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# EXPLANATORY PROOFS IN MATHEMATICS: NONEISM, SOMEISM, AND ALLISM*

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ABSTRACT

In recent times philosophers of mathematics have generated great interest in explanations in mathematics. They have focused their attention on mathematical practice and searched for special cases that seem to own some kind of explanatory power. Two main views can be identified, namely noneism and someism: the first is the view that no proof is explanatory, whereas the second is the view that some proofs are explanatory while others are not. The present paper aims to discuss the plausibility of the latter view. I first point out the main difficulties involved in this kind of research and I focus on a recent someist account, namely Frans and Weber's mechanistic model. Their approach seems promising, but further research is needed before accepting this someist model as a proper one. I then outline a general assessment on someism which doesn't turn out to be so convincing. I therefore suggest another view that I call allism, i.e. the view that all proofs are explanatory, at least in some sense, and I argue for its plausibility.

## 1. Proof and Explanation

In recent times philosophers of mathematics have aroused great interest in the issue of explanation within mathematics. They have focused their attention on mathematical practice and searched for special proofs that own some kind of explanatory power.[1] Some mathematicians have also talked about explanations. For instance, Giancarlo Rota emphasized a fundamental distinction between proofs and explanations in mathematics. He proclaimed:

> Not all proofs give satisfying reasons why a conjecture should be true. Verification is proof, but verification might not give reason (Rota 1997, pp. 186-187).

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[1] See Mancosu (2008, 2011), Cellucci (2014), Pincock and Mancosu (2012), Molinini (2013a). These authors support this position by means of several examples drawn from the history of mathematics.

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A similar remark has been suggested by the Fields Medal winner Timothy Gowers, who stated:

> For mathematicians, proofs are more than guarantees of truth: they are valued for their explanatory power, and a new proof of a theorem can provide crucial insights (Gowers and Nielsen 2009, p. 879).

Various other examples could be found in the literature.[2] For this reason several contemporary philosophers have seriously taken into consideration this issue, hence raising a number of new questions.[3]
Two main views can be identified. The first supports the idea that mathematicians have never explained anything and mathematics would just be about proving theorems without explaining them. For instance, Grosholz states:

> mathematical truths are constructed or demonstrated, not explained (Grosholz 2000, p. 81).

Resnik and Kushner support such position and talk about:

> […] the mistaken idea that there is an objective distinction between explanatory and non-explanatory proofs (Resnik and Kushner 1987, p. 154).

I refer to this view as *noneism*, i.e. the view that no proof, either formal or informal, is explanatory.[4] The second supports instead the idea that there exists a proper subset of proofs that own some kind of explanatory power so we can make a distinction between proofs that explain and proofs that merely prove. I refer to this view as *someism*, i.e. the view that some proofs are explanatory, while others are not.

In this paper I aim to discuss the plausibility of the latter view since someism has been supported by several contemporary philosophers. I first present the main difficulties that we have to face when we search for explanatory proofs in mathematics. In the third section I focus on a recent someist account, namely Frans and Weber's mechanistic model. Their approach seems promising, but further research is needed before accepting this someist model as a proper one. In the fourth section I shift the focus to other someist accounts that have been suggested so far and I outline a general assessment on someism which doesn't turn out to be so convincing. Finally, in the last section I suggest another view that I call *allism* and I argue for its plausibility.

[2] See Mancosu (2000) and Molinini (2014) for an historical account. See also Molinini (2012, 2013a), Hafner and Mancosu (2005), Mancosu (2001, 2008).

[3] A few philosophers have argued that not all explanations within mathematics come in the form of proofs, but I raise a few doubts on such account. However, I'm not going to discuss this issue in this paper because I'm only going to deal with proofs that show some kind of explanatory power. See Mancosu (2008, p. 142), Resnik and Kushner (1987, p. 152), Colyvan (2012, p. 97), Molinini (2014, p. 98).

[4] We borrow the term "noneism" from Lewis (1990), where it is used with a different meaning.

## 2. A Cumbersome Research

When we look for the word "explanation", in the dictionary, we find definitions such as "a statement, fact or situation that gives a reason for something, that makes something easier to understand."[5] As you can see, such a definition makes use of expressions like "gives a reason for", and "easier to understand", but their meaning is not so clear, so is the meaning of "explanation". However, there exist specific cases where a sort of explanation can be identified, at least as an intuitive notion. Consider, for instance, the following example: many people know that if one pours a bit of oil in a glass of water, after a while, the oil will be placed upon water as a second layer. Nevertheless, not everyone knows why this happens. Acknowledging that a double layer will appear after that operation does not imply that we know the reason behind such physical phenomenon. It seems hence reasonable to recognize two kinds of knowledge: the first is about *what* we know to be as such, whereas the second concerns *why* such a fact is like that. The latter is connected with the notion of explanation and in mathematics this kind of knowledge can be related with proofs that answer the following why-questions Q:

