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ON DISTINGUISHING PROOF-THEORETIC CONSEQUENCE FROM DERIVABILITY

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1. Introduction

According to the common conception of *logical consequence* (see, for example, the recent book [20]), it can be defined[1] in two main ways:

Model-theoretically: For a suitable notion of a model, consequence is taken, following Tarski's definition [32], to be *preservation* (also called *propagation*[2] *or transmission*) of truth over all models. After defining $\mathcal{M} \models \varphi$, namely truth (or satisfaction) of a formula φ in a model $\mathcal{M}$ (usually by recursion on the structure of φ), and after naturally extending to collections[3] (most often sets, as in classical logic) of formulas Γ by

$$\mathcal{M} \models \Gamma \text{ iff } \mathcal{M} \models \varphi \text{ for every } \varphi \in \Gamma \tag{1}$$

one defines logical consequence, also denoted '$\models$', by

$$\Gamma \models \varphi \text{ iff for every model } \mathcal{M}, \text{ if } \mathcal{M} \models \Gamma \text{ then } \mathcal{M} \models \varphi \tag{2}$$

This propagation of truth is taken to be endowed with the following characteristics:

necessity: Here manifested by the universal quantification over *all* models.

formality: The truth in models is in virtue of (logical) *form*, depending[4] on the *logical constants* and their arrangement.

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[1] For simplicity, I will not consider here *multiple-conclusion* (Scott) consequence.

[2] I will ignore here for simplicity dual definitions as backward transmission of falsity, though in certain cases it gives rise to different conceptions of logical consequence (cf. [34]).

[3] I consider only *finite* Γs here.

[4] The idea may be applied also to more general notions of consequence, that do not depend on the logicality of terms. See [30] for an argument to this effect in a model-theoretic setting.

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Proof-Theoretically: For a suitable proof-system $\mathcal{N}$ (which I will take here as a natural-deduction proof-system), consequence is taken as *derivability* in $\mathcal{N}$, denoted '$\vdash_{\mathcal{N}}$'; after defining derivations $\mathcal{D}$ in $\mathcal{N}$, one defines

$$\Gamma \vdash_{\mathcal{N}} \varphi \text{ iff there exists an } \mathcal{N}-\text{ derivation } \mathcal{D} \text{ of } \varphi \text{ from (open) assumptions } \Gamma \tag{3}$$

Typically, (strong) soundness and completeness theorems, that is

$$\Gamma \vdash_{\mathcal{N}} \varphi \text{ iff } \Gamma \vDash \varphi \tag{4}$$

establish the coextensiveness of those two notions of consequence. For example, this relation holds in classical 1st-order logic, where $\mathcal{N}$ is taken as Gentzen's *NK* system, and for Intuitionistic logic (say, with Kripke models), where $\mathcal{N}$ is taken as Gentzen's *NJ* system. According to this view, logical consequence is closely related[5] to the *validity* of the argument from the assumptions Γ to the conclusion φ.

The idea that logical consequence involves preservation of *something*, not necessarily of truth, has been suggested by many. Some examples:

Information: For example, in [19] and [1], propagation of the information (in a situation) is underlying consequence of Relevant Logic. Also, in [2], propagation of structural features of information underlies logical consequence.

Ambiguity: This notion[6] is due to [3], suggesting to treat a proposition p *ambiguously* as two different propositions, p_t and p_f. A *measure* of ambiguity of an inconsistent Γ is defined as the minimal number of proposition in Γ the treatment of which as ambiguous renders Γ consistent. In [3], propagation of ambiguity is used for defining consequence for paraconsistent logics. In [6], consequence for Relevant Logics is defined by preservation of ambiguity.

Precisification: In the context of *vagueness*, there is an appeal to *super-truth* (i.e., truth in all *precisifications*) in defining logical consequence (see [34]). One formulation of logical consequence is by preservation[7] (or propagation) of super-truth.

A natural question arising is, what is common to all the "things" being suggested as preserved, or propagated, by the various consequence relations

[5] A notable exception is [31].

[6] Note that 'ambiguity' here a technical term, different in meaning from the usual use of this word, namely *having* two (or more) meanings *in* some theory of meaning. Thus, ambiguity of p here is not an intrinsic attribute of p.

[7] As shown in [34], there are several natural variants for the meaning of preservation in this context.

mentioned above. I want to argue that they all serve (either explicitly or implicitly) as *central concepts on which theories of meaning are based*.

Two of the main theories of meaning are the following.

