating the equivalence thesis, which lead either to paraconsistent or to paracomplete positions. The first way is a consequence of allowing for the possibility of alethic paraconsistency and paracompleteness (see e.g. [8, 9]).

- **Managing truth-value gaps and gluts**

Paraconsistent option. Suppose we accept the possibility of *truth-value gluts*. Then, we are in a position to accept both φ and the negation of φ, so that, if acceptance and rejection are exclusive, we reject neither φ nor $\neg\varphi$. Thus, we can accept φ without rejecting $\neg\varphi$, while the rejection of φ can either imply (strong version) or not imply (liberal version) the acceptance of $\neg\varphi$. Hence

- Paraconsistency: $\mathbf{A}\varphi \to \mathbf{R}\neg\varphi$ is not valid. For, $\mathbf{A}\varphi \wedge \mathbf{A}\neg\varphi$ is admissible and it is assumed that $\mathbf{A}\varphi$ excludes $\mathbf{R}\varphi$. (i) Strong version: $\mathbf{R}\varphi \to \mathbf{A}\neg\varphi$ is valid, since $\mathbf{R}\varphi$ is assumed to be stronger than $\mathbf{A}\neg\varphi$. (ii) Liberal version: $\mathbf{R}\varphi \to \mathbf{A}\neg\varphi$ is invalid, since $\mathbf{R}\varphi$ is not considered stronger than $\mathbf{A}\neg\varphi$.

Paracomplete option. Suppose we accept the possibility of *truth-value gaps*. Then, we are in a position to reject both φ and the negation of φ, so that, if acceptance and rejection are exclusive, we accept neither φ nor $\neg\varphi$. Thus, we can reject $\neg\varphi$ without accepting φ, while the acceptance of φ can either imply (strong version) or not imply (liberal version) the rejection of $\neg\varphi$.

- Paracompleteness: $\mathbf{R}\varphi \to \mathbf{A}\neg\varphi$ is not valid. For, $\mathbf{R}\varphi \wedge \mathbf{R}\neg\varphi$ is admissible and it is assumed that $\mathbf{A}\varphi$ excludes $\mathbf{R}\varphi$. (i) Strong version: $\mathbf{A}\varphi \to \mathbf{R}\neg\varphi$ is valid, since $\mathbf{A}\varphi$ is assumed to be stronger than $\mathbf{R}\neg\varphi$. (ii) Liberal version: $\mathbf{R}\varphi \to \mathbf{A}\neg\varphi$ is invalid, since $\mathbf{A}\varphi$ is not considered stronger than $\mathbf{R}\neg\varphi$.

The second way of negating **ET** emerges when we try to merge different sources of information.

- **Managing sources of information**

Suppose that we have two equally trustworthy sources of information about a certain domain. These sources can agree or disagree on the valuation of specific propositions, but they are not inconsistent, even if they can be silent on the truth value of some proposition. Hence, it is possible for us to receive nine kinds of report on the same proposition, where each report consists in a pair of truth-values or blanks. Suppose that we decide to adopt a prudent approach, accepting a proposition only if the sources agree on its truth-value, but rejecting it when one source reports it as false and the other source is silent. Then, we obtain a strongly paracomplete position. Suppose, in contrast, that we decide to adopt an optimist approach, accepting a proposition when one source reports it as true and the other source is silent, but rejecting it only if the sources agree on its truth-value. Then, we obtain a strongly paraconsistent position.

	Report on φ	Prudent approach	Optimistic approach
Case 1	(1, 1)	$\mathbf{A}\varphi \wedge \mathbf{R}\neg\varphi$	$\mathbf{A}\varphi \wedge \mathbf{R}\neg\varphi$
Case 2	(1, −)	$\mathbf{R}\neg\varphi$	$\mathbf{A}\varphi$
Case 3	(1, 0)	$\mathbf{R}\varphi \wedge \mathbf{R}\neg\varphi$	$\mathbf{A}\varphi \wedge \mathbf{A}\neg\varphi$
Case 4	(−, 1)	$\mathbf{R}\neg\varphi$	$\mathbf{A}\varphi$
Case 5	(−, −)	−	−
Case 6	(−, 0)	$\mathbf{R}\varphi$	$\mathbf{A}\neg\varphi$
Case 7	(0, 1)	$\mathbf{R}\varphi \wedge \mathbf{R}\neg\varphi$	$\mathbf{A}\varphi \wedge \mathbf{A}\neg\varphi$
Case 8	(0, −)	$\mathbf{R}\varphi$	$\mathbf{A}\neg\varphi$
Case 9	(0, 0)	$\mathbf{A}\neg\varphi \wedge \mathbf{R}\varphi$	$\mathbf{A}\neg\varphi \wedge \mathbf{R}\varphi$

In case of many sources of information of different quality, more interesting approaches can be adopted in view of the preservation of the correctness and completeness of the data (see [1] for an introduction).

1.2. *The perspectival character of acceptance*

The main problem now is how cases of epistemic paraconsistency and paracompleteness can be handled in an appropriate way. We propose to exploit the intuition that acceptance and rejection are perspective dependent, where a perspective is a triple $i = (\sigma, \mathbf{i}, \mathbf{s})$ in which

1. σ is a member of a set Σ of subject matters;
2. $\mathbf{i}$ is a member of a set $\mathbf{I}$ of sources of information;
3. $\mathbf{s}$ is a subset of a set $\mathbf{S}$ of specific sets of epistemic standards.

The central idea underlying such an approach is that acceptance and rejection are connected to epistemic justifications, where a justification is intended as an epistemic ground for assuming a proposition p as a solution to a specific problem relative to a subject matter $\sigma \in \Sigma$: a solution that derives from a source of information $\mathbf{i} \in \mathbf{I}$ and satisfies a set of standards $\mathbf{s} \in \mathbf{S}$.[1] Hence, we will model acceptance and rejection in accordance with the intuition that an agent has a perspective on the world, which provides her with an epistemic book, written in such a way that the results of a research based on the given perspective are recorded in it. Furthermore, it is assumed that the book is associated with an epistemic cell, which is a set of scenarios, thought of as complete representations of the world that are consistent with the content of the book. Finally, it is assumed that each epistemic book is subdivided into two separate parts: a yes-box, where all the accepted propositions are to be inserted, and a no-box, where all the rejected propositions are to be inserted. Thus, we can introduce the following definitions

1. accepting := agent-dependent writing in the yes-box of the epistemic book.
2. rejecting := agent-dependent writing in the no-box of the epistemic book.

