Logique & Analyse **234** (2016), 227-263

FORMAL PHILOSOPHY AND LEGAL REASONING: THE VALIDITY OF LEGAL INFERENCES

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ABSTRACT

The aim of the present paper is to introduce a method to test the validity of legal inferences. We begin by presenting the rationale of our method and then we expose the philosophical foundations of our analysis. If formal philosophy is to be of help to legal discourse, then it must first reflect upon the law's fundamental characteristics that should be taken into account. Our analysis shows that (Canadian) legal discourse possesses three fundamental characteristics which ought to be considered if one wants to represent the formal structure of legal arguments. These characteristics are the presupposed consistency of legal discourse, the fact that there is a hierarchy between norms and obligations to preserve this consistency and the fact that legal inferences are subjected to the principle of deontic consequences. We present a formal deontic logic which is built according to these characteristics and provide the completeness results. Finally, we present a semi-formal method (based on the proposed deontic logic) to test the validity of legal inferences. This paper contributes to the literature insofar as it provides a method that covers a portion of the intuitive validity of legal inferences which is not covered by other frameworks.

Keywords: Legal obligations, Legal discourse, Normative consistency, Deontic consequence, Non Kripkean semantics, Deontic logic, Contingency.

1. Introduction

Deontic logic began with the work of von Wright (1951) and has since been interpreted in many different ways.[1] Although the approaches that we find nowadays within the literature vary significantly, most share a common feature: the use of possible world semantics and the interpretation of deontic logic as a modal logic.[2] Among these approaches the reader will find monadic, dyadic, temporal, non-monotonic, first-order, dynamic and *stit*

[1] There was also a previous proposal made by Mally (1926), but it did not have as much impact as von Wright's at the time.

[2] Some approaches do not use possible world semantics, see for instance Makinson and van der Torre (2000).

doi: 10.2143/LEA.234.0.3159742.

deontic logics (for an overview, see Peterson 2011; 2014d).[3] The modal interpretations are usually characterized by the fact that the truth value of a normative proposition depends upon the truth value of the descriptive proposition in the scope of the deontic operator. Modal interpretations are usually of the *Ougth-to-Be* type, as opposed to the *Ought-to-Do* interpretation of deontic modalities (for the distinction, see von Wright 1999). In the modal interpretations, a deontic proposition $O\varphi$ usually expresses that a specific description φ of the world *ought to be* the case, or that the world *ought to be* in a specific state. However, one can nonetheless find some modal interpretations of the *Ought-to-Do* type, which are characterized by the fact that the proposition in the scope of a deontic operator is instead interpreted as the name of an action. This kind of interpretation can be found notably within the dynamic approach to deontic logic[4], where an action is forbidden when its performance implies that the world is in a state within which there is a violation V.[5] It can also be found in deontic logics based upon a Boolean algebra.[6]

The present paper does not aim at criticizing these approaches but rather aims at providing a new one, based on different philosophical foundations. We propose an interpretation of deontic logic based upon the analysis of (Canadian) legal norms. Our goal is not to provide an analysis of how we use 'ordinary deontic language' or how 'ordinary deontic reasonings' work, as Castañeda (1981, p. 38) would say, but is rather to show how a normative reasoning *should* work. The objective is to define a proper consequence relation according to some basic properties that govern a 'correct' use of a (legal) normative inference. We begin by exposing the rationale behind our framework and present the philosophical foundations of our system. The formal deontic logic and the completeness results are then presented. On these grounds, a semi-formal method that can be used to analyze the

[3] See for example Castañeda (1981), Schotch and Jennings (1981), Jones and Pörn (1985, 1986), Hansson (1990) or Jones (1991) for monadic deontic logic; Al-Hibri (1978), Chellas (1974), Mott (1973) and van Fraassen (1972) for dyadic deontic logic; Chellas (1969), McKinney (1977), Åqvist and Hoepelman (1981), Thomason (1981), van Eck (1982a, 1982b) or Bailhache (1991, 1993) for temporal deontic logic and Nute (1997) for non monotonic deontic logic. See also Broersen (2011a), Carmo and Pacheco (2001), Carmo and Jones (2002), Horty (2001) for multi-modal deontic logics, and Belnap and Perloff (1988), Broersen (2008, 2009, 2011a, 2011b), Carmo and Pacheco (2000, 2001), Horty and Belnap (1995), Horty (2001), Pacheco and Carmo (2003), Pacheco and Santos (2004) and Xu (1995) for *stit* logics. The reader may consult Wolenski (1990) for a sketch of possible world semantics in deontic logic.

[4] See Meyer (1987; 1988) for the introduction of dynamic deontic logic and, among others, see Royakkers (1998), Broersen (2004), Hughes and Royakkers (2008), Segerberg (2009), Anglberger (2009), Demolombe (2014), Segerberg (2012) or Prisacariu and Schneider (2012).

[5] This type of reduction is inspired by the work of Anderson (1958).

[6] See for instance Segerberg (1982) and Trypuz and Kulicki (2009, 2010).

validity of legal inferences is introduced. The method consists in testing the validity of an argument through its graphical representation. Some insights regarding the analysis of the premises of a legal argument are given and the paradoxes of deontic logic are discussed. We conclude in the last section by summarizing the limitations of our approach and we present avenues for future research.

2. Philosophical assumptions

Contra the modal interpretation of deontic logic, we do not consider deontic propositions as being of the *Ought-to-Be* type. Following Solt (1984, p. 350), the truth value of a normative proposition $O\varphi$ for the actual world does not depend upon the truth value of φ at any 'deontic alternative'. The truth value of a deontic proposition $O\varphi$ does not rely on the performance value of φ in every accessible 'deontically perfect world'. The fact that φ is obligatory depends upon the existence of a norm, which is established by some authority (Alchourrón and Bulygin 1981, pp. 97,102) and aims to guide one's actions (Weinberger 2001, p. 134). This is consistent with the legal adage *nullum crimen sine lege*: there is no crime without law, nor any obligation without a norm. Therefore, the truth value of a normative proposition $O\varphi$ does not depend upon the truth value of the descriptive proposition φ in the scope of the deontic operator but depends upon the fact that there is a norm which makes φ obligatory.[7] Following the same reasoning, the truth value of a deontic proposition does not depend upon the fact that there is a 'violation' in every state where the action is performed.

We do not wish to analyze legal obligations within the framework of modal logics since the operator O_i does not behave as a modality of the type $\Box_i$. Although O_i marks the property of an action, this operator is not interpreted as a predicate since it can be applied to combinations of actions, which are expressed by molecular compounds of descriptive formulas. The operator O_i is not interpreted as a modality of type $\Box_i$ since we do not wish to obtain formulas such as $O_i(A \vee \neg A)$ or $O_i A \supset O_i(B \vee \neg B)$, which are dubious from a legal point of view.

It is noteworthy that we are not considering legal discourse as a *normative system*, that is a "set of agents (human or artificial) whose interactions can fruitfully be regarded as norm-governed (Carmo and Jones 2002, p. 265)". This point is important: we are not trying to *describe* how agents interact within a normative system. Rather, we wish to define a semantical consequence relation for the validity of legal inferences. Our approach is *normative* rather than *descriptive*. It concerns the analysis of legal reasoning

[7] This position is also consistent with the legal literature. See Baudoin (2010).

from a critical point of view. This is mainly why we will not use *stit* or dynamic logic: we are not focusing on the notion of *agency*. Rather, we are analyzing the semantical consequence relation within a legal argument.

According to the semantical dichotomy between facts and norms (cf. Jørgensen 1937), descriptive and normative propositions are not true in the same conditions. While a descriptive proposition is true or false with regard to the world it describes, the truth of a normative proposition (which says how the world should be or how people should act, rather than how the world is) depends on the existence of a norm, which is established by some authority. We distinguish between a norm and a normative proposition: the former is neither true nor false while the latter can be true or false (the truth value of the normative proposition depending upon the existence of a norm). A normative proposition expresses that an action (either a specific action or a class of specific actions) possesses a deontic property. For example, from the Canadian Criminal Code we can conclude that the action 'stealing a red bicycle' possesses the propriety 'forbidden' or, equivalently, that the action negation 'not stealing a red bicycle' possesses the property 'obligatory'. Assuming that an action possesses a deontic property, we want to develop a basic logic that can represent how this property can be transmitted from one action to another.

One property of legal discourse is that there are no obligations without norms. Following Chellas (1974, p. 24), we want to be able to represent situations where there are no obligations, and thus our system must not include 'absolute' or 'unconditional' obligations. Likewise, following Jones and Pörn (1985, p.279), it is always possible for someone to act against one's obligations. Hence, tautologies are not obligations unless there is a norm that makes it so. This is the principle of normative contingency, first introduced by von Wright (1951). Formally, we do not want our system to validate theorems of the form $\vdash O_i\top$ or $\vdash A \supset O_i\top$ (with $\top$ a tautology). Also, since the meaning of a deontic operator changes when it is iterated (Jones and Pörn 1985, p. 286), it follows that, according to Castañeda (1981, p. 66), it must not be possible to iterate the same deontic operator. For now, our system will concentrate on propositions within which there is only one type of deontic operator (one authority) that cannot be iterated. Finally, we will not be considering mixed propositions (i.e., propositions composed with both descriptive and normative atoms). Indeed, it is unclear whether or not a material conditional is sufficient to represent the transmission of truth value between descriptive and normative propositions since the truth-value assignment for a descriptive proposition differs from the truth-value assignment of a normative one. The connective '$\supset$' does not preserve truth from a descriptive proposition to a normative one. When one tries to model deontic conditionals and contrary-to-duty reasoning, one faces the problems of augmentation and detachment (cf. Jones 1991). For example, it is possible to have a situation where $p \supset P_i q$ is true but $(p \wedge p') \supset P_i q$ is false.[8] Hence, the descriptive antecedent of a deontic conditional cannot be augmented, as the normative consequent cannot always be detached.[9] These are also arguments in favour of non-monotonic foundations for deontic logic.[10] This, however, will be the topic of another paper.[11] For now, we will not be considering mixed formulas and will only concentrate on normative ones.

The deontic logic we propose is based upon an analysis of Canadian legal discourse. Since legal norms are meant to guide one's actions, it follows that norms must be consistent since it would be impossible to act accordingly with an inconsistent set of norms. Thus, we assume the criterion of *normative consistency*: obligations are supposed to be consistent (at least after interpretation) since the legislator is presupposed to be rational, meaning that it is assumed that he thinks rationally and logically (Côté 2006, p. 387). The set of norms created by the legislator is therefore presupposed to be consistent, and thus legal obligations must not be interpreted as contradictory. Consistency is a rational criterion that ensures the accessibility, the authority and the equity of the law (Côté 2006, p. 387). Canadian legal discourse is considered as forming a 'logical system' which is *horizontally* and *vertically* consistent (after interpretation). As such, obligations from the same set of norms must be consistent with each other, as are the different sets of norms (Côté 2006, p. 388). In other words, the set of all legal laws is presupposed to form a consistent whole (Côté 2006, p. 433). Consistency is a rational criterion that enables one to judge the value of a set of norms, which can be examined through the consistency of the set of obligations that it generates.[12]

The relation of hierarchy between norms is meant to preserve a legal system's consistency. It is necessary to insure *vertical* and *horizontal* consistency. Since there can be (*a priori*) contradictions between different sets of laws or different laws within a set of norms, hierarchy enables us to preserve the consistency of the whole system by resolving potential conflicts of obligations. Moreover, laws are constructed in an hierarchical manner. For example, in Canada, no legislation can go against the Canadian

[8] If Paul has a driver's license, then he is permitted to drive, but if in addition he is drunk, then he is not.