Model-Theoretic Semantics (MTS): The central concept of MTS is *truth* (in arbitrary models). Meaning is defined[8] as truth-conditions.

Proof-Theoretic Semantics (PTS): The central concept of PTS is *proof*, or more generally, *canonical derivation* (explained below) in appropriate meaning-conferring proof-systems. Meaning is defined as *determined* by the rules of the meaning-conferring system. The definition is either implicit, or explicit as in [8]. For a detailed account of PTS see [12].

The other propagated "things" mentioned above have a similar role in theories of meaning for Relevant Logic, general paraconsistent logics and for languages with vagueness. Other kinds of theories of meanings include, for example, Game-Theoretic semantics and Category-Theoretic semantics, which I will not consider here.

Consequently, I suggest the following informal principle.

meaning-based logical consequence: In a theory of meaning $\mathcal{T}$, logical consequence is based on the propagation of the central concept of $\mathcal{T}$.

By being faithful to a theory of meaning I mean relating the notion of φ being a logical consequence (logically following from) Γ on the meanings of φ and Γ as determined by that theory of meaning.

In this paper I want to argue that, in spite of the coextensiveness for 1st-order logic (and many other logics) of derivability and preservation of truth in models (as in (4)), if one adheres to the proof-theoretic semantics theory of meaning then (3) is not the right definition of proof-theoretic consequence. While (2) is faithful to the usual model-theoretic conception of meaning, (3) is not faithful to the PTS conception of meaning.

I suggest another definition of proof-theoretic consequence that *is* faithful to PS, sometimes (i.e., for some logics) coextensive with derivability, and sometimes – not.

2. Meaning according to proof-theoretic semantics

Ever since Gentzen's casual remark in [17], p. 80)

> ... The introductions represent, as it were, the 'definitions' of the symbol concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. …

[8] I will not distinguish here general model theoretic meaning and truth-conditional meaning.

(where 'introduction' and 'elimination' refer to rules of a natural-deduction proof-system, to be abbreviated as *I*-rules and *E*-rules, respectively), the *meaning* of a logical constant (connective, quantifier, etc.) is taken to be determined (or fixed) *solely* in proof-theoretic terms, which is the essence of PTS. According to Gentzen, it is the *I*-rules that determine the proof-theoretic meanings, an approach adhered to by mainstream PTS, e.g. Dummett (for example, [4]), Prawitz (for example, [25]), Tennant (for example, [33]) and many others. Call this the *I*-view of PTS. There is also a dual approach, basing meaning on *E*-rules; see [8] for a more detailed discussion of those different views. I will retain the *I*-view in this paper.

In the PTS literature, meaning is conceived as *implicitly defined*[9] by the *I*-rules, not appealing to any proof-theoretic *semantic value* as an *explicit* definition. However, for several purposes, for example the construction of a grammar, such an explicit definition is needed. In model-theoretic semantics (MTS), where meaning is taken as truth-conditions (in arbitrary models), the explicit definition of meaning is usually taken as the following semantic value.

$$[\![\varphi]\!] = \{\mathcal{M} \mid \mathcal{M} \models \varphi\} \tag{5}$$

Note that MTS is not interested in assigning semantic values to the logical constants themselves, so no definitions of $[\![\wedge]\!]$, $[\![\forall]\!]$ and the like are considered.

But what exactly can be taken as an *explicitly* defined *proof-theoretic* semantic value within PTS as the result of the determination of meaning via the meaning-conferring *I*-rules?

In [13], and subsequently in [8], such a proof-theoretic semantic value is proposed as an explicit definition of meaning. I recapitulate this proposal below.

An important concept in PTS is that of a *canonical proof* in $\mathcal{N}$, where a proof of some compound φ is a derivation of φ from no open assumptions (an empty Γ). I take here derivations to have the Prawitz-style [24] tree-structure.

Definition 1 (canonical proof). *A proof $\mathcal{D}$ in $\mathcal{N}$ is canonical iff the last rule applied in $\mathcal{D}$ is an I-rule.*

To define proof-theoretic consequence, there is a need to extend canonicity to arbitrary $\mathcal{N}$-derivations, including ones with open assumptions.

[9] There are approaches, though, like Engel [5], Hjortland [18] and Garson [15, 16] according to which the goal of PTS is viewed as the ability to reconstruct (or recover) the truth-tables (also for multi-valued logics) from the *ND*-rules. It is claimed by these authors that PTS achieves its goal of determining meaning *in virtue* of this ability to recover truth-conditions. Thus, we are back to truth in a model as the meaning determined. While seeing this feature of proof-system as interesting on its own, I consider this approach to the goal of PTS as beating the purpose of PTS and am not going to relate to it any further.