This initial model can be refined along different directions. In particular, we are interested in

- introducing different books
- introducing primary books among all the books
- distinguishing the no-box from the complement of the yes-box
- distinguishing the no-box from the set of negations in the yes-box

In any case, acceptance and rejection are assumed to be locally incompatible attitudes, and the fundamental principle is endorsed that *it is impossible for the same agent to both accept and reject the same proposition under the same perspective*.

2. Basic non-perspectival systems

In this section, we present two systems of logic for an agent whose epistemic space is based on a unique book. The first one is a basic system of acceptance, where rejection is defined in terms of acceptance. The second system extends the first by introducing rejection as a primitive epistemic attitudes.[2]

[1] We will not pursue here a development of this framework from an epistemological point of view. We only intend to propose a hierarchy of logical systems based on it and to show how these systems help us to model different interesting concepts of acceptance and rejection.

[2] In particular, we will treat implicit acceptance and rejection (see [4] ch. 9). In fact, it would be possible to extend the following systems so as to introduce both implicit and

Definition 1. Let P be a set of propositional variables. The initial language $\mathcal{L}$ is defined according to the following rules:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathbf{A}\varphi \mid \mathbf{R}\varphi$$

where $p \in P$. The other connectives and the dual modalities $\langle\mathbf{A}\rangle\varphi$ and $\langle\mathbf{R}\rangle\varphi$ are defined as usual. Here, $\mathbf{A}\varphi$ / $\mathbf{R}\varphi$ says that the agent implicitly accepts / rejects φ.

2.1. *The basic system of acceptance*

We first consider an agent with a unique book and a unique positive epistemic attitude of acceptance. In this context, a unique epistemic cell is sufficient for modeling the agent. The attitude of rejection can then be introduced by definition in at least two different ways.

1. classical rejection: $\mathbf{R}\varphi := \mathbf{A}\neg\varphi$.
2. reflexive rejection: $\mathbf{R}\varphi := \mathbf{A}\neg\mathbf{A}\varphi \wedge \neg\mathbf{A}\varphi$.

The first definition is a straightforward implementation of **ET**, and thus the no-box of the book comes to coincide with the set of the negations of the propositions contained in the yes-box. By contrast, the idea underlying the second definition is that rejecting φ is stronger than not accepting φ, but not so strong as to imply $\mathbf{A}\neg\varphi$, so that rejecting φ is interpreted as the active acceptance of not accepting φ. Hence, rejecting coincides with excluding the case that φ is in the yes-box of the epistemic book, without incurring in the problem of identifying the no-box with the complement of the yes-box, so that any proposition that is not accepted is rejected.

Definition 2. (*model*): A basic model for $\mathcal{L}$ is a tuple $M = \langle W, R, V\rangle$ where

W is a set of epistemic worlds.

$R : W \to \wp(W)$ is an accessibility function.

$V : P \to \wp(W)$ is a valuation function.

In addition, R satisfies the condition $R(w) \neq \emptyset$, for all w. Hence, no inconsistency is allowed within the epistemic book.

Definition 3. (*truth in a model*):

$M, w \vDash p \Leftrightarrow w \in V(p)$

$M, w \vDash \neg\varphi \Leftrightarrow M, w \nvDash \varphi$

$M, w \vDash \varphi \wedge \psi \Leftrightarrow M, w \vDash \varphi$ and $M, w \vDash \psi$

$M, w \vDash \mathbf{A}\varphi \Leftrightarrow \forall v(v \in R(w) \Rightarrow M, v \vDash \varphi)$

explicit attitudes and to show that the results obtained in the present setting carry over to the explicit attitudes as well.

Theorem 1. *the logic* **A1** *of basic acceptance, given by classical propositional logic plus the following group of axioms*

$\mathbf{R_A}$: $\varphi/\mathbf{A}\varphi$

$\mathbf{K_A}$: $\mathbf{A}(\varphi \to \psi) \to (\mathbf{A}\varphi \to \mathbf{A}\psi)$

$\mathbf{D_A}$: $\neg(\mathbf{A}\varphi \land \mathbf{A}\neg\varphi)$

is sound and complete with respect to the class of basic models.

Connection between rejection and acceptance.

classical	reflexive
$\vdash_{\mathbf{A1}} \mathbf{R}\varphi \to \neg\mathbf{A}\varphi$	$\vdash_{\mathbf{A1}} \mathbf{R}\varphi \to \neg\mathbf{A}\varphi$
$\vdash_{\mathbf{A1}} \mathbf{A}\varphi \to \mathbf{R}\neg\varphi$	$Con_{\mathbf{A1}}(\mathbf{A}\varphi \land \neg\mathbf{R}\neg\varphi)$
$\vdash_{\mathbf{A1}} \mathbf{R}\varphi \to \mathbf{A}\neg\varphi$	$Con_{\mathbf{A1}}(\mathbf{R}\varphi \land \neg\mathbf{A}\neg\varphi)$

Stronger systems can be obtained by introducing further conditions.

	Axioms on **A**	Conditions on R
$\mathbf{3_A}$	$\mathbf{A}\varphi \to \neg\mathbf{A}\neg\mathbf{A}\varphi$	$\exists v(v \in R(w)$ and $R(v) \subseteq R(w))$
$\mathbf{4_A}$	$\mathbf{A}\varphi \to \mathbf{A}\mathbf{A}\varphi$	$v \in R(w) \Rightarrow R(v) \subseteq R(w)$
$\mathbf{5_A}$	$\neg\mathbf{A}\neg\mathbf{A}\varphi \to \mathbf{A}\varphi$	$v \in R(w) \Rightarrow R(w) \subseteq R(v)$

If $\mathbf{3_A}$ is assumed, then reflective rejection comes to coincide with $\mathbf{A}\neg\mathbf{A}\varphi$. If $\mathbf{4_A}$ is assumed, then $\mathbf{3_A}$ can be derived by $\mathbf{D_A}$. Finally, if both $\mathbf{4_A}$ and $\mathbf{5_A}$ are assumed, we obtain a standard system of ideal acceptance, but the distinction between reflexive rejection and non-acceptance is lost, since $\mathbf{A}\neg\mathbf{A}\varphi$ and $\neg\mathbf{A}\varphi$ turn out to be equivalent. Hence, even if the basic system of acceptance is appropriate for representing paraconsistent and paracomplete positions, when the notion of reflexive rejection is assumed, it is not robust in doing that, since the addition of the principles of positive and negative introspection, $\mathbf{4_A}$ and $\mathbf{5_A}$, destroys the distinction between classical and reflexive rejection.