[9] This is why dyadic deontic logics were introduced. See Loewer and Belzer (1983) and Vorobej (1986) for a discussion.

[10] As a result, we will not be considering defeasible reasoning (see for example Horty (1994), Sartor (1994), van der Torre and Tan (1997) or Governatori and Rotolo (2004)).

[11] See Peterson (2014b, 2014c).

[12] The fact that many deontic logics that satisfy the axiom schema (D) cannot represent conflicting obligations is often seen as an argument in favour of non-monotonic deontic logic (see for example Horty 1997). Since we only concentrate on one authority, normative consistency can be assumed, otherwise the legal authority would force us to act within an impossible frame.

Constitution. The fact is that there are sets of laws that are superior to others by construction, although these relations of hierarchy can also be explicitly mentioned in the law itself (Côté 2006, p. 45 and p. 450).[13]

Legal reasoning is characterized by the fact that it is hypothetical: a legal conclusion cannot be drawn without a legal premise. This is consistent with semantical dichotomy. A valid legal inference implies that there is a set of hypothetical norms (or obligations) from which legal conclusions are drawn. Thus, an obligation is conditional to a set of norms (or principles), which is established by some authority. The fact that obligations are derived from norms implies that normative inferences are governed by a specific criterion of validity. Since there is a finite number of norms, it would seem that there is also a finite number of obligations that will derive directly from these norms. For example, we can derive from the law that it is forbidden to steal. However, norms are often formulated in order to be applied to a class of specific actions. The obligation that derives directly from the law is to *not steal*, but clearly it is meant to be applied to a class of actions: not stealing a car, not stealing money, not stealing Paul's money, not stealing Peter's car, not stealing Paul's car and Peter's bicycle, etc. In other words, there is a much greater number of obligations that can be derived from a specific norm. But how do we legitimately conclude them since they are not *explicitly* mentioned by the norms?

To answer this question, we follow Alchourrón and Bulygin (1981, p. 102) and distinguish between *fixed* and *derived* obligations. A fixed obligation stems from a norm and can be general in order to be applied to a class of specific actions. A finite number of norms will hence entail a finite number of fixed obligations. A derived obligation can be inferred from a fixed one. From *it is forbidden to steal* we can derive *Paul has the obligation to not steal Peter's money*. The rule which governs this type of inference is the *principle of deontic consequences* (Castañeda 1968, p. 13)[14]: if A is an obligation and A implies B, then B is also an obligation. The mere formulation of the law implies that there are derived obligations. In *French Civil Law* it is obvious since the law is formulated in a general way and applies implicitly to each particular case that falls within its scope. Even though it seems to be the contrary in *Common Law*, where the law is constructed upon each particular judgment, the same principle applies nonetheless since a judgment applies to a class of particular actions.[15] Even if a judgment comes from a particular action, the case law applies to other similar cases.

[13] The use of hierarchy to solve normative conflicts was advocated by Alchourrón and Makinson (1981).

[14] See also van Fraassen (1972, p. 421).

[15] Note that Quebec's legislation (in Canada) is composed of both French Civil Law and Common Law.

Put differently, there are actions that are obligatory (or forbidden) even though they are not mentioned explicitly by the law (or the case law). The law is not an enumeration of all possible cases. The law says that it is forbidden to steal. What it means is that it is forbidden to do any action that implies (or *counts-as*) stealing.[16] Furthermore, normative consistency implies that legal obligations are governed by the principle of deontic consequences. Indeed, assuming that the legislator is rational, it is possible to deduce from the law certain things that logically follow (Côté 2006, p. 422). The aim of the present paper is to define this consequence relation.

To sum up, the analysis of Canadian legal norms brought to light three fundamental characteristics of legal discourse, namely its presupposed consistency, the fact that hierarchy is meant to preserve that consistency and the validity of the principle of deontic consequences, which enables one to infer derived obligations from fixed ones. If one is to apply formal philosophy to law, then one must take these characteristics into account.

3. The logic of legal obligations

According to these characteristics, the logic we propose relies upon a consequence relation which is defined as a function of normative consistency and deontic consequences. For now, we do not need to include hierarchy within the formal definition since we are only considering one type of deontic operator which cannot be iterated. However, as we will see, hierarchy will play a role for the analysis of the soundness of an argument. The basic idea of our approach is to define a consequence relation which can represent how the property 'obligatory' can be transmitted from an action (or a combination of actions) to another. We follow Castañeda (1981, p. 46) and assume a distinction between different types of obligations, where the type of an obligation depends upon the set of norms from which it can be derived (Alchourrón and Bulygin 1981, p. 120).

According to the semantical dichotomy between facts and norms, the idea is to develop a framework which distinguishes between factual and normative truth. As such, the language $\mathcal{L}$ will be divided in two sub-languages $\mathcal{L}_{PL}$ and $\mathcal{L}_{OL}$, respectively representing the language of descriptive propositions expressed by propositional logic (*PL*) and the language of normative propositions expressed by the logic of (unconditional) obligations *OL*.[17] Similarly,

[16] We are not pretending that logic can resolve the problem of interpreting the law. We only say that laws are formulated in order to include different kinds of actions, which has nothing to do with deciding whether the action falls within the scope of the law or not.

[17] Since we assumed that every obligation derives from a norm (established by some authority), it follows that every obligation is 'conditional' in nature (i.e., obligations are conditional to the existence of norms). This type of conditionality, however, is different from

the semantical model $\mathfrak{M}$ will be divided in two sub-models $\mathcal{M}$ and $\mathcal{N}$, the first being descriptive and the other normative. We assume that the model $\mathcal{M}$ of $\mathcal{L}_{PL}$ is a standard model of propositional logic. A proposition is said to be derivable in $\mathcal{L}$ if and only if either it is a consequence of *PL* or it is a consequence of *OL*. Similarly, a proposition A is said to be *valid* for $\mathfrak{M}$ if and only if either it is *descriptively* or *normatively* valid. For the rest of this paper, we concentrate only upon the normative fragments $\mathcal{L}_{OL}$ and $\mathcal{N}$.

3.1. *Syntax*

The formulas of *OL* have the form O_iA, where O_i is a single type of obligation and A is a proposition that refers to an action or a combination of actions. We assume a language

$$\mathcal{L} = \{(,), Prop, \neg, \supset, O_i\}$$

where $Prop = \{p_1, \ldots, p_n, \ldots\}$ is a denumerable set of propositional descriptive atoms and descriptive propositions are understood as descriptions of actions (not any descriptions). The other logical connectives $\wedge$, $\vee$, $\equiv$, F_i and P_i are defined as usual, with:

$$F_i A =_{def} O_i \neg A \qquad \text{(Interdiction)}$$

$$P_i A =_{def} \neg O_i \neg A \qquad \text{(Weak permission)}$$

We use A, B and C as meta-variables. The set $WFF_{\mathcal{L}}$ of well-formed formulas of $\mathcal{L}$ is defined recursively by:

$$p_i \in WFF_{\mathcal{L}_{PL}} \text{ for all } p_i \in Prop \qquad (1)$$

$$\text{if } A, B \in WFF_{\mathcal{L}_{PL}}, \text{ then } \neg A, A \supset B \in WFF_{\mathcal{L}_{PL}} \qquad (2)$$

$$\text{if } A \in WFF_{\mathcal{L}_{PL}}, \text{ then } O_iA \in WFF_{\mathcal{L}_{OL}} \qquad (3)$$

$$\text{if } A, B \in WFF_{\mathcal{L}_{OL}}, \text{ then } \neg A, A \supset B \in WFF_{\mathcal{L}_{OL}} \qquad (4)$$

$$\text{if } A \in WFF_{\mathcal{L}_{PL}} \text{ or } A \in WFF_{\mathcal{L}_{OL}}, \text{ then } A \in WFF_{\mathcal{L}} \qquad (5)$$

Thus defined, $WFF_{\mathcal{L}}$ does not contain any mixed formulas or any formula where there is an iteration of a deontic operator. The axiom schema for normative consistency is represented by (A1), which is propositionally equivalent to the axiom (D) of standard deontic logic.

$$\neg(O_iA \wedge O_i\neg A) \qquad \text{(A1)}$$

the distinction between *conditional* and *unconditional* obligations. A conditional obligation is formulated in such a way that it specifies in which context the obligation holds. An unconditional obligation is understood as an obligation that always holds, unless specified by another norm in some specific context. Note that in such a context, *OL* would not apply anymore and we would need a logic for conditional normative reasoning.

The principle of deontic consequences is represented by the rule of inference (R1) and its use is restricted by two conditions.

$$
\begin{array}{r|ll}
1 & \vdots & \\
2 & O_i A_1 \wedge \cdots \wedge O_i A_n & (n \geq 1) \\
3 & (A_1 \wedge \cdots \wedge A_n) \supset B & \\
4 & O_i B & \text{R1 l.2, 3}
\end{array} \tag{R1}
$$

First, $\{A_1, \ldots, A_n\}$ cannot be the empty set ($n \geq 1$), otherwise *B* would be a theorem of *PL* and there would be a set of absolute obligations. Since every obligation is conditional to a norm, it follows that *B* must be a consequence that depends upon $A_1 \wedge \cdots \wedge A_n$. This condition is meant to preserve the principle of normative contingency. Put differently, (R1) cannot be used if there is no normative hypothesis. Second, the conditional

$$(A_1 \wedge \cdots \wedge A_n) \supset B$$

must be either a theorem of *PL* or a hypothesis. As a result, (R1) cannot be applied to *any* material conditional but can only be applied to a conditional which is either a theorem of *PL* or a hypothesis. For instance, the following use would be an *incorrect* application of (R1).

$$
\begin{array}{r|ll}
1 & p & \text{H} \\
2 & \underline{O_i q} & \text{H} \\
3 & p \supset (q \supset p) & \textit{PL} \\
4 & q \supset p & \text{MP l,3} \\
5 & O_i p & \text{INCORRECT USE OF } \mathrm{R}_1
\end{array}
$$

Let us note that since (R1) can be applied to an hypothesis, it follows that this rule is invalid in a standard system such as *KD*. We introduce *OL* in a natural deduction system. We say that *A* is a theorem of *OL*, written $\vdash_{OL} A$, when there is a proof of *A* without the use of any hypothesis. In addition to (R1), we assume the rules of *PL* (and hence also the derived rules of *PL*) of Garson (2006, p. 35), that is:

1. hypothesis (H);
2. reiteration (Reit);
3. detachment ($\supset$ *out*);
4. conditional proof ($\supset$ *in*);
5. double negation (DN).