Following [7] and [10], I suggest the following definition. Its justification is discussed in detail in [10].

Definition 2 (canonical derivation from open assumptions). *A $\mathcal{N}$-derivation $\mathcal{D}$ for $\Gamma \vdash \psi$ (for a compound ψ) is* canonical *iff it satisfies one of the following two conditions.*

- *The last rule applied in $\mathcal{D}$ is an I-rule (for the main operator of ψ).*
- *The last rule applied in $\mathcal{D}$ is an assumption-discharging E-rule, the major premise of which is some φ in Γ, and its encompassed sub-derivations $\mathcal{D}_1$, ..., $\mathcal{D}_n$ are all canonical derivations of ψ.*

Denote by $\vdash^c_{\mathcal{N}}$ canonical derivability in $\mathcal{N}$. Let $[\![\varphi]\!]^c_\Gamma$ the (possibly empty) collection of canonical derivations of φ from Γ, and $[\![\varphi]\!]^*_\Gamma$ the (also possibly empty) collection of all derivations of φ from Γ.

For Γ empty, the definition reduces to Definition 1 of a canonical proof. Note the recursion involved in this definition. The important observation regarding this recursion is that it always terminates via the first clause, namely by an application of an *I*-rule. I refer to such an application of an *I*-rule as an *essential* application, the outcome of which is propagated throughout the canonical derivation. Note that each sub-derivation $\mathcal{D}_i$ may end with an essential application of an *I*-rule, thus having “parallel” essential applications of that rule.

Note also that, similarly to the case of canonical proofs, there are no canonical derivations for *an atomic sentence*, which by definition has no introducible operators. Traditionally, the PTS programme views the meaning of an atomic sentence to be *given*, possibly from outside the meaning-conferring proof system. To overcome this non-specificity, I take[10] here the rule of assumption $\Gamma, p \vdash p$ to constitute the canonical way an atomic sentence is introduced into a derivation.

To realize the role of canonicity in the forthcoming definition of reified proof-theoretic meanings (according to the *I*-view), consider the following example derivation in, say, intuitionistic propositional logic (see Figure 1).

$$\frac{\alpha \quad (\alpha \rightarrow (\varphi \wedge \psi))}{\varphi \wedge \psi} (\rightarrow E) \tag{6}$$

This is a derivation of a conjunction – but not a canonical one, as it does not end with an application of $(\wedge I)$, nor does it have an essential application of it. Thus, the conjunction here was *not* derived according to its meaning! As far as this derivation is concerned, it could mean anything, e.g., disjunction.

[10] This possibility was suggested to me by Andreas Fjellstad (p.c.).

On the other hand, the following example derivation *is* according to the conjunction's meaning, being canonical.

$$\dfrac{\dfrac{\alpha \ \ (\alpha \rightarrow \varphi)}{\varphi}(\rightarrow E) \quad \dfrac{\beta \ \ (\beta \rightarrow \psi)}{\psi}(\rightarrow E)}{\varphi \wedge \psi}(\wedge I) \tag{7}$$

I now turn to the definition of what I take to be the reified proof-theoretic meaning determined by a meaning-conferring *ND*-system (see [8] for a more detailed discussion). The definition of meaning presented below corroborates, and makes precise, a common observation about PTS (cf. for example, [21, 23] for recent discussions): sentential meanings, while being compositional, are not *directly* compositional. The reason *here* is, that on the *I*-based approach, an *I*-canonical $\mathcal{N}$-derivation may have as *immediate* sub-derivations, premises of the essential application of the *I*-rule, *arbitrary* $\mathcal{N}$-derivations, not just canonical derivations. Due to the recursive nature of *L*, where sentences may be constituents of other sentences, there is an "interfacing" function applied to the meaning of a sentence, a collection of *I*-canonical derivations, yielding the collection of *all* the derivations (from the same context Γ) of that sentence. However, this does not mean that sentential meanings do not depend on their component phrases: only that the dependence is somewhat indirect. This is reflected here by having the *ND*-rules determining *two* semantic values, matching Dummett's distinction ([4], p. 48) between *(assertoric) contents* and *ingredient sense*. The *content* of φ is the meaning of φ "in isolation", on its own. The *ingredient sense* of φ is what φ contributes to the meaning of an ψ, in which φ occurs as a sub-expression, a component. See [9] for a more detailed discussion of this issue.