2.2. *The symmetric system of acceptance and rejection*

Let us consider now an agent with a unique book and distinct attitudes of acceptance and rejection. In this context, two epistemic modalities are required for modeling the agent, since rejection is a primitive concept. As a consequence, the connection between acceptance and rejection has to be characterized by introducing specific axioms (see [7, 6] for further discussion).

Definition 4. (*model*): A symmetric model for $\mathcal{L}$ is a tuple $M = \langle W, R_1, R_2, V \rangle$ where

W is a set of epistemic worlds.

$R_1 : W \to \wp(W)$ is used to model the positive attitude.

$R_2 : W \to \wp(W)$ is used to model the negative attitude.

$V : P \to \wp(W)$ is a valuation function.

In addition, R_1 and R_2 satisfy the condition $R_1(w) \cap R_2(w) \neq \emptyset$, for all $w \in W$. The condition on R_1 and R_2 implies that $R_1(w) \neq \emptyset$ and $R_2(w) \neq \emptyset$, for all $w \in W$, so that inconsistencies in accepting and rejecting propositions are prevented. A more general system can be obtained by dropping this condition, but the principle of incompatibility between acceptance and rejection results then invalid.

Definition 5. (*truth in a model*): beside the standard cases, we have

$M, w \vDash \mathbf{A}\varphi \Leftrightarrow \forall v(v \in R_1(w) \Rightarrow M, v \vDash \varphi)$

$M, w \vDash \mathbf{R}\varphi \Leftrightarrow \forall v(v \in R_2(w) \Rightarrow M, v \vDash \neg\varphi)$

Theorem 2. *the logic* **A2** *of symmetric acceptance, given by classical propositional logic plus the following group of axioms*

$\mathbf{R_A}$*:* $\varphi/\mathbf{A}\varphi$

$\mathbf{K_A}$*:* $\mathbf{A}(\varphi \to \psi) \to (\mathbf{A}\varphi \to \mathbf{A}\psi)$

$\mathbf{R_R}$*:* $\neg\varphi/\mathbf{R}\varphi$

$\mathbf{K_R}$*:* $\mathbf{R}\neg(\varphi \to \psi) \to (\mathbf{R}\psi \to \mathbf{R}\varphi)$

$\mathbf{D_{AR}}$*:* $\neg(\mathbf{A}\varphi \wedge \mathbf{R}\varphi)$

is sound and complete with respect to the class of symmetric models.

A2 is a conservative extension of **A1**: just consider that any model $M_1 = (W, R_1, V)$ for **A1** can be transformed into a model $M_2 = (W, R_1, R_2, V)$ for **A2**, satisfying the same **A1**-formulas, by putting $R_2(w) = W$, for every $w \in W$, and that any model $M_2 = (W, R_1, R_2, V)$ for **A2** can be transformed into a model for **A1**, satisfying the same **A1**-formulas, by simply deleting R_2.

Facts. (proofs are left as an exercise)

$\vdash_{\mathbf{A2}} \mathbf{A}\varphi \to \neg\mathbf{A}\neg\varphi$	$\vdash_{\mathbf{A2}} \mathbf{R}\varphi \to \neg\mathbf{R}\neg\varphi$
$\vdash_{\mathbf{A2}} \mathbf{A}\varphi \wedge \mathbf{A}\psi \leftrightarrow \mathbf{A}(\varphi \wedge \psi)$	$\vdash_{\mathbf{A2}} \mathbf{R}\varphi \wedge \mathbf{R}\psi \leftrightarrow \mathbf{R}(\varphi \vee \psi)$
$Con_{\mathbf{A2}}(\neg\mathbf{A}\varphi \wedge \neg\mathbf{A}\neg\varphi)$	$Con_{\mathbf{A2}}(\neg\mathbf{R}\varphi \wedge \neg\mathbf{R}\neg\varphi)$

It is not difficult to see that $Con_{\mathbf{A2}}(\mathbf{A}\varphi \wedge \neg \mathbf{R} \neg \varphi)$ and $Con_{\mathbf{A2}}(\mathbf{R}\varphi \wedge \neg \mathbf{A} \neg \varphi)$, and so that in this system both paraconsistency and paracompleteness are representable. In addition, the connection between **A** and **R** can be strengthened in several ways.

Axioms on **A**, **R**	Conditions on R and S
$\mathbf{A}\varphi \to \mathbf{R}\neg\varphi$	$R_2(w) \subseteq R_1(w)$
$\mathbf{R}\varphi \to \mathbf{A}\neg\varphi$	$R_1(w) \subseteq R_2(w)$
$\mathbf{R}\varphi \leftrightarrow \mathbf{A}\neg\varphi$	$R_2(w) = R_1(w)$

Hence, system **A1** with classical rejection can be viewed as the special case of **A2** where axiom $\mathbf{R}\varphi \leftrightarrow \mathbf{A}\neg\varphi$ holds. Analogously, a system corresponding to **A1** with reflexive rejection might be obtained by adding an axiom to the effect that $\mathbf{R}\varphi \leftrightarrow \mathbf{A}\neg\mathbf{A}\varphi \wedge \neg\mathbf{A}\varphi$.

In conclusion, introducing two epistemic attitudes constitutes a significant advance in modeling paraconsistent and paracomplete positions. In the next section, we go on in analyzing these attitudes and propose a way for defining different notions of acceptance and rejection and different kinds of negation of the equivalence thesis.