Since negation is taken as a primitive, we also need a rule to govern its introduction:

$$
\begin{array}{l|ll}
1 & \underline{A} & \text{H} \\
2 & \vdots & \\
3 & B & \\
4 & \neg B & \\
\multicolumn{1}{l}{5} & \neg A & (\neg in)
\end{array}
$$

Although *OL* contains only normative propositions, the proofs of the theorems are done in $\mathcal{L}$. Thus constructed, *OL* (and moreover $\mathcal{L}$) behaves at the propositional level according to the rules of *PL*. We write '*PL*' as a justification in a proof when the formula is a theorem of *PL*.[18]

3.2. *Semantics*

Let $Act = \{\Gamma_A : \Gamma_A$ is a positive action$\}$ be a denumerable set of *positive* actions (e.g., walking, talking, stealing, etc.), where Γ_A stands for the action described by the proposition A, and $\overline{Act} = \{\overline{\Gamma}_A : \overline{\Gamma}_A$ is a negative action$\}$ a denumerable set of *negative* actions (e.g., not walking, not talking, not stealing, etc.). The set $\mathbb{A} = Act \cup \overline{Act}$ is thus the denumerable set of all possible actions (actions in Act do not need to be atomic).[19] Each action in $\mathbb{A}$ can be described by a proposition of WFF_{PL}.

Let the sequence $s^{\alpha} = < s_1^{\alpha}, \ldots, s_n^{\alpha}, \ldots >$ be an arbitrary enumeration of Act and the sequence $\overline{s}^{\alpha} = < \overline{s}_1^{\alpha}, \ldots, \overline{s}_n^{\alpha}, \ldots >$ an enumeration of $\overline{Act}$, where $\overline{s}_i^{\alpha}$ refers to the negation of the action s_i^{α}. A propositional variable p_i (or a molecular compound A_i) refers to an action s_i^{α} member of an arbitrary sequence s^{α}, while $\neg p_i$ (or $\neg A_i$) refers to $\overline{s}_i^{\alpha}$. If A_i refers to s_i^{α}, then s_i^{α} is the action Γ_{A_i} (described by A_i), that is the i^{th} member of s^{α}. An arbitrary sequence s^{α} (and its counterpart $\overline{s}^{\alpha}$) is an interpretation of the language $\mathcal{L}$. It assigns a descriptive proposition of WFF_{PL} to each object of the domain (i.e., each action).[20]

[18] One the one hand, it should be noted that our syntactical system is similar to that of von Wright (1951) (see the axiomatization in von Wright (1967)). However, the two are not equivalent, since $\nvdash_{OL} O_i(A \supset A) \supset O_i(A \vee \neg A)$. On the other hand, even if (R1) may look like the Rule 0 in Alchourrón (1990), they differ from the important fact that $\{A_1, \ldots, A_n\}$ cannot be the empty set. Also, his syntactical system is equivalent to KD, which is not equivalent to OL since (R1) is invalid in the standard system.

[19] Note that our understanding of 'action' includes both action types and action tokens.

[20] To be precise, it assigns an equivalence class of propositions to an equivalence class of actions.

In order to formalize semantically the principle of deontic consequences, we assume that the set $\mathbb{A}$ is pre-ordered by '$\sqsubseteq$'. For example, if we suppose that

p_1 = Peter steals a red bicycle
p_2 = Peter steals a bicycle
p_3 = Peter steals

then we have $\Gamma_{p1} \sqsubseteq \Gamma_{p2} \sqsubseteq \Gamma_{p3}$. This allows us to give a semantical account of the relation of implication between actions. If an action Γ_A implies another action Γ_B, then $\Gamma_A \sqsubseteq \Gamma_B$. Thus, if we assume that 'stealing implies not violating the law' is true in some (descriptive) interpretation, then we assume that 'stealing' $\sqsubseteq$ 'not violating the law' in some normative model. Let $\mathcal{M}$ be a standard model of *PL*. In other words, if $\vDash_{\mathcal{M}} p_1 \supset p_2$ (i.e., $p_1 \supset p_2$ is assumed to be true in $\mathcal{M}$), then $\Gamma_{p1} \sqsubseteq \Gamma_{p2}$ holds in the normative interpretation. We do not assume the converse because it would lead us to ideality. (It is not because an entailment between actions is assumed in a normative interpretation that it is necessarily true in the descriptive one. For instance, one can assume that 'it is obligatory that *if one is in a public place*, *then one is not naked*' holds in a normative model while the conditional in the scope of the operator is false in a descriptive one.)

Let $\mathcal{N} = \langle W, \mathbb{A}, \sqsubseteq, a \rangle$ be a normative model, where $W \neq \varnothing$ is the universe of discourse and contains normative propositions (which are members of $WFF_{\mathcal{L}_{OL}}$), $\mathbb{A}$ is a denumerable set of actions pre-ordered by '$\sqsubseteq$' and $a\colon W \longrightarrow \{\top, \bot\}$ a function which assigns truth values to propositions in W. Let $\mathcal{O}$ be a proper subset of $\mathbb{A}$ ($\mathcal{O} \subsetneq \mathbb{A}$). Informally, $\mathcal{O}$ is a set of actions which have the property 'obligatory', meaning that $\mathcal{O}$ is the extension of the concept 'obligatory' within a normative interpretation.[21] The set has to satisfy three conditions:

$$\text{If } \Gamma_A \in \mathcal{O}, \text{ then } \overline{\Gamma}_A \notin \mathcal{O} \tag{C1}$$

$$\text{If } \Gamma_A \in \mathcal{O}, \text{ then } \Gamma_B \in \mathcal{O} \text{ for any } \Gamma_A \sqsubseteq \Gamma_B \text{ such that } \nvDash_{PL} B \tag{C2}$$

$$\text{If } \Gamma_A \in \mathcal{O} \text{ and } \Gamma_A \sqsubseteq \Gamma_{B \supset C}, \text{ then } \Gamma_B \sqsubseteq \Gamma_C \tag{C3}$$

The first condition represents normative consistency and the second implies that $\mathcal{O}$ is closed 'upwards' when B is not a tautology of the (classical) propositional calculus. The second part of C2 insures that normative contingency is respected. The third condition says that if the action described by A is semantically linked to the action described by $B \supset C$, then the action described by B is semantically linked to the action described by C

[21] If there are more than one authority, $\mathcal{O}$ can be indexed by $_i$.

insofar as the action described by A is obligatory. C2 and C3 allow us to represent the principle of deontic consequences. For a normative interpretation $\mathcal{N}$, $a_{\mathcal{N}}(A) = \top$ if and only if there is a sequence $s^\alpha = <s^\alpha_1, \ldots, s^\alpha_n, \ldots>$ which satisfies A. We now define satisfaction recursively.

Definition 1. *For any* A, s^α satisfies A *if and only if*

1. *If* A *is* $O_i B$, *then*
   (a) *If* B *is* p_i, *then* s^α *satisfies* A *iff* s^α *satisfies* p_i *iff* $s^\alpha_i \in \mathcal{O}$ *(i.e. the* i^{th} *member of* s^α *is a member of* $\mathcal{O}$*).*
   (b) *If* B *is* $\neg C$, *then* s^α *satisfies* A *iff* $\overline{s}^\alpha$ *satisfies* C.
      i. $\overline{s}^\alpha$ *satisfies* p_i *iff* $\overline{s}^\alpha_i \in \mathcal{O}$
      ii. $\overline{s}^\alpha$ *satisfies* $\neg D$ *iff* $\overline{\overline{s}}^\alpha = s^\alpha$ *satisfies* D
      iii. $\overline{s}^\alpha$ *satisfies* $D \supset E$ *iff* s^α *satisfies* D *and* $\overline{s}^\alpha$ *satisfies* E
   (c) *If* B *is* $C \supset D$, *then* s^α *satisfies* A *iff either*
      i. *there is* $s^\alpha_m = \Gamma_{C \supset D}$ *such that* $s^\alpha_m \in \mathcal{O}$ *or*
      ii. s^α *satisfies* D *(provided that* Γ_C *is not* Γ_D*).*[22]
2. *If* A *is* $\neg B$, *then* s^α *satisfies* A *iff* s^α *does not satisfy* B.
3. *If* A *is* $B \supset C$, *then* s^α *satisfies* A *iff* s^α *does not satisfy* B *or* s^α *satisfies* C.[23]

We say that a normative proposition A is *normatively valid*, written $\vDash A$, when it is true for any normative interpretation $\mathcal{N}$.[24]

3.3. *Completeness*

The following lemmas will be useful. As a notational convention, we will write s^α_A to refer to the action Γ_A (i.e., the action described by the descriptive proposition A), and which is the m^{th} member of s^α. We also write $s^\alpha_{\neg A}$ instead of $\overline{s}^\alpha_A$.

Lemma 1. *If* s^α *satisfies* $O_i A$, *then* $s^\alpha_A \in \mathcal{O}$.

Proof. We proceed inductively on the length of the formula. Suppose that s^α satisfies $O_i A$ but that $s^\alpha_A \notin \mathcal{O}$. (The inductive step (HI) is that if the property holds for $l = n$, then it also holds for $l = n + 1$).

[22] Informally, the first condition means that the conditional represents a fixed obligation while the second represents a derived obligation. The condition that Γ_C is not Γ_D is meant to preserve normative contingency.

[23] Let us note that it would be incorrect to infer from the fact that $\overline{s}^\alpha$ satisfies p_i that it also satisfies $A \supset p_i$ from condition 1c). Rather, if $\overline{s}^\alpha$ satisfies p_i, then s^α satisfies $\neg p_i$, and thus s^α satisfies $A \supset \neg p_i$.

[24] There is an equivalence of notation between $\vDash_{\mathcal{N}} A$ and $a_{\mathcal{N}}(A) = \top$. Also, from this definition it follows that $a_{\mathcal{N}}(A) = \top \Leftrightarrow a_{\mathcal{N}}(\neg A) = \bot$.