For a formula φ, the one semantic value, corresponding to Dummett's content, is its *contributed* value $[\![\varphi]\!]$, serving also as its *meaning*; the other, auxiliary, semantic value, corresponding to Dummett's ingredient sense, is its *contributing* semantic value $[\![\varphi]\!]^*$, used when φ is part of some larger expression ψ. As shown below, $[\![\varphi]\!]^*$ can be recovered from $[\![\varphi]\!]$, but the presentation seems clearer if the two are thought of as if independent.

I take the *I*-based sentential meanings (i.e., contributed sentential semantic values) of compound sentences in *L* to originate from canonical $\mathcal{N}$-derivations. On the other hand, the contributing semantic sentential value (for both atomic and compound sentences) is taken as the collection of *all* (not just canonical) $\mathcal{N}$-derivations. I emphasize once again, that it is *sentential meanings* that are *directly* explicitly defined by the *ND*-system, whereas meanings of connectives are *extracted* from compound sentential meanings as shown in [14]. In the sequel, I refer to functions from contexts to collections of *ND*-derivations as *contextualized* functions. To avoid cluttering the notation, I overload the meaning of $[\![\ldots]\!]$, leaving it to context to determine the variant meant. The notation used is $\lambda\Gamma.[\![\ldots]\!]$, meaning the appropriate (possibly empty) collection of derivations from an argument context Γ.

Definition 3 (sentential semantic values). *For a compound $\varphi \in L$, its* meaning *(contributed semantic value) $[\![\varphi]\!]$, is given as follows.*

$$[\![\varphi]\!] = \lambda\Gamma.[\![\varphi]\!]^{c}_{\Gamma} \tag{8}$$

For an arbitrary (atomic or compound) $\varphi \in L$, its *contributing semantic value* is given as follows.

$$[\![\varphi]\!]^{*} = \lambda\Gamma.[\![\varphi]\!]^{*}_{\Gamma} \tag{9}$$

In order to relate the auxiliary contributing semantic value to the contributed semantic value, the meaning, the following auxiliary function, **ex** (for 'exportation') is defined, retrieving the collection of all derivations of φ from Γ (if any) from the canonical ones.

Definition 4 (exportation function).

$$\mathbf{ex}([\![\varphi]\!]) =^{\text{df.}} \lambda\Gamma.[\![\rho([\![\varphi]\!])]\!]^{*}_{\Gamma} \tag{10}$$

Here[11] $\rho([[\varphi]]) = \varphi$.

The following immediately holds, relating the auxiliary semantic value to meanings.

$$[\![\varphi]\!]^{*} = \mathbf{ex}([\![\varphi]\!]) \tag{11}$$

By abuse of notation for convenience, I use $\rho([\![\varphi]\!])$ also for atomic φ, even though $[\![\varphi]\!]$ is not based on application of *I*-rules. Note that the denotational meaning of φ is *a proof-theoretic object*, a contextualized function from contexts to the collection of (*I/E*-canonical) derivations of φ from that context, not to be confused with model-theoretic denotations (of truth-values, in the propositional logic case).

3. Proof-theoretic consequence

The definition of *proof-theoretic consequence* (pt-consequence) rests on the notion of *grounds for assertion* for φ, closely related to $[\![\varphi]\!]$, the reified meaning of φ.

[11] I assume here that there is at most one φ s.t. $[\![\varphi]\!] = \lambda\Gamma.\emptyset$

Definition 5 (grounds for assertion).

$$GA\,[\![\varphi]\!] = \{\Gamma \mid \Gamma \vdash^{c} \varphi\} \tag{12}$$

Thus, any Γ that canonically derives φ serves as grounds for assertion of φ. For the methodological role of this concept in the theory of meaning adhered to by PTS, see [4]. The notion of grounds considered here is different than, though in the same spirit as, the grounds considered by Prawitz in [26]. The grounds here are formal constructs, collections of sentences (assumptions) canonically deriving a sentence (conclusion). On the other hand Prawitz considers grounds as *mental* counterparts, associated with *possession* of the formal grounds and justifying the *epistemic acts* of inference, assertion (and in the bilateral case [29], also denial).

Definition 3 yields a *very fine-grained* notion of meaning. One may wish to coarsen it on certain occasions, still retaining its spirit. This is discussed in detail in [11], where a relaxation by means of an equivalence relation induced by *sameness of grounds* is suggested. If we let $\varphi \equiv_{GA} \psi$ iff $GA\,[\![\varphi]\!] = GA\,[\![\varphi]\!]$, we have for example $\varphi \vee \psi \equiv_{GA} \psi \vee \varphi$, and $\varphi \vee (\psi \vee \chi) \equiv_{GA} (\varphi \vee \psi) \vee \chi$. Still, $\varphi \rightarrow \psi \not\equiv_{GA} \neg\varphi \vee \psi$, for example, So sameness of meaning does not collapse to boolean equivalence.