Q) *Why is the statement T of this theorem true?*

Observe that it is a very broad characterization of explanations in mathematics, but it seems to be a good starting point. Since antiquity, philosophers have attempted to clarify what an explanation is and we can find in literatures several ways of referring to this notion. Hafner and Mancosu have pointed out some of them:[6]

(a) a deep/satisfying/true reason
(b) the reason why
(c) an understanding of the essence
(d) a better understanding
(e) an account of the fact
(f) the causes of

It shows how difficult figuring out a satisfactory notion of explanation is. The situation is well depicted in Resnik and Kushner:

> Although from Aristotle onwards empirical science has acknowledged the production of explanations as one of its major goals and accomplishments, this is not an acknowledged goal of mathematical research. Mathematicians rarely describe themselves as explaining […] Given such evidence that the practice of explaining mathematical phenomena has been barely acknowledged, one could

[5] See, for instance, Oxford Wordpower Dictionary (2006).
[6] See Hafner and Mancosu (2005, p. 218).

> hardly expect that testing descriptive or normative accounts of it would be an easy task (Resnik and Kushner 1987, p. 151).

For this reason, any philosophical investigation on explanation has to deal with the issue concerning what we repute a reasonable way of identifying an explanation. In other words, we have to establish a criterion according to which we convince ourselves that we are facing exactly what we are searching for. Such a criterion seems to be a crucial point affecting the whole debate and we might refer to it as *the problem of evidence*.[7] Indeed, it concerns which evidence allows us to assert that we are dealing with a genuine explanation, avoiding to face a circularity, i.e. we are facing an explanation because it is an explanation.

The problem of evidence has been taken into account by some philosophers. A few of them suggest that looking at the works of mathematicians is the right way to deal with this issue.[8] Indeed, they argue, if a mathematician has claimed that a specific proof is actually an explanation, then it would be reasonable to accept it as such. However, this argument do not seem to be well grounded. It seems plausible that only a proper training in mathematics allows one to establish any difference between proofs and to identify explanations among them, but few objections may be raised. Firstly, what means to be "a mathematician" does not seem to be well-defined. One might think that having a bachelor degree in mathematics is enough to be labelled as a mathematician, others may suggest that only a master degree allows one to possess such title. And still others may object that only a full professor working for a university with several publications in his curriculum is allowed to be defined a mathematician. Moreover, it might be the case that a philosopher may be used to deal with mathematics every day, hence reaching a fair mathematical training. But even if it might make sense to identify a "subset of mathematicians" the fact that a mathematician has ascribed to a specific proof an explanatory power is not sufficient to claim that it actually is an explanation. It might be the case that they don't discuss the topic of explanation with the same philosophical care and rigour as a philosopher would expect. Finally, even if they prove to be good philosophers, we should take into consideration the entire community of mathematicians and not just few of them as the supporters of this approach seems to do. There might be a disagreement. Hence, their proposal does not seem to work. For these reasons the problem of evidence is still open and we have to face a troublesome situation. The following example might help to clarify. Suppose we are going to explore a garden and we wish to describe a special flower from Asia named "F". It might be the case that we already

[7] See Molinini (2013a).
[8] See Hafner and Mancosu (2005, 2008), Molinini (2012).

know in which place in the garden the flower F is kept. Thus our aim here is to define a model able to describe it in a proper way making it clearly distinguishable from all other flowers. Alternatively, another situation can be such that we do not know if this flower actually exists and which characteristics it has to possess. However, we try to recognize it among all flowers by means of several hypotheses. For example, we might suppose that it has to be similar to other well-known Asian flowers and so it needs to possess some peculiar properties that we can deduce from observing those Asian flowers. Hence, we aim to define a model capable of identifying the flower F, but it might still be the case that the flower doesn't exist. These two situations seem to be rather different, and our previous discussion suggests that the investigation on explanations in mathematics is likely to fall into the latter troublesome scenario. It might be the case that explanations do not exist at all.[9]

Nonetheless, some philosophers have tried to grasp the notion of explanation in mathematics, following their intuition regarding what means "to be explanatory".

## 3. Frans and Weber's someist view

### 3.1. *The model*

Frans and Weber aim to identify explanations in mathematics by means of a mechanistic approach.[10] Specifically, they consider a mechanistic model of explanation in science and try to develop it within mathematics.[11] A mechanistic explanation in science aims to explain why a system or a class of systems owns a specific capacity or regularity. For instance, a neuroscientist might wonder: "Why does this drug help people to feel better?". In this example, a mechanistic explanation succeeds if it manages to reveal which underlying mechanism is responsible for such phenomenon. An empirical study using neuroimaging techniques might reveal which parts of the brain are involved during the assumption of the drug. That allows the scientist to identify which parts are constituting the mechanism and consequently to give a mechanistic explanation of the phenomenon.