1. A is p_i, thus s^α satisfies p_i and $s^\alpha_{p_i} \in \mathcal{O}$.
2. A is $\neg B$, thus $\overline{s}^\alpha$ satisfies B.
   (a) B is p_i, thus $\overline{s}^\alpha$ satisfies p_i and $s^\alpha_{\neg p_i} \in \mathcal{O}$.
   (b) B is $\neg C$, thus $\overline{s}^\alpha$ satisfies $\neg C$, meaning that s^α satisfies C and by (HI) $s^\alpha_C \in \mathcal{O}$, and thus $s^\alpha_{\neg\neg C} \in \mathcal{O}$ by C2 since $\vDash_{PL} C \supset \neg\neg C$ and $\Gamma_C \sqsubseteq \Gamma_{\neg\neg C}$.
   (c) B is $C \supset D$, thus $\overline{s}^\alpha$ satisfies $C \supset D$, meaning that s^α satisfies C and $\overline{s}^\alpha$ satisfies D. But by (HI) $s^\alpha_C \in \mathcal{O}$ and $s^\alpha_{\neg D} \in \mathcal{O}$, and since $\vDash_{PL} C \supset (\neg D \supset \neg(C \supset D))$, we have $s^\alpha_C \sqsubseteq s^\alpha_{\neg D \supset \neg(C \supset D)}$, thus by C3 we have $s^\alpha_{\neg D} \sqsubseteq s^\alpha_{\neg(C \supset D)}$, and by C2 we obtain $s^\alpha_{\neg(C \supset D)} \in \mathcal{O}$ since $s^\alpha_{\neg D} \in \mathcal{O}$.
3. A is $B \supset C$, thus either
   (a) s^α satisfies $\Gamma_m = B \supset C$ and thus $s^\alpha_{B \supset C} \in \mathcal{O}$
   (b) s^α satisfies C ($\Gamma_B \neq \Gamma_C$) and by (HI) $s^\alpha_C \in \mathcal{O}$ and by C2 $s^\alpha_{B \supset C} \in \mathcal{O}$ since $\vDash_{PL} C \supset (B \supset C)$ and thus $\Gamma_C \sqsubseteq \Gamma_{B \supset C}$. □

Lemma 2. *If $s^\alpha_A \in \mathcal{O}$, then s^α satisfies O_iA.*

Proof. We proceed inductively on the length of the formula. Suppose that $s^\alpha_A \in \mathcal{O}$ but that s^α does not satisfy O_iA.

1. A is p_i, thus $s^\alpha_{p_i} \in \mathcal{O}$ and so s^α satisfies p_i.
2. A is $\neg B$, thus $\overline{s}^\alpha$ does not satisfy B.
   (a) B is p_i, thus $s^\alpha_{\neg pi} \in \mathcal{O}$ and so $\overline{s}^\alpha$ satisfies p_i
   (b) B is $\neg C$, thus $\overline{s}^\alpha$ does not satisfy $\neg C$, meaning that s^α does not satisfy C. Since $\vDash_{PL} \neg\neg C \supset C$, we obtain $s^\alpha_{\neg\neg C} \sqsubseteq s^\alpha_C$ and by C2 we have $s^\alpha_C \in \mathcal{O}$, and by (HI) s^α satisfies C.
   (c) B is $C \supset D$, thus $\overline{s}^\alpha$ does not satisfy $C \supset D$, meaning that either s^α does not satisfy C or $\overline{s}^\alpha$ does not satisfy D. By hypothesis, we have $s^\alpha_{\neg(C \supset D)} \in \mathcal{O}$, and by PL we obtain $s^\alpha_{\neg(C \supset D)} \sqsubseteq s^\alpha_C$ and $s^\alpha_{\neg(C \supset D)} \sqsubseteq s^\alpha_{\neg D}$ since $\vDash_{PL} \neg(C \supset D) \supset C$ and $\vDash_{PL} \neg(C \supset D) \supset \neg D$. Therefore, by C2 we obtain $s^\alpha_C \in \mathcal{O}$ and $s^\alpha_{\neg D} \in \mathcal{O}$, which by (HI) implies that s^α satisfies C and $\overline{s}^\alpha$ satisfies D.
3. A is $B \supset C$, thus s^α does not satisfy $\Gamma_m = B \supset C$ and either s^α does not satisfy C or $s^\alpha_B = s^\alpha_C$. However, by definition if s^α does not satisfy Γ_m, it implies that $s^\alpha_m \notin \mathcal{O}$, that is $s^\alpha_{B \supset C} \notin \mathcal{O}$. □

Lemma 3. *If $\Gamma_A \sqsubseteq \Gamma_B$, $\nvDash_{PL} B$ and s^α satisfies O_iA, then s^α satisfies O_iB.*

Proof. Assume $\Gamma_A \sqsubseteq \Gamma_B$, $\nvDash_{PL} B$ and s^α satisfies O_iA but s^α does not satisfy O_iB. By lemma 1 we have $s^\alpha_A \in \mathcal{O}$, thus by C2 $s^\alpha_B \in \mathcal{O}$ and by lemma 2 s^α satisfies O_iB. □

Lemma 4. *(A1) preserves validity.*

Proof. Suppose that (A1) does not preserve validity. Thus, we have $\vdash_{OL} \neg(O_iA \wedge O_i\neg A)$ and $\nvDash_{\mathcal{N}} \neg(O_iA \wedge O_i\neg A)$. However, if $\nvDash_{\mathcal{N}} \neg(O_iA \wedge O_i\neg A)$, then $\vDash_{\mathcal{N}} O_iA \wedge O_i\neg A$. It follows that s^α satisfies O_iA and $O_i\neg A$, meaning that s^α satisfies A and it satisfies $\neg A$. By lemma 1, it implies that both $s^\alpha_A \in \mathcal{O}$ and $s^\alpha_{\neg A} \in \mathcal{O}$, which contradicts C1. □

Lemma 5. *(R1) preserves validity.*

Proof. Assume that (R1) does not preserve validity. This means that we have a situation where $O_iA \vdash_{OL} O_iB$ is obtained by the use of (R1) but $O_iA \nvDash_{\mathcal{N}} O_iB$, that is $\vDash_{\mathcal{N}} O_iA$ and $\nvDash_{\mathcal{N}} O_iB$, and so s^α satisfies O_iA but does not satisfy O_iB. By the use of (R1), we know that $A \supset B$ is true by hypothesis and that $\nvDash_{PL} B$, and moreover that $\Gamma_A \sqsubseteq \Gamma_B$. Therefore, by lemma 3 s^α satisfies O_iB. □

Theorem 1. (Adequacy). *If $\vdash_{OL} A$, then $\vDash A$.*

Proof. Since OL is based upon PL, it suffices to show that (A1) and (R1) preserve validity, which follows from lemmas 4 and 5. □

Lemma 6. (Lindenbaum's lemma). *OL has a maximally consistent extension.*

Suppose $K = \bigcup_0^\infty K_i$, with $K_0 = OL$ the smallest set of wffs of $\mathcal{L}$ closed under the rules of PL, (A1) and (R1). Let $A_1, \ldots, A_n, \ldots$ be an arbitrary enumeration of OL's wffs. If $\vdash_{K_{n-1}} \neg A_n$, then $K_n = K_{n-1}$, else $K_n = K_{n-1} \cup \{A_n\}$. This way, we have K_i extension of OL for all $i \geq 0$. It is obvious that K is maximal since by construction either $A_i \in K$ or $\neg A_i \in K$ for all i. We now show that K is consistent.

Proof. Assume $K \vdash \perp$. Thus, there is a finite proof of $\perp$ from a finite subset of K, meaning that there is K_n such that $K_n \subset K$ and $K_n \vdash \perp$. If K_n is inconsistent, it implies that by construction the proposition A_n added to K_{n-1} broke K_n's consistency. However, such a situation is impossible. Indeed, this means that $K_n \vdash A_n \wedge \neg A_n$, with $K_n = K_{n-1} \cup \{A_n\}$ and $K_{n-1} \vdash \neg A_n$. But if $K_{n-1} \vdash \neg A_n$, then $K_n = K_{n-1}$, thus $K_n \nvdash \perp$. And if $K_{n-1} \nvdash \neg A_n$, then $K_n = K_{n-1} \cup \{A_n\}$, thus $K_n \nvdash \perp$ since $K_n \nvdash \neg A_n$. Therefore, K is consistent. □

Lemma 7. *$\mathcal{N}$ is a model of K.*

Proof. Assume that $\mathcal{N}$ is not a model of some maximally consistent extension K. It follows that there is a proposition A such that $\vdash_K A$ and $K \nvDash_{\mathcal{N}} A$. But if $\vdash_K A$, it means that there is a finite proof of A from a finite subset of K, thus $\vdash_{K_n} A$ and $K_n \nvDash_{\mathcal{N}} A$. However, since every subset of K is an extension of OL, it implies that $K_n \vdash_{OL} A$. By theorem 1, if $K_n \vdash_{OL} A$, then $K_n \nvDash_{\mathcal{N}} A$, which contradicts our first hypothesis. Therefore, $\mathcal{N}$ is a model of K. □

Theorem 2. (Completeness)**.** *If* $H \vDash C$, *then* $H \vdash_{OL} C$.

Proof. Assume that there is a maximally consistent extension of OL where $H \vDash_{\mathcal{N}} C$ and $H \nvdash_K C$. Since K is maximally consistent, it follows that $H \vdash_K \neg C$, and since $\mathcal{N}$ is a model of K it implies that $H \vDash_{\mathcal{N}} \neg C$. However, if $H \vDash_{\mathcal{N}} C$, then $H \nvDash_{\mathcal{N}} \neg C$, which contradicts our first hypothesis. Therefore, if $H \vDash_{\mathcal{N}} C$, then $H \vdash_K C$ for any $\mathcal{N}$. □

The reader may note that the deduction theorem does not need to be proven in a natural deduction system since it follows immediately from the conditional proof rule.

4. Validity of legal reasoning

4.1. *Graphical representation*

The idea that lies behind this approach is that, according to the semantical dichotomy, a scenario w is divided in two parts: one descriptive ($\mathfrak{D}$) and the other normative ($\mathfrak{N}$), which contains a set of obligations $\mathcal{O}$.[25] A scenario w is inconsistent if either $\mathfrak{D}$ or $\mathfrak{N}$ is. The scenario can be seen as a model $\mathfrak{M}$ where the descriptive part represents $\mathcal{M}$ and the normative one $\mathcal{N}$. The validity of an argument is tested via the notion of a *counterexample*, that is, a scenario w in which the premises are assumed to be true but the conclusion is false. Since a valid argument does not possess any counterexample, it follows that if the scenario is consistent, then the argument is invalid. As such, if the scenario is consistent, there is a truth-value assignment which makes the premises true but the conclusion false. Otherwise, if the scenario is inconsistent (i.e., it is impossible for the premises to be true while the conclusion is false), then the argument is valid. The propositional rules (figure 1) representing schematically truth conditions for complex propositions are quite straightforward (cf. Garson 2006, p. 91).

$A \supset B$	$\neg(A \supset B)$	$A \wedge B$	$\neg(A \wedge B)$	$A \vee B$	$\neg(A \vee B)$	$\neg\neg A$
$\neg A$ | B	A $\neg B$	A B	$\neg A$ | $\neg B$	A | B	$\neg A$ $\neg B$	A

Figure 1: Propositional rules

25 This section summarizes the results presented in Peterson (2013b).

The semantical dichotomy implies that normative and descriptive atoms are not true in the same conditions. Otherwise, complex normative formulas behave schematically as complex descriptive formulas since both are composed of the same logical connectives (figure 1). The difference between the truth value of normative and descriptive formulas can be seen at the atomic level. While the truth of a descriptive proposition can be represented by the fact that it belongs to the descriptive part of a scenario, the truth of a normative proposition depends upon a norm established by some authority. A normative proposition A is obligatory if and only if there is a norm which makes the action described by A (i.e., Γ_A) an obligation. As such, if a normative proposition $O_i A$ is true for a scenario w, then there is a norm (established by authority i) which makes Γ_A obligatory and the action described by A pertains to a set of obligations. Considering that the logical connectives can (classically) be reduced to compositions of $\neg$ and $\supset$, we only represent schematically the truth conditions for $O_i p$, $O_i \neg A$ and $O_i(A \supset B)$ (figure 2). Note, however, that derived rules could easily be constructed from the lemmas that the reader will find at the end of this section. These rules, combined with conditions C1, C2 and C3, allow us to test the validity of a normative inference. Their construction is possible according to lemmas 1 and 2.