Next, I need an extension of the definition of the grounds for assertion of a single sentence to grounds for the collective assertion of a finite, non-empty collection of sentences, say Δ. There are two[12,13] natural possibilities here, distinguished by the way assumptions are combined.

common grounds:

$$GA_c\,[\![\Delta]\!] =^{\text{df.}} \cap_{\psi \in \Delta} GA\,[\![\varphi]\!] \tag{13}$$

This is a *conjunctive* combination of the grounds of the individual assumptions in Γ, the same as the one in (1) in the MTS case.

joint grounds:

$$GA_j\,[\![\Delta]\!] =^{\text{df.}} \circ_{\psi \in \Delta} GA\,[\![\varphi]\!] \tag{14}$$

where '$\circ$' is *fusion*, known also as intensional conjunction. Since fusion is known to be commutative and associative, it is well-defined to apply it to a set of formulas, yielding the fusion of all formulas in the set; if the set is empty, the unit elemen $\top$ results. In the formulation of

[12] In contrast to (1) in MTS, being traditionally the only way, the conjunctive way, of combining assumptions.

[13] A similar distinction between ways of combining assumptions can be found in [22], in the interpretation of a (multiple-conclusions) sequent as an object-language formula.

multiplicative rules below, in order to avoid notational clutter I keep the context as Γ, but interpreted as $\circ_\Gamma =^{df.} \circ_{\psi\in\Gamma}\psi$.

In both cases, $GA[\![\{\varphi\}]\!] =^{df.} GA[\![\varphi]\!]$.

The difference between conjunction and fusion originates in the *I*-rules for conjunction being *additive* (or shared context), while the *I*-rules for fusion are *multiplicative* (context free), as evident from their *I*-rules.

$$\frac{\Gamma\vdash\varphi \quad \Gamma\vdash\psi}{\Gamma\vdash\varphi\wedge\psi}\,(\wedge I) \qquad \frac{\Gamma_1\vdash\varphi \quad \Gamma_2\vdash\psi}{\Gamma_1\circ\Gamma_2\vdash\varphi\circ\psi}\,(\circ I) \tag{15}$$

By the above remark on the interpretation of the context, it is the fusion of all the formulas in the context, and a unity formula T if the context is empty.

Recall that conjunction has two general-elimination rules (GE-rules), given by

$$\frac{\Gamma\vdash\varphi\wedge\psi \quad \Gamma,\varphi\vdash\chi}{\Gamma\vdash\chi}\,(\wedge GE_1) \qquad \frac{\Gamma\vdash\varphi\wedge\psi \quad \Gamma,\psi\vdash\chi}{\Gamma\vdash\chi}\,(\wedge GE_2) \tag{16}$$

that can be simplified (assuming the structural rule of *weakening*) to the more familiar

$$\frac{\Gamma\vdash\varphi\wedge\psi}{\Gamma\vdash\varphi}\,(\wedge E_1) \qquad \frac{\Gamma\vdash\varphi\wedge\psi}{\Gamma\vdash\psi}\,(\wedge E_2) \tag{17}$$

Thus, conjunction projects each conjunct *separately*.

On the other hand, fusion has a single GE-rule, given by

$$\frac{\Gamma\vdash\varphi\circ\psi \quad \Gamma,\varphi,\psi\vdash\chi}{\Gamma\vdash\chi}\,(\circ GE) \tag{18}$$

which does not simplify, hence projecting *both* conjuncts *simultaneously*. When I want to remain neutral regarding this difference in combining grounds, I will speak generically of “collective grounds”, with a generic notation $GA[\![\Delta]\!]$ (without a qualifying superscript).

Based on the definitions of grounds combination, I define two notions of proof-theoretic *consequence* (pt-consequence).

Definition 6 (proof-theoretic consequences). *Let* Γ, $\psi \in L$.

conjunctive pt-consequence: ψ *is a* conjunctive proof-theoretic consequence *of* $\Gamma(\Gamma \Vdash_c \psi)$ *iff* $GA_c[\![\Gamma]\!] \subseteq GA[\![\psi]\!]$.

fused pt-consequence: ψ *is a* fused proof-theoretic consequence *of* $\Gamma(\Gamma \Vdash_j \psi)$ *iff* $GA_j[\![\Gamma]\!] \subseteq GA[\![\psi]\!]$.