[9] Note that the search for explanations in science is rather different. Indeed, here there is at least a reasonable subset of all possible claims in science that are recognized as being an explanation of a physical phenomenon, and our investigations aim at defining a model(s) that fits all these explanations.

[10] See Frans and Weber (2014).

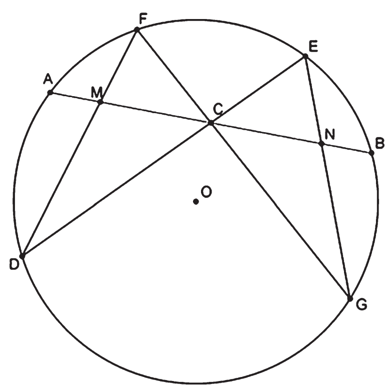
[11] Recently, a few authors have taken into consideration a mechanistic approach to scientific explanation. They claim that such approach has been neglected in the literature, even if it proves to be an interesting alternative, especially in life sciences, such as biology or neuroscience. See Machamer et al. (2000), Bechtel and Abrahamsen (2005).

Frans and Weber admit that such an account should be clarified. They give the following definitions:

> A *capacity* ascribed to a system is a systematic connection between inputs and outputs, i.e. a system has a capacity if it is able to produce specific outputs given specific inputs.
>
> A *mechanism* is a collection of entities and activities that are so organized that they realize the capacity.
>
> A *mechanistic explanation* of a capacity is a description of the underlying mechanism.[12]

Afterwards, Frans and Weber consider a proof of a theorem taken from geometry, namely the Butterfly Theorem:[13]

**Theorem 1.** *Let C be the midpoint of a chord AB of a circle. Consider the chords FG and ED that go through point C, and the chords FD and EG that cut AB respectively at M and N. Then C is the midpoint of MN. The picture below shows the situation.*



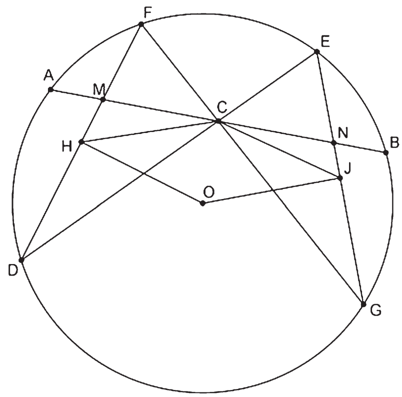

*Proof.* Consider the inscribed angles *EDF* and *EGF*. Recall that if two inscribed angles of a circle intercept the same arc, then the angles are congruent. Hence, the angles *EDF* and *EGF* are equal. Similarly, the angles *DFG* and *DEG* are equal. It follows that the angles *FCD* and *GCE* are equal as well. Then, the triangles *FCD* and *EGC* are similar, since they have equal angles. By the properties of similar triangles, we obtain

$$\frac{DF}{FC} = \frac{EG}{CE}$$

[12] The notions of system, entity, activity and organization, are not sharply defined.

[13] The curious name is clearly linked with the spatial configuration of the figure set by the problem, similar to a Butterfly.

Now draw a perpendicular to *DF* through the centre *O* and construct the point *H*. Similarly, construct the point *J*.




Recall that if a radius of a circle is perpendicular to a chord, then the radius bisects the chord. Hence, $DF = 2HF$ and $EG = 2EJ$. By substitution into the previous relation, we obtain

$$\frac{HF}{FC} = \frac{EJ}{CE}$$

It easily follows that

$$\frac{HF}{EJ} = \frac{FC}{CE}$$

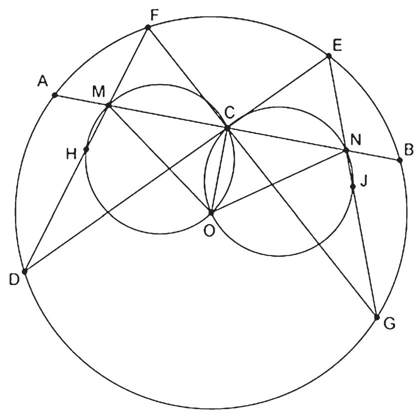
Hence, the triangles *FCH* and *EJC* are similar (by the property on similar triangles: if an angle of one triangle is congruent to an angle of a second triangle and the lengths of the sides spanning these angles are proportional, then the triangles are similar). It follows that the corresponding angles *CHF* and *EJC* are equal.