As an example, let us test (R1)'s validity with $n = 1$ (figure 3). Assume that w is a counter-example for (R1). Then, place the assumptions in their respective descriptive or normative part of w. According to the rules for normative atoms, Γ_p is in $\mathcal{O}$. Since $p \supset q$ is assumed to be true in the descriptive part of w, it follows that $\Gamma_p \sqsubseteq \Gamma_q$ holds in its normative part. By C2 we obtain Γ_q in O,

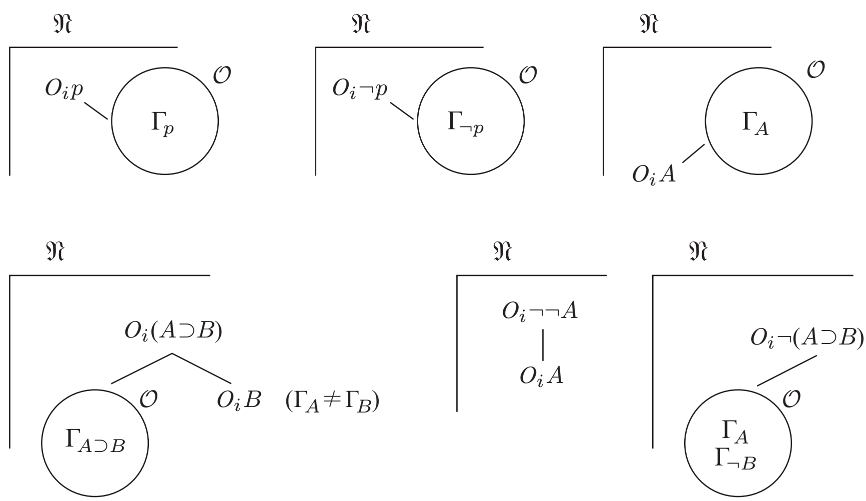

Figure 2: Truth conditions of normative atoms

hence Oq can be concluded in the normative part of w. The only branch in the normative part closes (schematically represented by $\times$) since it contains a contradiction (i.e., $O_i q$ and $\neg O_i q$). Therefore, there is no possible scenario in which the premises of (R1) are true while its conclusion is false, hence the proof of (R1)'s validity.

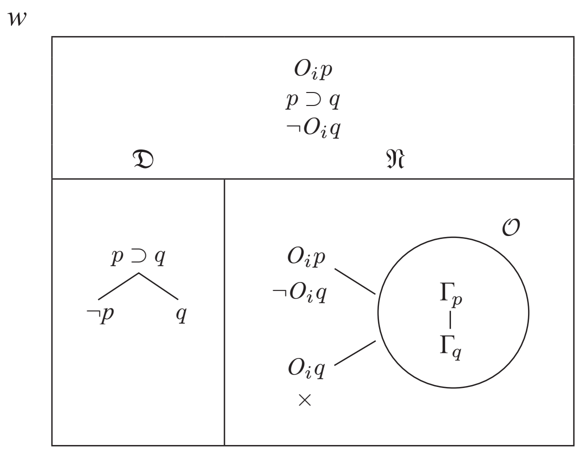

Figure 3: Test of (R1)'s validity

We now list some semantical properties of the model. These properties are useful for both the formal semantical proofs and the graphical representation. The following lemmas can be used to construct derived rules.

Lemma 8. *If* $\Gamma_A \in \mathcal{O}$, *then* $\Gamma_{B \supset A} \in \mathcal{O}$ *for any* B.

Proof. Assume that $\Gamma_A \in \mathcal{O}$. By *PL* we know that $A \supset (B \supset A)$ is true in $\mathfrak{D}$, thus $\Gamma_A \sqsubseteq \Gamma_{B \supset A}$ holds in $\mathfrak{N}$. Hence, by C2 $\Gamma_{B \supset A} \in \mathcal{O}$. □

Lemma 9. *If* $\Gamma_A \in \mathcal{O}$ *and* $\Gamma_B \in \mathcal{O}$, *then* $\Gamma_{A \wedge B} \in \mathcal{O}$.

Proof. Assume $\Gamma_A \in \mathcal{O}$ and $\Gamma_B \in \mathcal{O}$. By *PL* we have $A \supset (B \supset (A \wedge B))$ true in $\mathfrak{D}$, thus $\Gamma_A \sqsubseteq \Gamma_{B \supset (A \wedge B)}$ holds in $\mathfrak{N}$. By C2, we obtain $\Gamma_{B \supset (A \wedge B)} \in \mathcal{O}$, and by C3 $\Gamma_B \sqsubseteq \Gamma_{A \wedge B}$. Therefore, by C2 we have $\Gamma_{A \wedge B} \in \mathcal{O}$. □

Lemma 10. *If* $\Gamma_{A \wedge B} \in \mathcal{O}$, *then* $\Gamma_A \in \mathcal{O}$.

Proof. Assume $\Gamma_{A \wedge B} \in \mathcal{O}$. By *PL* we have $(A \wedge B) \supset A$ true in $\mathfrak{D}$, thus $\Gamma_{A \wedge B} \sqsubseteq \Gamma_A$ holds in $\mathfrak{N}$, and by C2 $\Gamma_A \in \mathcal{O}$. □

Lemma 11. (Deontic detachment)**.** *If* $\Gamma_A \in \mathcal{O}$ *and* $\Gamma_{A \supset B} \in \mathcal{O}$, *then* $\Gamma_B \in \mathcal{O}$.

Proof. Assume that $\Gamma_A \in \mathcal{O}$ and $\Gamma_{A \supset B} \in \mathcal{O}$. Then by lemma 3 $\Gamma_{A \wedge (A \supset B)} \in \mathcal{O}$. Since by *PL* $(A \wedge (A \supset B)) \supset B$ is true in $\mathfrak{D}$, we have $\Gamma_{A \wedge (A \supset B)} \sqsubseteq \Gamma_B$ in $\mathfrak{N}$, and by C2 $\Gamma_B \in \mathcal{O}$. Equivalently, assume that $\Gamma_A \in \mathcal{O}$ and $\Gamma_{A \supset B} \in \mathcal{O}$. By C3 $\Gamma_A \sqsubseteq \Gamma_B$ holds in $\mathfrak{N}$ and by C2 $\Gamma_B \in \mathcal{O}$. □

Lemma 12. (De Morgan)**.** $\Gamma_{A \wedge B} \in \mathcal{O}$ *if and only if* $\Gamma_{\neg(\neg A \vee \neg B)} \in \mathcal{O}$.

Proof. ($\Rightarrow$) Assume $\Gamma_{A \wedge B} \in \mathcal{O}$. Since by *PL* we have $(A \wedge B) \supset \neg(\neg A \vee \neg B)$ true in $\mathfrak{D}$, we have $\Gamma_{A \wedge B} \sqsubseteq \Gamma_{\neg(\neg A \vee \neg B)}$ in $\mathfrak{N}$, thus $\Gamma_{\neg(\neg A \vee \neg B)} \in \mathcal{O}$ by C2.

($\Leftarrow$) Assume $\Gamma_{\neg(\neg A \vee \neg B)} \in \mathcal{O}$. Since by *PL* we have $\neg(\neg A \vee \neg B) \supset (A \wedge B)$ true in $\mathfrak{D}$, we have $\Gamma_{\neg(\neg A \vee \neg B)} \sqsubseteq \Gamma_{A \wedge B}$ in $\mathfrak{N}$, thus $\Gamma_{A \wedge B} \in \mathcal{O}$ by C2.

Lemma 13. *The next lemmas are consequences of (De Morgan) or lemma 8.*

$$\textit{If } \Gamma_A \in \mathcal{O}, \textit{ then } \Gamma_{A \vee B} \in \mathcal{O} \tag{6}$$

$$\textit{If } \Gamma_A \in \mathcal{O}, \textit{ then } \Gamma_{\neg(\neg A \wedge \neg B)} \in \mathcal{O} \tag{7}$$

$$\textit{If } \Gamma_{\neg A} \in \mathcal{O}, \textit{ then } \Gamma_{\neg A \vee \neg B} \in \mathcal{O} \tag{8}$$

$$\textit{If } \Gamma_{\neg A} \in \mathcal{O}, \textit{ then } \Gamma_{\neg(A \wedge B)} \in \mathcal{O} \tag{9}$$

$$\textit{If } \Gamma_{\neg(A \vee B)} \in \mathcal{O}, \textit{ then } \Gamma_{\neg A} \in \mathcal{O} \tag{10}$$

$$\textit{If } \Gamma_{\neg(\neg A \vee \neg B)} \in \mathcal{O}, \textit{ then } \Gamma_A \in \mathcal{O} \tag{11}$$

4.2. *Soundness of legal inference*

Formal logic is relevant to legal discourse and critical thinking insofar as it enables one to have a well-defined concept of validity, and hereby to have a method to prove that inferences are valid or not. However, each formal framework that tries to model the validity of our natural language possesses its limits, the natural language being vast and rich and, as we will see below, ours possesses its limitations too. Hence, it might be useful to present some informal considerations one must take into account when analyzing normative inferences. A *sound* inference is valid and has true (or acceptable) premises (cf. Peterson 2013a). As such, not all valid inferences are sound. Without pretending to be exhaustive, we provide the reader with some philosophical insights regarding the acceptability of the premises within a legal argument (see also Peterson 2013b).

A sound inference must be valid. Note, though, that our framework possesses its limits (e.g., it cannot model deontic conditionals), and therefore it is not because an argument is invalid within *OL* that it must be rejected. When the logical form of the argument falls within the scope of *OL*, the second step after verifying its validity is to determine whether or not its premises are acceptable. Since the truth of a normative proposition depends upon a norm, which is established by some authority, the first thing to do is to contextualize the normative propositions to a legal authority: is it true according to the Constitution? the Civil Code? the Criminal Code? the Canadian Charter of Rights and Freedom? It is noteworthy that some inferences can be fallacious when the meaning of 'ought' is not the same throughout the argument.