Thus, both pt-consequences are based on *grounds propagation*: every collective grounds for collectively asserting all of Γ (depending on the mode of combination of grounds employed) are already grounds for asserting ψ.

By this definition, ψ is a pt-consequence of Γ *according to ψ's meaning* as pt-consequence involves canonical derivability.

A difference between the two ways of combining grounds is that the second captures a notion of "relevant grounds", while the first does not. Thus, the pt-consequence '$\Vdash_c$', resting on common grounds, leads to a classical-like consequence relation, whereby a contradicting Δ (with empty common grounds) has every φ as its consequence (*explosion*), and a valid ψ is a pt-consequence of every Δ. On the other hand, pt-consequence '$\Vdash_j$', resting on joint grounds, leads to a relevant-like consequence relation. For a detailed discussion of this point in MTS see [28]. I personally prefer the second definition. However, this point is orthogonal to the main purpose of this paper, the separation of pt-consequence from derivability.

Proposition 1 (properties of conjunctive pt-consequence). *pt-consequence satisfies the following expected properties.*

1. $\varphi \Vdash_c \varphi$
2. *If* $\Gamma \Vdash_c \varphi$ *and* $\varphi, \Gamma' \Vdash_c \psi$ *then* $\Gamma, \Gamma' \Vdash_c \psi$.
3. *For every* Γ', *if* $\Gamma \Vdash_c \varphi$ *then also* $\Gamma, \Gamma' \Vdash_c \varphi$.

In other words, conjunctive pc-consequence is reflexive and transitive since inclusion is[14]. In addition, conjunctive pt-consequence is monotonic.

As for the fused pt-consequence, while also being reflexive and transitive, it is *not monotonic*. For example, $\varphi \Vdash_j \varphi$, but $\varphi, \varphi \nVdash_j \varphi$, since '$\circ$' is known not to be idempotent.

As is expected from a notion of consequence, pt-consequence has a modal force of necessity, as manifested by the universal quantification on grounds, the counterpart of the universal quantification over models in the MTS definition of consequence.

Example 1 (pt-consequence:). *I show that in intuitionistic propositional logic (specified below)*

$$\varphi \to (\psi \to \chi) \Vdash \varphi \wedge \psi \to \chi \tag{19}$$

As we are dealing with a single assumption, the mode of combining assumptions does not matter here.

[14] Note that the reflexivity is in virtue of the reflexivity of inclusion, and *not* in virtue of axioms of the form $\varphi \vdash_{\mathcal{N}} \varphi$ which are not canonical derivations (except for atomic φ).

Suppose that $\Gamma \in GA[\![\varphi \rightarrow (\psi \rightarrow \chi)]\!]$. *A canonical derivation of* $\varphi \rightarrow (\psi \rightarrow \chi)$ *from* Γ *ends with* $(\rightarrow I)$. *Therefore*

$$\Gamma, \varphi \vdash \psi \rightarrow \chi \tag{20}$$

which, in turn, implies

$$\Gamma, \varphi, \psi \vdash \chi \tag{21}$$

But then, $\Gamma, \varphi \wedge \psi \vdash \chi$ *too, by which* $\Gamma \vdash^c \varphi \wedge \psi \rightarrow \chi$, *implying* $\Gamma \in GA[\![\varphi \wedge \psi \rightarrow \chi]\!]$; *i.e. the grounds* Γ *have been propagated from assumption to conclusion, establishing pt-consequence.*

A simple example of fused pt-consequence is the following.

Example 2 (fused pt-consequence):

$$\varphi, \psi \Vdash_j \varphi \circ \psi \tag{22}$$

If $\Gamma_1 \in GA[\![\varphi]\!]$ *and* $\Gamma_2 \in GA[\![\psi]\!]$, *then* $\Gamma_1 \circ \Gamma_2 \in GA[\![\varphi, \psi]\!]$. *But* $\Gamma_1 \circ \Gamma_2 \in GA[\![\varphi \circ \psi]\!]$ *by one application of* $(\circ I)$ *above.*

Definition 7 (smoothness). *An ND-system* $\mathcal{N}$ *is* proof-theoretically smooth *iff for every* Γ *and every* compound φ:

$$\Gamma \vdash_{\mathcal{N}} \varphi \text{ iff } \Gamma \vdash^c_{\mathcal{N}} \varphi \tag{23}$$

That is, a compound φ is $\mathcal{N}$-derivable from Γ iff it is canonically derivable. In other words, in a smooth $\mathcal{N}$, derivability *is* coextensive with pt-consequence. This is another formulation of Dummett's *Fundamental Assumption* (*FA*) [4], extended from proofs to derivations from open assumptions. It is also yet another view of *harmony* and *stability* [4], the properties seen by PTS as required for the ND-system qualifying as meaning conferring.