Now consider the convex quadrilateral *MCOH*.[14] Since *C* is the midpoint of the chord *AB*, then the angle *MCO* is right (by the property on circles: the line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord). Hence, the sum of the angles *OHM* and *MCO* equals 180° and it follows that the convex quadrilateral *MCOH* is cyclic[15] (by the theorem: a convex quadrilateral is cyclic if the opposite angles are

[14] A convex polygon is a not self-intersecting polygon in which no line segment between two points on the boundary ever goes outside the polygon.

[15] A quadrilateral is called cyclic if its vertices lie on a single circle.

supplementary). Similarly, the convex quadrilateral *CNJO* is cyclic. The picture below shows the situation.[16]



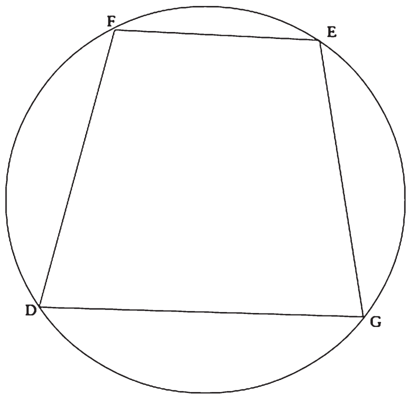

The inscribed angles *CHM* and *COM* are congruent since they intercept the same arc (now we are referring to the circle which inscribes the quadrilateral *MCOH*). For the same reason, the angles *NJC* and *NOC* are equal. As shown above, the angles *CHM* and *NJC* are equal.[17] Then, the angles *COM* and *NOC* are equal as well. Finally, we conclude that the triangles *MCO* and *CNO* are congruent (by the theorem: if two angles and the included side of one triangle are congruent to two angles and the included side of a second triangle, then the two triangles are congruent). This implies *MC* = *CN*, that is *C* is the midpoint of *MN*. □

Frans and Weber claim that the proof is explanatory. Specifically, they firstly identify a system with a peculiar capacity. Secondly, they recognize in the proof the entities and the activities that belong to the system and discuss how these elements are organized. The system is a quadrilateral inscribed into a circle, and it owns the capacity to connect the following inputs/output:[18]

[16] The authors note that these last two steps are not always possible, as in the case the point *H* and the point *M* coincide. At any rate, the theorem can be proved in another way. Here, we do not discuss it since it does not seem relevant to the present discussion. Indeed, we may restrict the original theorem to a specific case and reformulate all these considerations such that Frans and Weber's account is not undermined. See Frans and Weber (2014, p. 9).

[17] Note that the angles *CHF* and *CHM* are exactly the same. Similarly, the case of *EJC* and *NJC*.

[18] Below we show a specific case (the same quadrilateral that can be recognized from the previous proof), but their account regards any quadrilateral inscribed into a circle. Indeed, recall that the Butterfly theorem applies to any possible geometric construction that satisfies its hypothesis.




Inputs

*C* is the midpoint of a chord *AB* of the circle
*FG* and *DE* are chords that go through point *C*
*DF* is a chord that cuts *AB* at *M*
*EG* is a chord that cuts *AB* at *N*

Output

*C* is the midpoint of *MN*

Then, they claim that the proof identifies several entities such as chords, triangles, midpoints and angles. Differently, rather than speaking about activities they prefer to speak about dependence between properties of entities. For instance, it is possible to recognize the following entities:

- the line passing through the centre of the circle and intersecting the chord *DF*;
- the chord *DF*;
- the triangle *FCD*;

and the related dependency between properties:

- if the line passing through the centre of the circle and intersecting the chord *DF* has the property of being perpendicular to the chord *DF*, then the chord *DF* has the property of being cut at its midpoint;
- if the triangle *FCD* has the property of having two angles equal to two angles of another triangle *T*, then the triangle *FCD* has the property of being similar to the triangle *T*.

They further claim that these elements might be manipulated in order to formulate what-if-things-had-been-different questions. Specifically, it is possible to vary the properties of an entity and see how other properties of the same entity or of another entity would change in response. They give the following example of what-if-things-had-been-different question:

- What if the line passing through the centre of the circle and intersecting the chord *DF* was not perpendicular?
- What if the triangle *FCD* has not two angles equal to two angles of another triangle *T*?

These questions allow us to recognize which entities and properties (of entities) are relevant so that the proof works. The theorem might collapse if one changes one of such elements. For this reason, the authors refer to these relevant elements as difference-makers.