In addition to the normative propositions, the descriptive propositions within the argument should also be contextualized to some authority. For instance, the meaning of 'Paul respects his neighbour' will vary depending whether the legal authority is the Civil Code or the Criminal Code or the Canadian Charter of Rights and Freedoms. Then, one must determine what legally counts as a lack of respect from the Civil Code's point of view. In other words, one must answer the question 'what action counts-as *A* from the authority's standpoint?'.[26]

The context of the argument must also be considered to establish the truth of the normative premises. For example, if 'ought' is understood legally, the proposition 'Paul ought to tell the truth' is not true in general but is true if Paul is providing testimony in a court of justice. Furthermore, the descriptive propositions used together with (R1) must be analyzed in terms of necessity and sufficiency. One must determine whether or not the conditional represents a semantical entailment between actions and see if the consequent is necessarily entailed by the action described in the antecedent.

Also, since our framework is not able to deal properly with contrary-to-duty reasoning and deontic conditionals, it might happen that conflicting obligations arise within a given situation. For example, consider the following argument (and assume a context in which premises 2 and 3 hold).

1. Paul ought to rescue Peter from drowning.
2. If Paul rescues Peter, then Paul calls 911.
3. If Paul calls 911, then Paul breaks into Sam's house.

∴ Paul ought to break into Sam's house.

Assuming that breaking and entering is legally forbidden, this yields a conflict of obligations when we add the following implicit premise.

4. Paul ought to not break into Sam's house.

Alchourrón and Makinson (1981) suggested that conflicts of obligations can be resolved by introducing a relation of hierarchy (or priority) between obligations. This idea was used by many logicians in non-monotonic deontic logic. Among others, van der Torre and Tan (1997) provided an insightful analysis of how conflicts of obligations can emerge in various situations. According to their analysis, the aforementioned example is a case of *weak*

[26] For an analysis of 'count as', see Jones and Sergot (1996) and Boella and van der Torre (2006).

overridden defeasibility, where the contradiction appears because there is a conflict between the *prima facie* obligation to not break into Sam's house and the obligation to rescue Peter under the circumstances that he is drowning. Even though Paul has in general the obligation to not break into Sam's house, there might be specific situations in which it can be excused (but not allowed nor justified), namely if it is necessary to save Peter's life.

The assumption that premise 3 holds implies that Paul could argue that he should not be punished for breaking and entering since it was necessary to break into Sam's house to save Peter's life. Indeed, under specific circumstances, the defense of necessity can be invoked to excuse (but not justify) the violation of the law.[27] In the aforementioned example, Paul's actions are not morally voluntary given that he has no realistic choice: all things considered, it is reasonable to expect that Paul will break into Sam's house to call 911 if it is indeed the only possible course of action he has to save Peter's life. In such a context, the obligation to not break into Sam's house is *overshadowed* (as opposed to *cancelled*) by the obligation to break into Sam's house so that Paul can save Peter's life. Hence, in this context, Paul can be excused to break into Sam's house. Even though the obligation to not break into Sam's house is still in force (i.e., it does not simply disappear when a conflict arises), it is not *actual* in the sense that it should not be guiding Paul's actions in that context. Having this information at hand, one can thus reject premise 4 and the argument can be formalized within our framework while avoiding the conflict.

At last, the premises of a normative inference can be analyzed in the light of the ought-implies-can principle. Instead of the aforementioned example, consider the following argument.

1. Paul ought to rescue Peter from drowning.
2. If Paul rescues Peter from drowning, then Paul jumps in the water to retrieve Peter.

$\therefore$ Paul ought to jump in the water to retrieve Peter.

If for some reasons Paul cannot swim (e.g., if Paul has two broken arms), then premise 2 is a violation of the ought-implies-can principle. Thus, it must be rejected since in this case 'rescuing Peter from drowning' does not entail that Paul jumps in the water to retrieve him.

4.3. *The paradoxes of deontic logic*

Having presented some insights regarding the analysis of the soundness of legal inferences, let us now examine how our framework deals with the

[27] This has been accepted by the Supreme Court. See *Perka c. La Reine*, *[1984] 2 R.C.S. 232*.

notorious *paradoxes* of deontic logic. Following Åqvist's (2002) presentation, a formula A is a paradox for a deontic logic Δ either if it is derivable in Δ but its translation does not seem derivable within the natural normative language, or it is *not* derivable in Δ but its translation seems derivable within the natural normative language. In other words, a formula is a paradox either when it can be derived but it should not, or when it cannot be derived but it should.

4.3.1. Ross's paradox

Ross's (1941) disjunctive paradox appeared in reaction to Jørgensen's (1937) dilemma and aimed to show that a logic for imperatives does not behave similarly to propositional logic. For instance, the satisfaction of the imperative *Mail the letter!* does not imply that *Mail the letter or burn it!* is also satisfied.

Theorem 3. $\vdash_{OL} O_i p \supset O_i(p \vee q)$

Proof. Assume $\vDash_{\mathcal{N}} O_i p$ but $\vDash_{\mathcal{N}} \neg O_i(p \vee q)$. Thus, s^α satisfies $O_i p$ but does not satisfy $\mathcal{O}_i(p \vee q)$, which by definition is $O_i(\neg p \supset q)$. Thus, s^α satisfies p and $s^\alpha_p \in O$ but s^α does not satisfy $\neg p \supset q$, and so s^α does not satisfy $\Gamma_m = \neg p \supset q$. By *PL*, we have $\Gamma_p \sqsubseteq \Gamma_m$, and so by C2 s^α satisfies Γ_m.

1	$O_i p$	H
2	$p \supset (p \vee q)$	*PL*
3	$O_i(p \vee q)$	(R1) 1,2
4	$\vdash_{OL} O_i p \supset O_i(p \vee q)$	$\supset$ *in* 1-3

□

4.3.2. Prior's paradox

Prior's (1954) paradox of derived obligations was an objection against von Wright's (1951) initial approach to deontic logic. It aimed to show that von Wright's notion of *commitment*, represented by $O(p \supset q)$, fails in contexts of derived obligations: it is not because p is forbidden that doing p *commits* one to do q.

Theorem 4. $\vdash_{OL} F_i p \supset O_i(p \supset q)$

Proof. Assume $\vDash_{\mathcal{N}} F_i p$ but $\vDash_{\mathcal{N}} \neg O_i(p \supset q)$. By definition it means that s^α satisfies $O_i \neg p$, thus s^α satisfies $\neg p$, meaning that $\overline{s}^\alpha$ satisfies p, and s^α does not satisfy $O_i(p \supset q)$, thus does not satisfy $p \supset q$. Therefore, s^α does not satisfy $\Gamma_m = p \supset q$ and $s^\alpha_m \notin \mathcal{O}$. However, we have $\overline{s}^\alpha_p \sqsubseteq \Gamma_m$ since $\vDash_{PL} \neg p \supset (p \supset q)$, and so by C2 $s^\alpha_m \in \mathcal{O}$.

1	$F_i p$	H
2	$O_i \neg p$	def F1
3	$\neg p \supset (p \supset q)$	*PL*
4	$O_i(p \supset q)$	(R1) 2,3
5	$\vdash_{OL} O_i p \supset O_i(p \supset q)$	$\supset in$ 1-4 $\square$

Theorem 5. $\vdash_{OL} O_i p \supset O_i(q \supset p)$

Proof. Assume $\vDash_{\mathcal{N}} O_i p$ but $\vDash_{\mathcal{N}} \neg O_i(q \supset p)$. Thus, s^α satisfies $O_i p$ but does not satisfy $O_i(q \supset p)$, meaning that s^α satisfies p, and so $s^\alpha_p \in \mathcal{O}$, but s^α does not satisfy $q \supset p$. Therefore, s^α does not satisfy $\Gamma_m = (q \supset p)$ and $s^\alpha_m \notin \mathcal{O}$. However, $s^\alpha_p \sqsubseteq s^\alpha_m$ since $\vDash_{PL} p \supset (q \supset p)$, and thus by C2 $s^\alpha_m \in \mathcal{O}$.

1	$O_i p$	H
2	$p \supset (q \supset p)$	*PL*
3	$O_i(q \supset p)$	(R1) 1,2
4	$\vdash_{OL} O_i p \supset O_i(q \supset p)$	$\supset in$ 1-3 $\square$

Although it might surprise the reader, we think that Ross's and Prior's paradoxes are both desirable consequences of *OL*. Both paradoxes can be compared with theorem 6 since there is an equivalence between $O_i(p \vee q)$ and $O_i \neg(\neg p \wedge \neg q)$, and between $O_i(p \supset q)$ and $O_i \neg(p \wedge \neg q)$.

Theorem 6. $\vdash_{OL} F_i p \supset F_i(p \wedge q)$

Proof. Assume $\vDash_{\mathcal{N}} F_i p$ but $\vDash_{\mathcal{N}} \neg F_i(p \wedge q)$. By definition it means that s^α satisfies $O_i \neg p$ but does not satisfy $O_i \neg(p \wedge q)$. Thus, s^α satisfies $\neg p$, meaning that $\overline{s}^\alpha$ satisfies p and so $\overline{s}^\alpha_p \in O$, but s^α does not satisfy $\neg(p \wedge q)$, which by definition is $p \supset \neg q$. Therefore, s^α does not satisfy $\Gamma_m = p \supset \neg q$, and so $s^\alpha_m \notin O$. However, $\overline{s}^\alpha_p \sqsubseteq \Gamma_m$ since $\vDash_{PL} \neg p \supset (p \supset \neg q)$, and so by C2 $s^\alpha_m \in O$.

1	$F_i p$	H
2	$O_i \neg p$	def F1
3	$\neg p \supset \neg(p \wedge q)$	*PL*
4	$O_i \neg(p \wedge q)$	(R1) 2,3
5	$F_i(p \wedge q)$	def F4
6	$\vdash_{OL} F_i p \supset F_i(p \wedge q)$	$\supset in$ 1-5 $\square$

Recall that *OL* aims to model how the property *obligatory* is transmitted from an action to another. Thus, while Ross's paradox says that if the action described by p is obligatory, then any conjunction of actions which includes the action described by $\neg p$ is forbidden, Prior's paradox says that if the action described by p is forbidden, then any conjunction of actions which includes the action described by p is also forbidden. Hence the comparison with theorem 6: if the action described by p is forbidden, then so is the action described by $p \wedge q$ for any q. When considered from a legal point of view, these theorems are desirable. If an action is legally forbidden, then any conjunction of action which includes the forbidden action is also forbidden, notwithstanding the nature of the other action. If it is forbidden to steal a car, then it is also forbidden to steal a car while eating ice cream.

4.3.3. Chisholm's paradox

Chisholm's (1963) contrary-to-duty paradox is the most damaging for monadic deontic logic. It shows that monadic frameworks are not able to properly represent conditional obligations. Although Prior's paradox motivated von Wright (1956, 1967) to introduce dyadic deontic logic (see Åqvist (2002) for an overview of dyadic deontic logic and Tomberlin (1981) for a discussion of conditional obligations), the need for a dydadic deontic logic arises when one considers Chisholm's puzzle. While the following sentences are perfectly consistent within the natural normative language, the conjunction of their translation is inconsistent in standard monadic deontic logic, i.e., the modal logic *KD* (cf. Åqvist 2002, p. 155).[28]

1. Paul ought to not steal.
2. It ought to be that if Paul does not steal, then he does not give back what was stolen.
3. If Paul steals, then he ought to give back what was stolen.
4. Paul steals.