Proposition 2 (smoothness of *NJ*). *The ND-system NJ (for propositional intuitionistic logic, Figure 1) is proof-theoretically smooth.*

Proof. Suppose $\Gamma \vdash_{NJ} \varphi$ for a compound φ. I show $\Gamma \vdash^c_{NJ} \varphi$ by induction on the derivation.

The basis of the induction are identity derivations $\Gamma, \varphi \vdash_{NJ} \varphi$. Here it is known[15] that every such derivation can be *expanded* to a canonical derivation, eliminating φ by *E*-rules and reconstructing it via *I*-rules.

For the induction step, only derivations ending with an application of an *E*-rule need to be considered.

[15] This is the η-expansion in the simply-typed λ-calculus.

$$\frac{}{\Gamma,\ \varphi\vdash_{NJ}\varphi}(Ax)$$

$$\frac{\Gamma,\ \varphi\vdash_{NJ}\psi}{\Gamma\vdash_{NJ}(\varphi\rightarrow\psi)}(\rightarrow I)\quad \frac{\Gamma\vdash_{NJ}\psi\ \ \Gamma\vdash_{NJ}(\psi\rightarrow\varphi)}{\Gamma\vdash_{NJ}\varphi}(\rightarrow E)$$

$$\frac{\Gamma\vdash_{NJ}\varphi\ \ \Gamma\vdash_{NJ}\psi}{\Gamma\vdash_{NJ}(\varphi\wedge\psi)}(\wedge I)\quad \frac{\Gamma\vdash_{NJ}(\varphi\wedge\psi)}{\Gamma\vdash_{NJ}\varphi}(\wedge_1 E)\quad \frac{\Gamma\vdash_{NJ}(\varphi\wedge\psi)}{\Gamma\vdash_{NJ}\psi}(\wedge_2 E)$$

$$\frac{\Gamma\vdash_{NJ}\varphi}{\Gamma\vdash_{NJ}(\varphi\vee\psi)}(\vee_1 I)\quad \frac{\Gamma\vdash_{NJ}\psi}{\Gamma\vdash_{NJ}(\varphi\vee\psi)}(\vee_2 I)\quad \frac{\Gamma\vdash_{NJ}(\varphi\vee\psi)\ \ \Gamma,\varphi\vdash_{NJ}\chi\ \ \Gamma,\psi\vdash_{NJ}\chi}{\Gamma\vdash_{NJ}\chi}(\vee E)$$

$$\frac{\Gamma\vdash_{NJ}\bot}{\Gamma\vdash_{NJ}\varphi}(\bot E)$$

Figure 1: The propositional intuitionistic *NJ*-system

conjunction: Suppose

$$\frac{\Gamma\vdash_{NJ}\varphi\wedge\psi}{\Gamma\vdash_{NJ}\varphi}(\wedge E_1)$$

By the induction hypothesis on the premise, there is a canonical derivation

$$\frac{\Gamma\vdash_{NJ}\varphi\ \ \Gamma\vdash_{NJ}\psi}{\Gamma\vdash_{NJ}\varphi\wedge\psi}(\wedge I)$$

and by the induction hypothesis on $\Gamma\vdash_{NJ}\varphi$ also $\Gamma\overset{c}{\vdash}_{NJ}\varphi$.
The proof for ($\wedge E2$) is similar.

implication: Suppose

$$\frac{\Gamma\vdash_{NJ}\psi\ \ \Gamma\vdash_{NJ}\psi\rightarrow\varphi}{\Gamma\vdash_{NJ}\varphi}(\rightarrow E)$$

By the induction hypothesis on the minor premise, we get $\Gamma\overset{c}{\vdash}_{NJ}\psi$. By the induction hypothesis on the major premise, there is a canonical derivation $\Gamma\overset{c}{\vdash}_{NJ}\psi\rightarrow\varphi$. Therefore, there is a canonical derivation $\Gamma,\ \psi\overset{c}{\vdash}_{NJ}\varphi$. By composing the canonical derivations $\Gamma\overset{c}{\vdash}_{NJ}\psi$ and $\Gamma,\ \psi\overset{c}{\vdash}_{NJ}\varphi$ we get a canonical derivation $\Gamma\overset{c}{\vdash}_{NJ}\varphi$.

disjunction: Suppose[16]

$$\frac{\Gamma\vdash_{NJ}\psi\vee\chi\ \ \Gamma,\psi\vdash_{NJ}\varphi\ \ \Gamma,\chi\vdash_{NJ}\varphi}{\Gamma\vdash_{NJ}\varphi}(\vee E)$$

[16] I thank the anonymous referee for suggesting this proof, simpler than my original one.