Finally, they point out that the steps in the proof are organized in a specific sequence which cannot be freely changed. We need to set the first steps at the beginning in order to proceed in the proof. Moreover, they argue that a particular spatial organization of all difference-makers is needed. If we apply the same difference-makers to a similar geometrical structure, we might be unable to prove the theorem.[19] According to Frans and Weber, all the above considerations are sufficient to establish that the proof describes how all the difference-makers are organized (mechanism) such that the given quadrilateral inscribed in a circle (system) satisfies the statement of the theorem (output) if certain conditions (inputs) hold. That is, the proof fits their mechanistic model of explanation. To gain greater precision they add that the proof allows one to answer several what-if-things-had-been-different questions, which in turn sheds light on why the theorem is true. Such information, they argue, is genuinely explanatory.

To conclude, they further claim that not all proofs are explanatory since at least the proofs using reductio ad absurdum always fail to be explanatory.

### 3.2. *Criticism*

This model has not yet been discussed in literature and I believe that my considerations will be useful. I am going to test Frans and Weber's model on a reductio proof in geometry, but we first need few clarifications.

The authors don't develop a careful account of a mechanistic description of a given proof so that it seems open to interpretation. We need to clarify what "give inputs to a system", means since giving inputs to a mathematical object does not seem to be a very clear notion. In the previous example, a reasonable way of understanding this notion is the following: given a quadrilateral inscribed into a circle, one draws all elements, that appear in the inputs, on this figure according to the specific properties that are described

[19] The authors don't clarify what they exactly mean with a "similar structure" although it does not seem to be a serious problem for their account.

in such inputs.[20] Then, among all possible chords inscribed into a circle, one needs to choose the right one, and so on. Another reasonable interpretation of "give inputs to a system", might be: given a system, one does a translation/rotation/dilatation/contraction on one or more components that form the system such that the conditions prescribed into the inputs are satisfied.[21] By component we mean a geometric object that can be visually recognized inside the system, where "inside" is left intuitive. Thus "give inputs to a system" might make sense if one means "draw all elements, that appear in the inputs, on the system according to the specific properties described by the inputs" or "do a translation/rotation/dilatation/contraction on one or more components that form the system such that the conditions prescribed into the inputs are satisfied".

Similarly, we have to clarify what "produce outputs" means. Here a reasonable interpretation might be that a system "produces outputs if specific inputs are given" if the system, as it appears to be after we have given the inputs, turns out to be a mathematical object with new properties that were lacking in the initial system.[22]

Furthermore, the notion of entity is not so clear. They claim that "the proof […] identifies certain entities […] such as chords, triangles, midpoints, and angles"[23], but what it exactly means to be "an entity of a system" is not clarified. A reasonable interpretation seems to be that an entity is a geometric object and it belongs to the system if it can be visually recognized inside the system, where "inside" again is left intuitive.

Another concern regards the choice of the system and its capacity. In their example, they regard a quadrilateral inscribed into a circle as the system, but only the circle and the four vertices of the quadrilateral are mentioned in the proof. Anyways, four points on a circle seem to be enough to identify a quadrilateral inscribed into a circle. Therefore we might assume that we must be able to identify the system in the proof although it need not be explicitly mentioned. Furthermore, they do not specify if this choice is unequivocal.[24] I believe their model aims to grasp a possible mechanistic interpretation in a proof, but there may be other justified ways to describe such proof as a mechanistic explanation.

[20] This remark is supported by their comment: "[…] the instructions used to construct the figure are the inputs […]" (p. 10).

[21] This remark is supported by their comment: "[…] identify entities and perform imaginary manipulations as discussed above […]" (p. 16).

[22] This remark is supported by their comment: "Within the set of quadrilaterals inscribed in a circle […] these specific conditions result in a specific output" (p. 11).

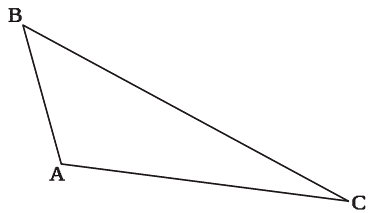
[23] See Frans and Weber (2014, p. 11).

[24] Although the notion of "identity between objects" in mathematics is troublesome, in this case it seems reasonable to speak about identity or difference between geometrical objects in terms of visual recognition on the paper. Two objects are not considered the same if we can distinguish them by visualization on the paper.

Finally, these remarks seem to provide a fair interpretation of Frans and Weber's account even if we are not arguing that such interpretation is the only one possible. Indeed, other different clarifications might make sense as well with our proposal representing just one.

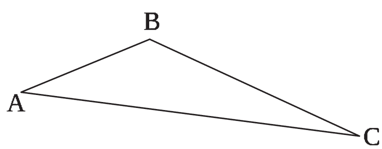
Let's now consider the following reductio proof:

**Theorem 2.** *Let ABC be a triangle. If the angle $\hat{A}$ is greater than* 90°*, then both the angle $\hat{B}$ and the angle $\hat{C}$ are less than* 90°.