These are translated in *KD* by:

$$O\neg p \tag{12}$$

$$O(\neg p \supset \neg q) \tag{13}$$

$$p \supset Oq \tag{14}$$

$$p \tag{15}$$

Propositions 13 and 14 must be translated differently since they are independent within the natural normative language (cf. Åqvist 2002). Translating 13 by

$$\neg p \supset O\neg q \tag{16}$$

[28] See Åqvist (1967), Decew (1981) or Tomberlin (1981; 1983) for a discussion of the paradox.

would make 16 a consequence of 15, as translating 14 by

$$O(p \supset q) \tag{17}$$

would make 17 a consequence of 12.

We agree with the literature that material conditionals cannot adequately model deontic conditionals. As such, we do not think that $O(p \supset q)$ is the appropriate formulation to model an obligation Oq conditional to a context p. Since mixed formulas are not available within our framework, it follows that Chisholm's paradox cannot be formulated appropriately. The available translations are:

$$O_i \neg p \tag{18}$$

$$O_i(\neg p \supset \neg q) \tag{19}$$

$$O_i(p \supset q) \tag{20}$$

$$p \tag{21}$$

$$O_i \neg p \tag{22}$$

$$O_i \neg p \supset O_i \neg q \tag{23}$$

$$O_i(p \supset q) \tag{24}$$

$$p \tag{25}$$

$$O_i \neg p \tag{26}$$

$$O_i(\neg p \supset \neg q) \tag{27}$$

$$O_i p \supset O_i q \tag{28}$$

$$p \tag{29}$$

$$O_i \neg p \tag{30}$$

$$O_i \neg p \supset O_i \neg q \tag{31}$$

$$O_i p \supset O_i q \tag{32}$$

$$p \tag{33}$$

It is noteworthy that each of these translations is consistent within our language. All imply $O_i \neg q$ but none implies $O_i q$ since $\Gamma_p \notin \mathcal{O}$. However, they fail to model Chisholm's paradox insofar as the propositions within these translations are not independent from each other. Indeed, 20 (=24) is a consequence of 18 (=22) in virtue of theorem 4, and 28 (=32) is a consequence of 26 (=30) and theorem 7.

Theorem 7. $\vdash_{OL} O_i\neg p \supset (O_i p \supset O_i q)$

Proof.

1	$O_i\neg p$	H
2	$\neg(O_i\neg p \wedge O_i\neg\neg p)$	(A1)
3	$O_i\neg p \supset \neg O_i\neg\neg p$	def $\wedge$ 2
4	$\neg O_i\neg\neg p$	$\supset$ *out* 1,3
5	$\neg O_i\neg\neg p \supset (O_i\neg\neg p \supset O_i q)$	*PL*
6	$O_i\neg\neg p \supset O_i q$	$\supset$ *out* 4,5
7	$\quad O_i p$	H
8	$\quad p \supset \neg\neg p$	*PL*
9	$\quad O_i\neg\neg p$	(R1) 5,6
10	$\quad O_i\neg\neg p \supset O_i q$	(Reit) 6
11	$\quad O_i q$	$\supset$ *out* 4,5
12	$O_i p \supset O_i q$	$\supset$ *in* 7-11

□

Chisholm's paradox is thus a limitation of our framework. However, the aim of this paper was not to model deontic conditionals but was rather to define a proper consequence relation for inferences within which there are no mixed formulas. The aim was to model the transmission of the property *obligatory* between *normative* propositions. The discussion of contrary-to-duty reasoning and deontic conditionals goes far beyond this paper and is an avenue for future research (see Peterson 2014b).

4.3.4. Detachment and Augmentation

Chisholm's paradox is related to the problems of *detachment* and *augmentation* of deontic conditionals (cf. Jones 1991).[29] The problem of (factual) detachment can be summarized as follows: although the detachment of a conditional obligation from its context seems suitable, as in Chisholm's paradox where we want to derive that Paul ought to give back what was stolen from the contrary-to-duty $p \supset Oq$ and the fact that Paul stole, there are situations where some extra conditions can make detachment undesirable. For instance, even though Paul has the conditional obligation to drive his wife to the hospital if she is near delivering, he does not have this obligation

[29] There is a distinction between *deontic* detachment, which is expressed by lemma 11, and *factual* detachment, which is the detachment problem we are discussing within this section. See Loewer and Belzer (1983) for the distinction.

under the conditions that she is near delivering *and* (for some reason) he is drunk. As such, only a form of *restricted* detachment seems desirable, namely when we can insure that no other information will thwart detachment.[30]

The problem of (unrestricted) detachment is related to the problem of augmentation. In a nutshell, the problem of augmentation comes from the fact that one cannot strengthen the antecedent of a deontic conditional. Hence,

$$\frac{p \supset Oq}{(p \wedge r) \supset Oq}$$

is not a valid inference pattern for deontic conditionals since the other conditions r might block the detachment of Oq. If one wants to model Chisholm's paradox and contrary-to-duty reasoning, then one will need to address these problems. This, which is also why we only concentrated upon the consequence relation of normative inferences in which there are no mixed formulas, will be the subject of another paper.[31]

4.3.5. Forrester's paradox

Forrester's (1984) paradox aims to show that (R1) is not an acceptable rule for normative reasoning. The paradox is similar to Prior's (1958) initial good Samaritan paradox and Nozick's and Routley's (1962) robbery paradox.[32]Assume a legal system in which a murder must be gentle. The original formulation of the paradox uses a conditional obligation (i.e., if Jones murders Smith, then he ought to do it gently)[33], but since in this respect it does not tell us more than what was already pointed out by Chisholm, we will only present the argument against (R1). The paradox follows from:

1. Jones ought to murder Smith gently.
2. If Jones murders Smith gently, then Jones murders Smith.
3. Jones ought to murder Smith.

According to Forrester, (R1) must be rejected since it allows one to conclude that Jones ought to murder Smith in a context where gentle murders are preferable to violent ones. The validity of this inference in our framework follows from lemma 14.

[30] On this point, see Decew (1981), Loewer and Belzer (1983), Vorobej (1986), Jones (1991), Alchourrón (1996) and Bonevac (1998).
[31] See Peterson (2014b).
[32] The good Samaritan paradox was reformulated by Åqvist (1967).
[33] The original paradox also insists on the contradiction between the assumption that *Jones ought to not murder Smith* and the conclusion from (R1) that *Jones ought to murder Smith*.

Lemma 14. *If* $\vDash_{\mathcal{N}} O_i p$ *and* $\vDash_{\mathcal{M}} p \supset q$, *then* $\vDash_{\mathcal{N}} O_i q$.

This represents the principle of deontic consequences. Notice that both $\vDash_{\mathcal{N}} O_i p$ *and* $\vDash_{\mathcal{M}} p \supset q$ *say that the propositions are true* (*respectively in a normative model* $\mathcal{N}$ *and a descriptive model* $\mathcal{M}$), *which does not mean that the propositions are valid*, *that is*, *true in every models.*

Proof. Assume $\vDash_{\mathcal{N}} O_i p$ and $\vDash_{\mathcal{M}} p \supset q$ but $\vDash_{\mathcal{N}} \neg O_i q$. Therefore, s^α satisfies $O_i p$ but does not satisfy $O_i q$, meaning that it satisfies p but not q, thus $s^\alpha_p \in \mathcal{O}$ but $s^\alpha_q \notin \mathcal{O}$. However, by hypothesis $\vDash_{\mathcal{M}} p \supset q$, so $\Gamma_p \sqsubseteq \Gamma_q$, and therefore s^α satisfies q by C2.

However, despite its validity, this argument is not sound. As it was argued by Jacquette (1986, p.762), the first premise is to be rejected. Of course, if one accepts a legal system where Jones ought to murder Smith gently, then one accepts a system where Jones ought to murder Smith! But the obligation of 'murdering gently' must be rejected in favour of the obligation of 'not murdering violently', and this does not lead to paradoxical results.[34]

4.3.6. Lemmon's paradox

Lemmon's (1962) paradox insists on the possibility of moral dilemmas and conflicting obligations. Although it is possible to face conflicting obligations in real life situations, a deontic logic which accepts (A1) as an axiom for normative consistency cannot model such situations. Moreover, when such a conflict happens, a deontic logic which satisfies (A1) will dictate that anything is obligatory insofar as any extension of classical propositional logic validates $\vdash \bot \supset A$ for any A (cf. Horty 1997).[35] Thus, from Paul's obligation to pick up his kids at the kindergarten by 5 p.m. and from his obligation to help the victim of a car accident that just happened, Paul can conclude that he ought to commit adultery, assuming that he picks up his kids by 5 p.m. if and only if he does not help the victim.

This, however, is not a paradox for *OL*. Indeed, axiom (A1) says that an authority i cannot state that both A and $\neg A$ are obligatory at the same time. This is acceptable insofar as norms are meant to guide one's actions, and it would be impossible to act within the frame dictated by inconsistent norms. That being said, the obligation to pick up his kids in the aforementioned example is not of the same type as his obligation to help the victim of the car accident. Hence, the conflict can be translated by $O_i p \wedge O_j \neg p$, which is not inconsistent in *OL*. In the eventuality that the conflict arises from

[34] For an analysis of Forrester's paradox, see Peterson and Marquis (2012).
[35] See also theorem 7 above.

contrary-do-duty reasoning or conditional obligations, it can be solved through the relation of hierarchy as we saw previously.

4.3.7. Aggregation

Aggregation of obligations is sometimes contested insofar as it can lead to conflicting obligations (see for instance Schotch and Jennings (1981) and Horty (1997)). These conflicts, however, often arise from different types of obligations. Since *OL* concerns only one type of obligation, it is acceptable to accept that if an authority makes *A* and *B* obligatory, then it makes both obligatory together. This is represented by theorem 8.[36]

Theorem 8. $\vdash_{OL} O_i(p \wedge q) \equiv (O_i p \wedge O_i q)$

Proof. It follows from theorems 9 and 10. □

Theorem 9. $\vdash_{OL} O_i(p \wedge q) \supset (O_i p \wedge O_i q)$

Proof. Assume $\vDash_{\mathcal{N}} O_i(p \wedge q)$ but $\vDash_{\mathcal{N}} \neg(O_i p \wedge O_i q)$. Thus, s^α satisfies $O_i(p \wedge q)$, which by definition means that s^α satisfies $\neg(p \supset \neg q)$, and so $\overline{s}^\alpha$ satisfies $p \supset \neg q$. Thus, s^α satisfies p and $\overline{s}^\alpha$ satisfies $\neg q$, meaning that s^α satisfies q. However, s^α does not satisfy $(O_i p \wedge O_i q)$, meaning that either (a) s^α does not satisfy $O_i p$ or (b) it does not satisfy $O_i q$.