3. Basic perspectival systems

In this section, we present systems of logic for an agent whose epistemic space is based on different books. The epistemic space is thus constituted by a set of epistemic cells, where each cell contains scenarios which are compatible with a specific book. The agent is in a position to write different books according to different perspectives, and any book only contains propositions about a specific subject matter, which are accepted on the basis of a single source in an epistemic situation characterized by specific standards.

3.1. *The basic system of perspectival acceptance*

Let us consider an agent with a set of books and a unique positive epistemic attitude of acceptance. In this context, many epistemic cells are necessary for modeling the agent, and more concepts of acceptance and rejection becomes definable.[3]

Definition 6. (*model*): A perspectival model for L is a tuple $M = \langle W, I, \{R_i\}_{i \in I}, V \rangle$ where

[3] These models are introduced in [4] for modeling local reasoning.

$W \neq \varnothing$ is a set of epistemic worlds.

$I \neq \varnothing$ is a set of epistemic perspectives.

$R_i : W \to \wp(W)$ is an accessibility function, for each $i \in I$.

$V : P \to \wp(W)$ is a valuation function.

In addition, R_i satisfies the condition: $R_i(w) \neq \varnothing$, for all w and $i \in I$, so that no inconsistency is allowed within the same epistemic book: no source is inconsistent in itself, but they may not be jointly consistent.

Definition 7. (*truth in a model*): beside the standard cases, we have

$$M, w \vDash \mathbf{A}\varphi \Leftrightarrow \exists i \in I \forall v (v \in R_i(w) \Rightarrow M, v \vDash \varphi)$$

The crucial characteristic of the present system is given by the truth definition of a modal formula. $\mathbf{A}\varphi$ is true precisely when φ is true across some cell and, since there are many cells and no constraint on their connections, it is possible for an agent to accept both a proposition and its negation without accepting their conjunction. Still, since no cell contains a scenario where contradictions are true, the agent accepts no contradiction. In fact, the following facts are provable in the class of perspectival models

1. $\mathbf{A}\varphi \wedge \mathbf{A}\psi \to \mathbf{A}(\varphi \wedge \psi)$ is not valid;
2. $\neg(\mathbf{A}\varphi \wedge \mathbf{A}\neg\varphi)$ is not valid;
3. $\neg\mathbf{A}(\varphi \wedge \neg\varphi)$ is valid.

Proof. Take $W = \{w, w_1, w_2\}$, $I = \{1, 2\}$, $R_1(w) = \{w_1\}$, $R_2(w) = \{w_2\}$.

$$\begin{tikzcd}
w_1 : p_1 & & w_2 : p_2 \\
& w \arrow[ul, "R_1"'] \arrow[ur, "R_2"] &
\end{tikzcd}$$

Let $V(p_1) = R_1(w)$ and $V(p_2) = R_2(w)$. Then $M, w \vDash \mathbf{A}p_1$; $M, w \vDash \mathbf{A}\neg p_2$; $M, w \vDash \mathbf{A}p_2$; $M, w \nvDash \mathbf{A}(p_1 \wedge p_2)$, and so the first two facts are proven. The last fact is a straightforward consequence of the definition of truth. □

The failure of $\mathbf{A}\varphi \wedge \mathbf{A}\psi \to \mathbf{A}(\varphi \wedge \psi)$, ensuring the possibility of accepting two propositions without accepting their conjunction, is due to the presence of non-nested cells. The failure of $\mathbf{A}\varphi \to \neg\mathbf{A}\neg\varphi$, ensuring the possibility of accepting contradictory propositions, is due to the presence of non-intersecting cells. Indeed, in frames where all pairs of cells have non-empty intersection $\mathbf{A}\varphi \to \neg\mathbf{A}\neg\varphi$ is valid, since a φ-cell intersects every other cell, and so in every other cell there is a φ-scenario. Hence, it is the separation of the epistemic cells that allows the agent to be consistent within each cell, while accepting inconsistent propositions.

Theorem 3. *the logic* **PA**, *axiomatized by the following group of axioms*

$\mathbf{R_A}$: $\varphi/\mathbf{A}\varphi$

$\mathbf{M_A}$: $\varphi \rightarrow \psi/\mathbf{A}\varphi \rightarrow \mathbf{A}\psi$

$\mathbf{D_A}$: $\neg\mathbf{A}(\varphi \wedge \neg\varphi)$

is sound and complete with respect to the class of perspectival models.

The present framework allows us to model an extremely general notion of acceptance. The connection between rejection and acceptance is as follows.

classical rejection	reflexive rejection
$Con_{\mathbf{PA}}(\mathbf{R}\varphi \wedge \mathbf{A}\varphi)$	$Con_{\mathbf{PA}}(\mathbf{R}\varphi \wedge \mathbf{A}\varphi)$
$\vdash_{\mathbf{PA}} \mathbf{A}\varphi \rightarrow \mathbf{R}\neg\varphi$	$Con_{\mathbf{PA}}(\mathbf{A}\varphi \wedge \neg\mathbf{R}\neg\varphi)$
$\vdash_{\mathbf{PA}} \mathbf{R}\varphi \rightarrow \mathbf{A}\neg\varphi$	$Con_{\mathbf{PA}}(\mathbf{R}\varphi \wedge \neg\mathbf{A}\neg\varphi)$

As we can see, when reflexive rejection is assumed, no particular connection between acceptance and rejection is derivable. Thus, perspectival acceptance is useful to model epistemic attitudes that are sensitive to a switch of standards. As an example, in a setting where we are interested in modeling *context-sensitive* epistemic agents (see e.g. [3, 12]), the set of propositions accepted in a certain context, determined by a given set of standards $\mathbf{s_1} \subseteq \mathbf{S}$, will typically be different from the set of propositions accepted in a context determined by more stringent standards $\mathbf{s_2} \subseteq \mathbf{S}$. Hence, we obtain a suitable model of *epistemic contextualism*. Similarly, in a setting where we are interested in modeling *contrast-sensitive* epistemic agents (see e.g. [10, 11]), the set of propositions accepted as answers to a certain question, determined by the contrast class of a given subject matter $\sigma_1 \in \Sigma$, will typically be different from the set of propositions accepted as answers to a question determined by the contrast class of a different subject matter $\sigma_2 \in \Sigma$. Hence, we obtain a suitable model of *epistemic contrastivism*. Finally, in a setting where we are interested in modeling *source-sensitive* epistemic agents, the set of propositions accepted as pieces of information on a certain domain provided by a given source $\mathbf{i}_1 \in \mathbf{I}$, will typically be different from the set of propositions provided by a different source $\mathbf{i}_2 \in \mathbf{I}$. Hence, we obtain a suitable model of local knowledge (see e.g. [2, 4]).