*Proof.* Suppose that $\hat{B} \geq 90$, then $\hat{A} + \hat{B} > 180$. Now recall that the sum of all angles in a triangle equals 180, i.e. $\hat{A} + \hat{B} + \hat{C} = 180°$. This implies that $\hat{C} = 180° - (\hat{A} + \hat{B})$. By the above we obtain that $\hat{C} < 0°$, so we reach an absurd. We conclude that the angle $\hat{B} < 90°$. A similar reasoning works for the angle $\hat{C}$, which turns out to be less than 90°. □

Here it seems reasonable to identify a triangle as the system, which owns the capacity to connect the following input/outputs:[25]




Input

$\hat{A}$ is greater than 90°

Output

$\hat{B}$ is less than 90°
$\hat{C}$ is less than 90°

As we have observed above "give these inputs", means "do a rotation on *AB* and a rotation plus dilation on *BC* such that $\hat{A}$ is greater than 90°" and

[25] Below we show a specific case, but our account regards any triangle. Indeed, the above theorem applies to any possible triangle that satisfies its hypothesis.

the outputs refer to two properties that this modified system has to possess. This proof does not identify many entities, but it is possible to recognize the following ones:

- the angle $\hat{B}$;
- the angle $\hat{C}$;

and the related dependencies between properties:

- if the angle $\hat{B}$ has the property of being greater than or equal to 90°, then the angle $\hat{B}$ has the property that its sum with an angle greater than 90° is greater than 180°;
- if the angle $\hat{C}$ has the property of being greater than or equal to 90°, then the angle $\hat{C}$ has the property that its sum with an angle greater than 90° is greater than 180°;

These elements further allow us to formulate the following what-if-things-had-been-different questions:

- What if the angle $\hat{B}$ was not greater than or equal to 90°?
- What if the angle $\hat{C}$ was not greater than or equal to 90°?

Finally, a specific organization of all these difference-makers can be identified. For instance, we need to suppose that $\hat{B} \geq 90°$ in order to claim that $\hat{A} + \hat{B} > 180°$. Similarly, we have firstly to consider the angle $\hat{C}$ and its property to be equal to 180° if summed with the angles $\hat{A}$ and $\hat{B}$ in order to claim that $\hat{C} < 0°$.

Thus the above considerations show that there exists a justified way to describe this proof as a mechanistic explanation. Since Frans and Weber endorse the view that reductio proofs are non-explanatory, the fact that this model seems able to fit also them makes Frans and Weber's view not so convincing.[26] It might be the case that their model needs to be revised or their remark on reductio proofs is mistaken. I'm not supporting any particular claim here, but our positive test on a reductio proof should motivate further research before accepting this account as a proper one.

## 4. A General Assessment on Someism

Other authors have tried to identify which proofs are explanatory among all proofs. Several someist models have been suggested so far, but they either have not been properly developed or have been significantly criticized by others.

[26] They clearly state: "This does not render all proofs explanatory, since certain types of proof, such as proofs using reductio ad absurdum fail to give either bottom-up or top-down explanations" (p. 17).

One of the most discussed models in literature has been offered by Steiner.[27] It is based on the intuitive idea that an explanatory proof is able to grasp the "essence of an entity". This account is vague and several concepts are left undefined. This lack of clarity makes it open to interpretation and hence it does not seem to be well founded.[28] Although these criticisms sound reasonable, it might be the case that Steiner's intuitions are on the right track, but underdeveloped. The other main objection is that Steiner's model does not ascribe explanatory value to proofs that mathematicians ascribe it to.[29] As pointed out earlier, this argument do not seem to be well grounded. Thus these objections are not enough to dismiss this model. Steiner's idea might still be a good intuition, even if his attempt needs further improvement and clarification. Anyways, we do not have stronger reasons to accept it than to discard it, hence we would rather remain neutral on its validity.[30]

Other philosophers have discussed a few models of scientific explanations within mathematics, but these alternate models have also been rejected. Hafner and Mancosu, for instance, take into consideration Kitcher's account of explanation, but they dismiss it since this model does not fit a proof regarded to be explanatory by a mathematician.[31] A similar investigation is carried out by Molinini with respect to Hempel-Oppenheim model.[32] These arguments do not seem to be well grounded for the same reasons I have discussed earlier, but I do not have any valuable reason to accept these models anyways.

Generality and unification have also been proposed as possible criteria of explanatoriness. With regard to generality, Steiner suggests that the explanatory power of a proof is connected with its being generalizable whereas Kitcher claims that explanatory proofs can be identified in specific

[27] See Steiner (1978).