By the induction hypotheses on the major premise, there is a canonical derivation $\Gamma\vdash^{c}_{NJ}\psi\vee\chi$. Thus, either there is a canonical derivation (∗) $\Gamma\vdash^{c}_{NJ}\psi$ or there exists a canonical derivation (∗∗) $\Gamma\vdash^{c}_{NJ}\chi$. Suppose w.l.o.g. that the former is the case. By the induction hypothesis on the first minor premise, there is a canonical derivation (∗∗∗) $\Gamma,\ \psi\vdash^{c}_{NJ}\varphi$. Composing (∗) with (∗∗∗) yields the required canonical derivation $\Gamma\vdash^{c}_{NJ}\varphi$.

$\bot$**:** Suppose that

$$\frac{\Gamma\vdash_{NJ}\bot}{\Gamma\vdash_{NJ}\varphi}(\bot E)$$

The proof proceeds by case analysis on φ.

$\varphi=\psi\wedge\chi$: The canonical derivation of φ is

$$\frac{\dfrac{\Gamma\vdash_{NJ}\bot}{\Gamma\vdash_{NJ}\psi}(\bot E)\qquad\dfrac{\Gamma\vdash_{NJ}\bot}{\Gamma\vdash_{NJ}\chi}(\bot E)}{\Gamma\vdash_{NJ}\psi\wedge\chi}(\wedge I)$$

$\varphi=\psi\rightarrow\chi$: The canonical derivation of φ is

$$\frac{\dfrac{\Gamma,\psi\vdash_{NJ}\bot}{\Gamma,\psi\vdash_{NJ}\chi}(\bot E)}{\Gamma\vdash_{NJ}\psi\rightarrow\chi}(\rightarrow I)$$

$\varphi=\psi\vee\chi$: The canonical derivation of φ is

$$\frac{\dfrac{\Gamma\vdash_{NJ}\bot}{\Gamma\vdash_{NJ}\psi}(\bot E)}{\Gamma\vdash_{NJ}\psi\vee\chi}(\vee I)$$

Recall that for *NJ* negation is not primitive. It is defined by $\neg\varphi =^{df.} \varphi\rightarrow\bot$. So the case of negation is covered by the the general case of implication.

□

As for classical logic, suppose we consider the version of *NK* obtained by adding to *NJ* the rule for double-negation elimination.

$$\frac{\Gamma\vdash_{NK}\neg\neg\varphi}{\Gamma\vdash_{NK}\varphi}(DNE)$$

Proposition 3 (non-smoothness of *NK*). *The ND-system NK (for propositional classical logic) is not proof-theoretically smooth.*

Proof. If $\Gamma\vdash_{NK}\varphi$ was obtained by an application of (*DNE*) as the last rule, there need not exists a canonical derivation $\Gamma\vdash^{c}_{NK}\varphi$. □

As there are harmonious and stable presentations of classical logic, for example [27] using multiple-conclusion ND, and [29], using bilateral ND, those systems are smooth too, once pt-consequence is naturally extended to multiple-conclusion and to bilateralism.

4. Conclusions

The paper argues that proof-theoretic consequence should be separated from derivability, in spite of the fact that those two notions occasionally extensionally coincide, as is the case for intuitionistic logic. The argument is based on a general principle according to which a consequence relation is associated with a theory of meaning, and is defined by preservation (or propagation) of the central concept of that theory. The standard model-theoretic notion of consequence is based on propagation truth in a model, the central concept of model-theoretic semantics. In analogy, proof-theoretic consequence should be based of canonical derivations (or grounds for assertion), the central concept of (one variant of) proof-theoretic semantics. This consequence relation may coincide with derivability for logics I called smooth, but need not coincide with derivability in general.

As I see matters, it remains a challenge to find a model-theoretic consequence that is sound and complete for those cases that the standard model theoretic consequence, even if coinciding with derivability, does not coincide with proof-theoretic consequence as defined here.

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