As might be expected, one of the key problems in a perspectival framework is how to put together data deriving from different perspectives. Here, we will focus on three solution strategies. The first strategy consists in requiring that agents accept a proposition just in case that proposition is acceptable under all perspectives, which leads to a notion of *absolute acceptance*. The second strategy consists in requiring that agents accept a proposition just in case that proposition is acceptable under a special perspective, selected in virtue of some positive characteristics. For example, perspectives could be selected because they are determined by very fine-grained subject matters in Σ, by very high standards in **S**, or by highly dependable sources in **I**. This strategy leads to a notion of *critical acceptance*. The final strategy consists in requiring that agents accept a proposition just in case that proposition is acceptable under a certain class of perspectives, as determined according to a given ordering. For example, perspectives could be ordered in view of the granularity of the subject matters, the highness of the standards, or the dependability of the sources. This strategy leads to a notion of *ordinal acceptance*.

3.2. *First strategy: absolute acceptance*

The language is extended by introducing a modality $\mathbf{A}^U$ for absolute, or universal, acceptance. The intended meaning of $\mathbf{A}^U \varphi$ is that φ is accepted independently of any perspective, so that φ holds in all the epistemic cells. The definition of truth is extended accordingly:

- $M, w \vDash \mathbf{A}^U \varphi \Leftrightarrow \forall i \in I \forall v (v \in R_i(w) \Rightarrow M, v \vDash \varphi)$

Theorem 4. *the logic* **PU***, obtained by adding to* **PA**

$\mathbf{R_U}$*:* $\varphi / \mathbf{A}^U \varphi$

$\mathbf{K_U}$*:* $\mathbf{A}^U (\varphi \to \psi) \to (\mathbf{A}^U \varphi \to \mathbf{A}^U \psi)$

$\mathbf{I_U}$*:* $\mathbf{A}^U (\varphi \to \psi) \to (\mathbf{A}\varphi \to \mathbf{A}\psi)$

is sound and complete with respect to the class of perspectival models.

Theorems. (by **IU**)

$\vdash_{\mathbf{PU}} \mathbf{A}^U \varphi \to \mathbf{A}\varphi$

$\vdash_{\mathbf{PU}} \mathbf{A}^U \varphi \to \neg\mathbf{A}\neg\varphi$

$\vdash_{\mathbf{PU}} \mathbf{A}^U \varphi \to \neg\mathbf{A}^U \neg\varphi$

$\mathbf{R_U}$ and $\mathbf{K_U}$ state that $\mathbf{A}^U$ is a normal modality. $\mathbf{I_U}$ is the crucial axiom and implies that what is absolutely accepted is both accepted in every cell and classically rejected in no cell. This is due to the fact that no cell is empty. In addition, $\mathbf{I_U}$ allows us to derive $\mathbf{M_A}$ from $\mathbf{R_U}$. In this setting, rejection is defined according to the classical definition.

Absolute acceptance is useful to model the set of propositions that are not touched by a perspective shift. As an example, supposing that we are interested in modeling epistemic agents assessing different empirical theories relative to a shared empirical basis, the propositions in the basis being absolutely accepted. In this case, perspectives are determined by theories, intended as sources of information, while the empirical basis is determined by our measuring systems, whose results are invariant under a theory shift, even though the interpretation of these results is not invariant. It is in general a matter of doubt whether there could be a stable set of absolutely acceptable propositions. Still, within any logical system, the set of logically valid propositions is a set of this kind.

If absolute acceptance is available, we get new options for interpreting the connection between rejection and acceptance within the classical framework. An intuitive possibility is to work with two concepts of rejection, where the classical one is associated with a strong concept of absolute rejection. What we get is a model of two epistemic attitudes: a *prudent* one, according to which the best thing to do is (i) to reject φ when its negation holds in some cell and (ii) to accept it only when it holds in all the cells; and an *optimistic* one, according to which it is better (i) to accept φ when it holds in some cell and (ii) to reject it only when its negation holds in all the cells. As to the relation between acceptance and rejection, we obtain then

Prudent version	Optimistic version
$\vdash_{\mathbf{PU}} \mathbf{R}^U \varphi \to \neg\mathbf{A}\varphi$	$\vdash_{\mathbf{PU}} \mathbf{R}\varphi \to \neg\mathbf{A}^U \varphi$
$Con_{\mathbf{PU}}(\mathbf{A}\varphi \wedge \neg\mathbf{R}^U \neg\varphi)$	$\vdash_{\mathbf{PU}} \mathbf{A}^U \varphi \to \mathbf{R}\neg\varphi$
$\vdash_{\mathbf{PU}} \mathbf{R}^U \varphi \to \mathbf{A}\neg\varphi$	$Con_{\mathbf{PU}}(\mathbf{R}\varphi \wedge \neg\mathbf{A}^U \neg\varphi)$
(strong paraconsistency)	(strong paracompleteness)

3.3. *Second strategy: critical acceptance*

We extend our language by introducing a modality $\mathbf{A}^C$ for critical acceptance. The intended meaning of $\mathbf{A}^C \varphi$ is that φ is accepted on the basis of the most critical perspective, if any, so that φ holds in the epistemic cell that represents this perspective. A model is a tuple $M = \langle W, I, R, \{R_i\}_{i\in I}, V\rangle$ such that

1. $\langle W, I, \{R_i\}_{i\in I}, V\rangle$ as before
2. $R(w) \cap R_i(w) \neq \varnothing$, for all $i \in I$

Here R represents the agent's critical point of view, and condition 2 is a constraint that ensures that any cell contains at least a critical scenario, which is a scenario admissible according to the critical perspective. Hence, not all critical scenarios are excluded in a less critical cell. The definition of truth is extended so that:

- $M, w \vDash \mathbf{A}^C \varphi \Leftrightarrow \forall v(v \in R(w) \Rightarrow M, v \vDash \varphi)$

Theorem 5. *the logic* **PC***, obtained by adding to* **PA**

$\mathbf{R_C}$: $\varphi / \mathbf{A}^C \varphi$

$\mathbf{K_C}$: $\mathbf{A}^C (\varphi \to \psi) \to (\mathbf{A}^C \varphi \to \mathbf{A}^C \psi)$

$\mathbf{I_C}$: $\mathbf{A}^C \varphi \to \mathbf{A}\varphi \wedge \neg\mathbf{A}\neg\varphi$

is sound and complete with respect to the previous class of models.