[28] See Resnik and Kushner (1987), Butchart (2001), Hafner and Mancosu (2005), Cellucci (2008) and Molinini (2012).

[29] See Hafner and Mancosu (2005), Molinini (2012).

[30] An improved version of Steiner's model has been suggested by Weber and Verhoeven (2002). It is an interesting proposal though I have some doubts on the idea that such proposal really represents a refinement. At any rate, I do not discuss the issue here, since this proposal does not appear to be a proper someist account. Indeed, the authors aim mainly to show how explanatory power might be related to the capacity of proofs to answer specific why-questions, so that several kind of proofs might turn out to be explanatory, such as existence proofs, uniqueness proofs or identity proofs. Their account seems to be based on a pluralistic view of explanatory power in mathematics. Pluralism does not necessarily exclude someism, but we should be able to identify clear examples of non-explanatory proofs. These proofs are not supposed to answer any possible why-questions and it doesn't seem easy to show. The authors do not deal with it, so it is not clear if their proposal can be regarded as a proper someist model.

[31] See Hafner and Mancosu (2008), Kitcher (1984, 1989).

[32] See Molinini (2013b), Hempel and Oppenheim (1948).

generalizations, which are characterized as being able to see old theories as special cases in the new account.[33] With regard to unification, some authors have argued that unification processes seem to grasp essential features of an explanation, illuminating connections between different areas of mathematics and shedding light on previous results.[34] For instance, Mancosu has shown how the original approach to complex analysis suggested by Pringsheim reveals that unification might play a crucial role.[35] Their observations are deep and insightful, but the authors don't develop any proper account based upon these concepts. It might be the case that in the future an interesting model will be suggested, or it might similarly be the case that no one has yet done it as it is too difficult to be carried out, or even worse it might not make sense at all.

Other interesting connections with explanatoriness can be found in the literature but they are, once again, not properly developed. For instance, Molinini has advanced that Bolzano's theorem on continuous functions and Euler's theorem on the existence of an instantaneous axis of rotation reveal a close connection between purity and explanatoriness, but no proper account has been suggested.[36]

To conclude, I have taken into account most of the someist proposals suggested by philosophers so far and my considerations haven't turn out to be so optimistic. The final assessment on Frans and Weber's someist account has not been so convincing, and the other someist models haven't been deemed so adequate. Therefore, no satisfactory solution has been found to the problem of evidence. It might be the case that *someism* has to revised, but it is not required. The difficulties encountered in elaborating a successful model might depend on the fact that such investigations are in a preliminary stage and in the future more interesting accounts will likely be proposed. Although my investigations are not enough to dismiss *someism*, I can at least argue that its plausibility is not so reliable. For this reason, in the next section a few modest suggestions will be put forward in order to suggest a compelling alternative view.

[33] See Steiner (1978, p. 146), Kitcher (1984, p. 208-209). In another paper, Steiner claims that the embedding of the real numbers in the complex plane allows us to find explanatory proofs of otherwise unexplained facts about the real numbers. See Steiner (1999, p. 137).

[34] See Colyvan (2012, p. 96), Kitcher (1989, p. 437), Friedman (1974, p. 15). It is noteworthy that Kitcher has never dedicated any paper explicitly to explanation within mathematics, whereas he has defended a unification account of explanation in science. At any rate his account has been investigated in the context of geometry by Hafner and Mancosu (2008).

[35] See Mancosu (2001, part III), Pringsheim (1925).

[36] See Molinini (2012, section 3), Molinini (2013a, p. 127). See also Mancosu (2001, p. 112). With regard to this issue, Detlefsen and Arana have focused on a specific notion of purity, which they have called *topical purity*. They argue that a topical pure proof shows some epistemic advantages, which seem to be a virtue of the proof. See Detlefsen and Arana (2011).

## 5. Another view: Allism

My aim here is to offer another view on explanations in mathematics. I will refer to it as *allism*, i.e. the view that all proofs are explanatory, at least in some sense.[37] In order to grasp the plausibility of this view, note that we are searching for proofs that are able to answer the following why question Q:

Q) *Why is the statement T of this theorem true?*

Following our intuition regarding what means "to be explanatory", we might claim that we can ascribe an explanatory power to any proof that satisfies the following condition $\sigma$:[38]

σ) *if we read (and understand) it, then we are able to give a proper answer to the why-question Q by reference to some explanatory elements of the proof.*

If we accept this condition, then a proof appears always to possess some kind of explanatory power. Indeed, if we read (and understand) a proof then we can easily formulate the following statement

*because so and so,*

where the expression "so and so" has to be replaced with the text of the given proof. The above statement therefore seems to be a proper answer to the why-question Q. Hence, the given proof satisfies the condition $\sigma$. This remark therefore suggests that a basic explanatory power might be ascribed to each proof, at least in this sense, and so *allism* seems to be a reasonable view.