(a) If s^α does not satisfy $O_i p$, it implies that it does not satisfy p, which is inconsistent with our hypothesis

(b) If s^α does not satisfy $O_i q$, it implies that it does not satisfy q which is also inconsistent with our hypothesis

1	$O_i(p \wedge q)$	H
2	$(p \wedge q) \supset p$	*PL*
3	$O_i p$	(R1) 1,2
4	$(p \wedge q) \supset q$	*PL*
5	$O_i q$	(R1) 1,4
6	$O_i p \wedge O_i q$	$\wedge$ *in* 3,5
7	$\vdash_{OL} O_i(p \wedge q) \supset (O_i p \wedge O_i q)$	$\supset$ *in* 1-6

□

[36] This principle can be contested if we consider contrary-to-duty reasoning and conditional obligations. As the reader will see, we will change our mind in Peterson (2014c) and reject the aggregation principle on the grounds that $\wedge$ does not adequately represent action conjunction.

Theorem 10. $\vdash_{OL} (O_i p \wedge O_i q) \supset O_i(p \wedge q)$

Proof. Assume $\vDash_{\mathcal{N}} O_i p \wedge O_i q$ but $\vDash_{\mathcal{N}} \neg O_i(p \wedge q)$. Thus, s^α satisfies $O_i p$ and satisfies $O_i q$, meaning that it satisfies p and it satisfies q, and so $s_p^\alpha, s_q^\alpha \in \mathcal{O}$, but it does not satisfy $O_i(p \wedge q)$. By definition, it implies that s^α does not satisfy $\neg(p \supset \neg q)$, thus s^α does not satisfy $p \supset \neg q$. Thus, either s^α does not satisfy p, which is inconsistent with our hypothesis, or $\bar{s}^\alpha$ does not satisfy $\neg q$, meaning that s^α does not satisfy q, which is also inconsistent with our hypothesis.

1	$O_i p \wedge O_i q$	H
2	$(p \wedge q) \supset (p \wedge q)$	*PL*
3	$O_i(p \wedge q)$	(R1) 1,2
4	$\vdash_{OL} (O_i p \wedge O_i q) \supset O_i(p \wedge q)$	$\supset$ *in* 1-3

□

4.3.8. Contingency

The principle of normative contingency was first advocated by von Wright (1951). It has since then been defended by Chellas (1974) and Jones and Pörn (1985). This principle implies that tautologies are neither unconditional nor derived obligations. As it is shown by the following lemmas, our approach respects contingency.

Lemma 15. $\nvdash_{OL} O_i p \supset O_i(\neg p \supset \neg p)$

Proof. We show that there is a model $\mathcal{N}$ such that $\vDash_{\mathcal{N}} \neg(O_i p \supset O_i(\neg p \supset \neg p))$. Suppose $\vDash_{\mathcal{N}} O_i p$ and $\vDash_{\mathcal{N}} \neg O_i(\neg p \supset \neg p)$. So s^α satisfies $O_i p$ and thus $s_p^\alpha \in \mathcal{O}$ but s^α does not satisfy $O_i(\neg p \supset \neg p)$, meaning that s^α does not satisfy $\neg p \supset \neg p$ and either s^α does not satisfy $\neg p$, thus $\bar{s}^\alpha$ does not satisfy p, or $\Gamma_{\neg p} = \Gamma_{\neg p}$. Both cases are consistent with C1–C3 and the definition of satisfaction. □

Lemma 16. $\nvdash_{OL} O_i p \supset O_i(p \vee \neg p)$

Proof. We show that there is a model $\mathcal{N}$ such that $\vDash_{\mathcal{N}} \neg(O_i p \supset O_i(p \vee \neg p))$. Assume $\vDash_{\mathcal{N}} O_i p$ and $\vDash_{\mathcal{N}} \neg O_i(\neg p \vee \neg p)$. Thus, s^α satisfies $O_i p$ but does not satisfy $O_i(p \vee \neg p)$. By definition, it means that s^α satisfies p but does not satisfy $\neg p \supset \neg p$ (theorem 15). □

Lemma 17. $\nvdash_{OL} O_i p \supset O_i(p \supset p)$

Proof. We show that there is a model $\mathcal{N}$ such that $\vDash_{\mathcal{N}} \neg(O_i p \supset O_i(p \supset p))$. Assume $\vDash_{\mathcal{N}} O_i p$ and $\vDash_{\mathcal{N}} \neg O_i(p \supset p)$. Thus, s^α satisfies $O_i p$ but does not satisfy $O_i(p \supset p)$, meaning that s^α satisfies p and $s_p^\alpha \in \mathcal{O}$, but s^α does not satisfy $p \supset p$. Therefore, s^α does not satisfy $\Gamma_m = p \supset p$, which is consistent with C2 since $\vDash_{PL} p \supset p$, and either s^α does not satisfy p, which is inconsistent with our hypothesis, or $\Gamma_p = \Gamma_p$, which is true. Therefore, it is possible that s^α satisfies $O_i p$ but does not satisfy $O_i(p \supset p)$. $\square$

Lemma 18. $\nvdash_{OL} O_i(p \vee \neg p)$

Proof. It follows from theorem 16. $\square$

Lemma 19. $\nvdash_{OL} O_i(\neg p \supset \neg p)$

Proof. It follows from theorem 15. $\square$

Also, it must be possible to represent that some actions are neither obligatory nor forbidden, which follows from the following lemma.

Lemma 20. $\nvdash_{OL} O_i p \vee O_i \neg p$

Proof. Assume $\vDash_{\mathcal{N}} \neg O_i p \wedge \neg O_i \neg p$. Thus, s^α does not satisfy $O_i p$ and does not satisfy $O_i \neg p$, meaning that s^α does not satisfy p and does not satisfy $\neg p$. Therefore, $\overline{s}^\alpha$ does not satisfy p and $s^\alpha_p \notin \mathcal{O}$ and $\overline{s}^\alpha_p \notin \mathcal{O}$, which is not inconsistent. $\square$

This would have been violated if we had assumed an ‘if and only if’ instead of a simple ‘if then’ in condition C1.

4.3.9. Disjunctive permission

A consequence of *OL* is that if an action is permitted, then a disjunctive action which includes said action is also permitted. Hence, if it is permitted to walk in the park, then it is permitted to either walk in the park or rob a bank. This reasoning, which follows from theorem 11, is also derivable in standard deontic logic (it is actually derivable in a normal deontic logic, in the sense of Åqvist (2002, p. 155), i.e., an extension of the modal logic *K*). But this paradox is no more puzzling than the fact that from p’s truth one can infer the truth of $p \vee q$ in *PL*. Moreover, theorem 11 appears as a desirable consequence when one considers theorem 12.

Theorem 11. $\vdash_{OL} P_i p \supset P_i(p \vee q)$

Proof. It follows from theorem 12 by contraposition. $\square$

Theorem 12. $\vdash_{OL} F_i(p \vee q) \supset F_i p$

Proof. Assume $\vDash_{\mathcal{N}} F_i(p \vee q)$ but $\vDash_{\mathcal{N}} \neg F_i p$. By definition it means that s^α satisfies $O_i \neg(p \vee q)$, thus s^α satisfies $\neg(p \vee q)$, which by definition is

$\neg(\neg p \supset q)$, and s^α does not satisfy $\neg p$, thus $\overline{s}^\alpha$ does not satisfy p and $\overline{s}^\alpha_p \notin \mathcal{O}$. Since s^α satisfies $\neg(\neg p \supset q)$, $\overline{s}^\alpha$ satisfies $\neg p \supset q$, thus s^α satisfies $\neg p$, and so $\overline{s}^\alpha$ satisfies p.

1	$F_i(p \vee q)$	H
2	$O_i\neg(p \vee q)$	def F 1
3	$\neg(p \vee q) \supset \neg p$	*PL*
4	$O_i\neg p$	(R1) 2,3
5	$F_i p$	def F 4
6	$\vdash_{OL} F_i(p \vee q) \supset F_i p$	$\supset$ *in* 1-5

$\Box$

Theorem 12 says that if a disjunction of actions is forbidden, then each member of the disjunction is forbidden too. This is a desirable consequence. For instance, if Paul tells his son Peter that he is forbidden to either take the car or the motorcycle, we do not expect Peter to answer ‘Fine! I’ll take the motorcycle then!’.

5. Conclusion

To sum up, we introduced a method to test the formal validity of legal inferences. Although the main contribution of this paper was to develop a formal method to test validity, we also provided the reader with a brief analysis of some important aspects of sound normative inferences. It is our view that if formal philosophy is to be of help to legal discourse, then it must first reflect upon the law’s fundamental characteristics that should be taken into account. We provided the reader with a brief analysis of Canadian legal discourse and we exposed three fundamental characteristics which ought to be considered if one wants to represent the formal structure of legal arguments. These characteristics are the presupposed consistency of legal discourse, the fact that there is a hierarchy between norms and obligations to preserve this consistency and the fact that legal inferences are subjected to the principle of deontic consequences. The formal logic that was built according to these characteristics is restricted to normative inferences in which there is only one type of obligation, no iteration of deontic operator and no mixed formulas (i.e., no conditional obligations). This is both a strength and a weakness of our approach. It is a strength insofar as the paradoxes which use these properties cannot be formulated in our framework. But it is a weakness since we cannot formulate contrary-to-duty reasoning and conditional obligations, which are of central importance in legal reasoning.

The main contribution of this paper is that our method covers a portion of the intuitive validity of legal inferences that was not covered by other monadic frameworks in the literature. For instance, the principle of deontic consequences is not valid in standard deontic logic. Although the principle of deontic consequences can be formulated (and validated) in a multi-modal deontic logic which includes a minimal necessity operator $\Box$, our aim was to develop a system which can easily be applied to test the validity of some basic legal inferences and where the principles of contingency and deontic consequences are handled with very little formalism. Our intention was to introduce an alternative to the modal logic K, which is usually used as a building block for the construction of deontic logics. This framework also provides a method to graphically represent legal arguments and test their validity. It is our view that our framework is better suited than K to formalize the transmission of the property 'legally obligatory' between actions.

For future research, we intend to extend this method to arguments in which there are different types of obligations which can be iterated and are linked by a relation of hierarchy to prevent conflicts of obligations. We will also work on the representation of mixed formulas, conditional obligations and contrary-to-duty reasoning. It would be interesting to see if (and how) this system can deal with agency and incorporate (or be incorporated in) the frameworks of dynamic and *stit* logics. Another avenue will be to redefine the propositions within the scope of the deontic operator O_i. Instead of using descriptive formulas that refer to actions, the satisfaction of normative atoms could be redefined in terms of an action algebra.[37] However, this would require to study carefully action logics and determine the proper formal framework to represent the structure of actions.[38] Finally, another avenue for future research will be to see how this approach can be reformulated within the framework of categorical logic.

Acknowledgements

The first author would like to thank Andrew Irvine, François Lepage and Yvon Gauthier for comments on a previous draft of this paper. Thanks also for the comments of anonymous referees. This research was financially supported by the Social Sciences and Humanities Research Council of Canada.

[37] See Peterson (2014c).
[38] See Peterson (2014a).

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