$\mathbf{R_C}$ and $\mathbf{K_C}$ state that $\mathbf{A}^C$ is a normal modality. $\mathbf{I_C}$ states that what is critically accepted is accepted and classically rejected in no cell. This is due to the fact that every cell intersects the critical cell. Note that $\mathbf{R_C}$ and $\mathbf{I_C}$ imply $\neg\mathbf{A}(\varphi \wedge \neg\varphi)$.

Critical acceptance is useful to model the set of propositions that are justified by the best theories at our disposal. As an example, in a setting where we are interested in modeling epistemic agents comparing different empirical theories against the same empirical basis, the theory that fits at best the empirical evidence is critically accepted.

As before, critical acceptance allows us to define two concepts of rejection and to introduce the distinction between prudent and optimistic epistemic attitudes, thus obtaining the following results on the relation between acceptance and rejection.

Prudent version	Optimistic version
$\vdash_{\mathbf{PC}} \mathbf{R}^C \varphi \to \neg\mathbf{A}\varphi$	$\vdash_{\mathbf{PC}} \mathbf{R}\varphi \to \neg\mathbf{A}^C \varphi$
$Con_{\mathbf{PC}}(\mathbf{A}\varphi \wedge \neg\mathbf{R}^C \neg\varphi)$	$Con_{\mathbf{PC}}(\mathbf{A}^C \varphi \wedge \neg\mathbf{R}\neg\varphi)$
$Con_{\mathbf{PC}}(\mathbf{R}^C \varphi \wedge \neg\mathbf{A}\neg\varphi)$	$Con_{\mathbf{PC}}(\mathbf{R}\varphi \wedge \neg\mathbf{A}^C \neg\varphi)$
(liberal paraconsistency)	(liberal paracompleteness)

The prudent attitude, according to which an agent only accepts what is critically legitimate and rejects what is rejected in at least one cell, seems to match the epistemic attitude underlying critical rationalism.

3.4. *Third strategy: ordinal acceptance*

Finally, we extend our language by introducing a modality $\leq$ for ordinal acceptance[4]. The intended meaning of $\varphi \leq \psi$ is that, for every perspective which accepts φ, there is a perspective accepting ψ that is at least as secure, so that ψ is to be accepted, provided φ is accepted. A model is a tuple $M = \langle W, I, \{\leq_w\}_{w \in \mathrm{W}}, \{R_i\}_{i \in I}, V\rangle$ such that

1. $\langle W, I, \{R_i\}_{i \in I}, V\rangle$ as before
2. $\leq_w \subseteq I \times I$ is reflexive and transitive for all $w \in W$

[4] It is worth noting that the ordering concerns the perspectives and not the propositions, as in standard approaches. See [5] for a discussion.

The definition of truth is extended so that:

- $M, w \vDash \varphi \leq \psi \Leftrightarrow$ for all j
  if $\forall v(v \in R_j(w) \Rightarrow M, v \vDash \varphi)$
  then $\exists_i(j \leq_w i$ and $\forall v(v \in R_i(w) \Rightarrow M, v \vDash \psi)$

Theorem 6. *the logic* **PO***, obtained by adding to* **PA**

$\mathbf{R_O}$: $\varphi \rightarrow \psi / \varphi \leq \psi$

$\mathbf{1_O}$: $\mathbf{A}\varphi \vee \varphi \leq \psi$

$\mathbf{2_O}$: $\varphi \leq \psi \rightarrow (\mathbf{A}\varphi \rightarrow \mathbf{A}\psi)$

$\mathbf{3_O}$: $\varphi \leq \psi \wedge \psi \leq \psi' \rightarrow \varphi \leq \psi'$

is sound and complete with respect to the previous class of models.

$\mathbf{R_O}$ encodes the intuitive assumption according to which consequences of φ are at least as acceptable as φ, so that any consequence of φ is to be accepted, if φ is accepted. $\mathbf{1_O}$ states that, if φ is not accepted, then no proposition is less acceptable than φ. $\mathbf{2_O}$ states that any proposition that is at least as acceptable as φ is accepted, provided that φ is accepted. Finally, $\mathbf{3_O}$ states that $\leq$ is transitive. Hence, since $\leq$ is also reflexive, by $\mathbf{R_O}$, $\leq$ turns out to be a preorder.

Introducing an ordering on the set of perspectives allows us to define a kind of critical acceptance.

Definition 8. induced critical acceptance.

$\mathbf{A}^C_{\leq}\varphi := T \leq \varphi$, where T is logical truth

Note that, semantically, $\mathbf{A}^C_{\leq}\varphi$ is true at a certain world precisely when, for every epistemic cell, there is a non-worse epistemic cell through which φ is true. Hence, accepting φ is a best option from every perspective. This notion obeys some significant logical principles:

$\varphi / \mathbf{A}^C_{\leq}\varphi$

$\varphi \rightarrow \psi / \mathbf{A}^C_{\leq}\varphi \rightarrow \mathbf{A}^C_{\leq}\psi$

$\varphi \leq \psi \rightarrow (\mathbf{A}^C_{\leq}\varphi \rightarrow \mathbf{A}^C_{\leq}\psi)$

$\mathbf{A}^C_{\leq}\varphi \rightarrow \mathbf{A}\varphi$, and so $\neg\mathbf{A}^C_{\leq}(\varphi \wedge \neg\varphi)$

Still, this notion of acceptance allows for accepting both a proposition and its negation. This is due to the possibility of infinite ascending chains of epistemic cells. To be sure, let us consider the following model:

$\langle W, I, \{\leq_w\}_{w \in W}, \{R_i\}_{i \in I}, V\rangle$ where $W = I = \mathbb{N}$ and

$\leq_w = \leq$ for each $w \in W$

$R_i(w) = \{i\}$, for each $w \in W$ and $i \in I$

$V(p) = 2\mathbb{N}$

Then, for each world w, and each j, if p_0 ($\neg p_0$) holds at every world in R_j (w), then $\neg p_0$ (p_0) holds at every world in $R_{j+1}(w)$, and so both $\mathbf{A}^C_{\leq} p$ and $\mathbf{A}^C_{\leq} \neg p$ are true at w. As a corollary, since every cell is consistent, the acceptance of two propositions does not imply the acceptance of their conjunction. As to the relation of acceptance and classical rejection, all combinations are now allowed.