It is worth pointing out that the above suggestion is just a special kind of answer to why-question Q, while there might exist several others. Moreover, one might ascribe an explanatory power to any proof that is able to answer a different why-question.[39] Finally, another kind of explanatory power may be assigned according to other criteria. For this reason it might be possible to ascribe additional explanatory power to any proof other than its basic explanatory power. My previous claim only aims to suggest that any proof owns a basic explanatory power. Thus any other investigation on explanatoriness is not necessarily undermined.

[37] We borrow the term "allism" from Lewis (1990), where it used with a different meaning.

[38] One might reject this claim by arguing that the condition σ does not properly represent the notion of explanation. I might agree with this, but in the contemporary debate on explanation within mathematics a preliminary investigation on the general notion of explanation is not usually taken into consideration. The search for models which seek to identify explanations within mathematics are usually based on the intuition regarding what means "to be explanatory". Hence, if we do not want to accept the condition σ then we need first a better clarification than the intuitive notion.

[39] Weber and Verhoeven have suggested a connection between explanatoriness and the ability to answer a why-question. I believe that their considerations can be successfully reconsidered in my view. See Weber and Verhoeven (2002). See also footnote 30.

Despite the plausibility of *allism*, it might be the case that this view is not so interesting. If all proofs were explanatory only in the above sense, then it could be argued that all the investigations on explanation are meaningless since there exists only one basic explanatory power that makes sense. In this case there would not be much difference between *allism* and *noneism*. Nonetheless, this is not the case I will defend and I will instead argue for a different interpretation of *allism*.

I believe that *allism* can be regarded as an interesting view if it is possible to ascribe a further explanatoriness to proofs other than their basic explanatory power, at least in quite a number of cases. Hence, it would be possible to distinguish a proof from another according its explanatory power. In this new framework the problem of evidence has clearly changed. The issue no longer concerns the way we aim to identify an explanation among all proofs since in this view all proofs are explanatory, at least in a basic sense. The problem of evidence is now about what we consider a reasonable way of classifying all proofs according to their different explanatory power.

Now I'm going to show that Frans and Weber's someist model can be reconsidered in this view as a satisfactory allist model able to reveal some kind of explanatory degree in a proof, at least in geometry. If a proof can be modelled as a mechanistic explanation, then it is able to answer several what-if-things-had-been-different questions, a kind of explanatory information. We might define *d* as the number of all answers to what-if-things-had-been-different questions that it provides. This number may be regarded as an explanatory degree owned by the proof. Furthermore, given the huge variety of proofs, it seems unlikely that this number always turns out to be the same. Indeed, the proofs vary depending on the number of entities that they make reference to, on the number of properties that they ascribed to the entities and on the complexity of geometrical constructions that they use.[40] For this reason the number of answers to what-if-things-had-been-different may be rather different in each proof. In addition, in each proof we might identify another legitimate mechanistic interpretation so that if new explanatory information is found they might increase the number *d*. Finally, note that one may argue that some proofs might show an infinite number of what-if-things-had-been-different questions. This would be a problem to be dealt with if it were the case. However, this situation seems unlikely since the number of all elements mentioned in a proof is finite and the wording of what-if-things-had-been-different questions is not entirely arbitrary. These questions have to be related to the text of the proof. This

[40] Note that the number of entities and properties mentioned in a proof may significantly vary on the amount of previous theorems that a proof makes reference to. When a previous result is mentioned instead of proved in carrying out the proof, several potential explanatory information may disappear.

number might be huge and it may be difficult to catch the number *d*. However, this last remark might suggest that degree *d* will be hard to grasp, though this doesn't undermine the fact that a different explanatory degree might be ascribed to each proof.

Now remember that Frans and Weber's someist model was able to fit a reductio proof. This fact suggested further investigation before accepting the model. Under my view, however, this does not represent any evident problem. *Allism* admits the possibility that reductio proofs might have an explanatory power to some degree. An *allist* therefore may reconsider Frans and Weber's model in this view and use it to grasp explanatoriness in reductio proofs.

For these reasons it seems that at least a proper model has been found. It doesn't mean that the problem of evidence has been solved. Several other models may be needed to classify all proofs according to their different explanatory power. Anyways, it represents a good starting point and an *allist* may hope for positive developments.[41] To conclude, *allism* seems to be a fruitful view and I believe that future investigations with respect to this view might shed light on the troublesome topic of explanation.

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