3.5. *Completeness of the perspectival systems*

In this final section, we will show that the system obtained by combining absolute, critical and ordinal acceptance is complete with respect to the perspectival semantics. To the best of our knowledge, the proof is original. It exploits a canonicity argument and it is such that proofs of soundness and completeness for the three component systems can be readily extracted from it.

The system **PUCO** of ordinal acceptance with absolute and critical modalities is a perspectival system characterized by rule $\mathbf{R_U}$, and the following axioms:

$\mathbf{K_U}$: $\mathbf{A}^U(\varphi \to \psi) \to (\mathbf{A}^U \varphi \to \mathbf{A}^U \psi)$	$\mathbf{I_C}$: $\mathbf{A}^C \varphi \to \mathbf{A}\varphi \wedge \neg\mathbf{A}\neg\varphi$
$\mathbf{1_U}$: $\mathbf{A}^U \varphi \to \mathbf{A}^C \varphi$	$\mathbf{1_O}$: $\mathbf{A}\varphi \vee \varphi \leq \psi$
$\mathbf{2_U}$: $\mathbf{A}^U(\varphi \to \psi) \to \varphi \leq \psi$	$\mathbf{2_O}$: $\varphi \leq \psi \to (\mathbf{A}\varphi \to \mathbf{A}\psi)$
$\mathbf{K_C}$: $\mathbf{A}^C(\varphi \to \psi) \to (\mathbf{A}^C \varphi \to \mathbf{A}^C \psi)$	$\mathbf{3_O}$: $\varphi \leq \psi \wedge \psi \leq \psi' \to \varphi \leq \psi'$

$\mathbf{R_A}$, $\mathbf{R_C}$, $\mathbf{R_O}$ are now derivable and $\mathbf{2_U}$ together with $\mathbf{2_O}$ implies $\mathbf{I_U}$.

Definition 9. A model for the previous system is a tuple

$\langle W, I, R, \{\leq_w\}_{w \in W}, \{R_i\}_{i \in I}\rangle$ where

(1) W is a non-empty set of epistemic worlds

(2) I is a non-empty set of perspectives

(3) $R : W \to \wp(W)$ satisfying $R(w) \cap R_i(w) \neq \varnothing$, for all $i \in I$

(4) $\leq_w$ is a preorder defined on I, for each $w \in W$

(5) $R_i : W \to \wp(W)$ is an accessibility function, for all $i \in I$

(6) $V : P \to \wp(W)$ is a modal valuation function

We get completeness by constructing a canonical model and proving a canonicity lemma and a truth lemma. Let

- W be the set of all the maximally **PUCO**-consistent sets of formulas;
- I is the set of all the **PUCO**-formulas.

Hence, indices in I and formulas coincide. For $w \in W$, let

(1) $w/U = \{\varphi \mid \mathbf{A}^U \varphi \in w\}$

(2) $w/C = \{\varphi \mid \mathbf{A}^C \varphi \in w\}$

(3) $w/i = \begin{cases} \{\varphi \mid w/U,\ i\ \varphi\} & \textit{if } \mathbf{A}i \in w \\ w/U & \textit{if } \mathbf{A}i \notin w \end{cases}$

Note that w/U, w/C and w/i are closed sets. Indeed, suppose $w/U \vdash_{\mathbf{PUCO}} \varphi$. Then $w \vdash_{\mathbf{PUCO}} \mathbf{A}^U \varphi$, and so $\mathbf{A}^U \varphi \in w$, since w is maximally consistent. Thus, $\varphi \in w/C$, by the definition of w/C. Similarly for w/C and w/i.

Lemma 1. $w/i = \cap \{v \in W \mid w/i \subseteq v\}$ *and* $w/C = \cap \{v \in W \mid w/C \subseteq v\}$.

Proof. Since both w/C and w/i are closed sets, they coincide with the intersection of all the maximally consistent sets that contain them.

Lemma 2. $\varphi \in w/i \Rightarrow \mathbf{A}\varphi \in w$.

Proof. Suppose $\varphi \in w/i$. If $\mathbf{A}i \in w$, then $\mathbf{A}^U (i \rightarrow \varphi) \in w$, by the definition of w/i. Thus, $(\mathbf{A}i \rightarrow \mathbf{A}\varphi) \in w$, by **IU**, and so $\mathbf{A}\varphi \in w$. If $\mathbf{A}i \notin w$, then $\mathbf{A}^U \varphi \in w$, and so $\mathbf{A}\varphi \in w$, again by $\mathbf{I_U}$.

Definition 10. The canonical model is the tuple

$\langle W, I, R, \{\leq_w\}_{w\in W}, \{R_i\}_{i\in I}, V\rangle$ such that

(i) $R(w)$ is such that $v \in R(w) \Leftrightarrow w/C \subseteq v$;

(ii) $\leq_w$ is such that $j \leq_w i \Leftrightarrow (j \leq i) \in w$;

(iii) $R_i(w)$ is such that $v \in R_i(w) \Leftrightarrow w/i \subseteq v$;

(iv) V is such that $w \in V(pi) \Leftrightarrow pi \in w$.

Some preliminary facts.

Fact 1. $\mathbf{A}\varphi \in w \Leftrightarrow \exists i(\varphi \in w/i)$.

Proof. Suppose $\mathbf{A}\varphi \in w$. Then $\varphi \in w/\varphi$, by definition of w/φ. Suppose $\exists i(\varphi \in w/i)$. Then $\mathbf{A}\varphi \in w$, by lemma 2. □

Fact 2. $\mathbf{A}^U \varphi \in w \Leftrightarrow \forall i(\varphi \in w/i)$.

Proof. Suppose $\mathbf{A}^U \varphi \in w$. Then $\varphi \in w/U$, and so $\varphi \in w/i$, by definition of w/U and w/i. Suppose $\forall i(\varphi \in w/i)$. Then $\varphi \in w/\top$, and so $\varphi \in w/U$, since $w/\top = w/U$ by the definition of $w/\top$. □

Fact 3. $\varphi \in w/i \Leftrightarrow (i \leq \varphi) \in w$.

Proof. Suppose $\varphi \in w/i$. Then $\mathbf{A}U\,(i \rightarrow \varphi) \in w$, by the definition of w/i, and so $(i \leq \varphi) \in w$, by $\mathbf{I_O}$. Suppose $\varphi \notin w/i$. Then $\mathbf{A}i \in w$, by $\mathbf{1_O}$, and so $(i \leq \varphi) \notin w$, by $\mathbf{2_O}$. □

Fact 4. $\exists i((\varphi \leq i) \in w \text{ and } \psi \in w/i) \Leftrightarrow (\varphi \leq \psi) \in w$.

Proof. Suppose $(\varphi \leq i) \in w$ and $\psi \in w/i$. Then $(\varphi \leq i) \in w$ and $(i \leq \psi) \in w$, by fact 2, and so $(\varphi \leq \psi) \in w$, by $\mathbf{3_O}$. Suppose $(\varphi \leq \psi) \in w$. Since $(\psi \leq \psi) \in w$, $\psi \in w/\psi$ by fact 2, and so $(\varphi \leq \psi) \in w$ and $\psi \in w/\psi$. □

PART I: Canonicity Lemma

The canonical model is a model for the logic. That $\leq_w$ is reflexive and transitive follows from $\mathbf{R_U}$, $\mathbf{2_U}$, and $\mathbf{3_O}$. That $R(w) \cap R_i(w) \neq \varnothing$, for all $i \in I$, is consequence of $\mathbf{I_C}$. Indeed, since $\mathbf{A}^C\,\varphi \rightarrow \neg\mathbf{A}\neg\varphi \in w$, by $\mathbf{I_C}$, $w/C \cup w/i$ is consistent, for all $i \in I$. Therefore $w/C \cup w/i$ is included in some maximally consistent set v, so that $v \in R(w) \cap R_i(w)$.

PART II: Truth Lemma

Case 1: $M, w \vDash \mathbf{A}\varphi \Leftrightarrow \mathbf{A}\varphi \in w$.
Proof. By the definition of $\vDash$ and IH:

$M, w \vDash \mathbf{A}\varphi \Leftrightarrow \exists i \in I \forall v(v \in R_i(w) \Rightarrow \varphi \in v)$

$M, w \vDash \mathbf{A}\varphi \Leftrightarrow \exists i \in I(\varphi \in w/i)$, by lemma 1

$M, w \vDash \mathbf{A}\varphi \Leftrightarrow \mathbf{A}\varphi \in w$, by fact 1 □

Case 2: $M, w \vDash \mathbf{A}^U\,\varphi \Leftrightarrow \mathbf{A}^U\,\varphi \in w$.
Proof. By the definition of $\vDash$ and IH:

$M, w \vDash \mathbf{A}^U\,\varphi \Leftrightarrow \forall i \in I\,\forall v(v \in R_i(w) \Rightarrow \varphi \in v)$

$M, w \vDash \mathbf{A}^U\,\varphi \Leftrightarrow \forall i(\varphi \in w/i)$, by lemma 1

$M, w \vDash \mathbf{A}^U\,\varphi \Leftrightarrow \mathbf{A}^U\,\varphi \in w$, by fact 2 □

Case 3: $M, w \vDash \mathbf{A}^C\,\varphi \Leftrightarrow \mathbf{A}^C\,\varphi \in w$.
Proof. By the definition of $\vDash$ and IH:

$M, w \vDash \mathbf{A}^C\,\varphi \Leftrightarrow \forall v(v \in R(w) \Rightarrow \varphi \in v)$

$M, w \vDash \mathbf{A}^C\,\varphi \Leftrightarrow \varphi \in w/C$, by lemma 1

$M, w \vDash \mathbf{A}^C\,\varphi \Leftrightarrow \mathbf{A}^C\,\varphi \in w$, by def. w/C □

Case 4: $M, w \vDash \varphi \leq \psi \Leftrightarrow (\varphi \leq \psi) \in w$.
Proof. By the definition of $\vDash$ and IH:

$M, w \vDash \varphi \leq \psi \Leftrightarrow \forall j(\varphi \in w/j \Rightarrow \exists i(j \leq_w i \text{ and } \psi \in w/i))$

$j \leq_w i \Leftrightarrow (j \leq i) \in w$, by def. $\leq_w$

$\psi \in w/i \Leftrightarrow (j \leq \psi) \in w$, by fact 3

$M, w \vDash \varphi \leq \psi \Leftrightarrow \forall j(\varphi \in w/j \Rightarrow \exists i((j \leq i) \in w \text{ and } (j \leq \psi) \in w))$

$M, w \vDash \varphi \leq \psi \Leftrightarrow \forall j(\varphi \in w/j \Rightarrow (j \leq \psi) \in w)$, by fact 4

$M, w \vDash \varphi \leq \psi \Leftrightarrow \forall j((j \leq \varphi) \in w \Rightarrow (j \leq \psi) \in w)$, by fact 3

$M, w \vDash \varphi \leq \psi \Leftrightarrow (\varphi \leq \psi) \in w$, by the reflexivity and transitivity of $\leq_w$ $\square$

4. Conclusion

In this paper, we have presented systems of perspectival attitudes where phenomena of paraconsistency and paracompleteness can be handled and shown that they are adequate with respect to a semantical framework in which the epistemic space is subdivided in different cells. We have also indicated how these systems can be used for interpreting some interesting epistemological theses. A more general study should take into account at least three kinds of developments. The first is an analysis of the dynamics of the systems, along the lines proposed in [6] for the two cells space. The second one concerns the role of the sources in generating epistemic states and the connection with justification logic. The final one concerns the role of subject matters in ordering the set of the epistemic cells and the connection with inquisitive logic. These developments are left for future work.

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