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ARITHMETIC WITH FUSIONS

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ABSTRACT

In this article, the relationship between second-order comprehension and unrestricted mereological fusion (over atoms) is clarified. An extension PAF of Peano arithmetic with a new binary mereological notion of "fusion", and a scheme of unrestricted fusion, is introduced. It is shown that PAF interprets full second-order arithmetic, Z_2.

1. Introduction

The weakest way of extending a theory T of objects with a single layer of sets (or collections) of those objects is to consider its monadic second-order extension, T_2, or a subsystem thererof. The properties of such extensions, both model-theoretic and proof-theoretic, are now fairly well understood (see, e.g., Simpson 1998; Leivant 1994). It has now become clear that T_2 is intimately related to two somewhat different kinds of extension of T. The first is the result of adding axioms for *plural quantification* over those objects (Boolos 1984, Linnebo 2012). The second is the result of adding a theory of *mereological fusions* of those objects, treating the original objects as mereological atoms.[1] To explain this, *mereology* is the theory of the "part-whole" relation, first proposed by Leśniewski (1916) as a kind of more innocent replacement for the emerging axiomatic set theory of Zermelo and Cantor. The ontological commitments of Cantorian set theory – intuitively, the axioms given by Zermelo represent a description of the transfinite cumulative hierarchy V – were considered extravagant from a nominalistic perspective. For a mereological theory, the basic notion is the the parthood relation between the aggregate and its parts: this relation is in some respects analogous to the subset relation $\subseteq$ amongst classes/sets.[2]

[1] For expositions of the various systems of mereology, see Hovda 2009 and Varzi 2011.

[2] In Lewis 1991, the connection between mereology and set theory is explored, with the parts of a class are its subsets, but also taking the concept of a *singleton* as primitive. This allows one to formulate a set theory more or less equivalent to usual ones. Lettting $\preceq$ mean "part of", the definition of $\in$ in terms of $\preceq$ and the singleton operation $\{.\}$ is:

$$x \in y \text{ iff } \{x\} \preceq y.$$

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Leśniewski's student, Tarski, established the formal connection between mereology and Boolean algebra: a complete Boolean algebra with its bottom element removed is a kind of standard model for a mereology (Tarski 1935).

The purpose of this article is to clarify the (syntactic) interpretability relationship between:

the *mereological fusion* extension
the *second-order* extension

given a base theory T.[3] Here, we focus on the result of extending Peano arithmetic PA with axioms for fusions of its objects: i.e., "fusions" of numbers, such as $0 \oplus 57 \oplus 10^{1000}$, $0 \oplus 2 \oplus 4 \oplus 6 \oplus \ldots$ and so on. We do not say, or even know, what such a "fusion" really is. However, as we show below, if the numbers are regarded as *atoms,* then the fusion of some numbers behaves *formally* just like the *set* of those numbers. Quite generally, we have the following:

> The *fusion* of all *atoms* x such that $\phi(x)$ in the fusion extension of T behaves much like the *class/set* $\{x \mid \phi(x)\}$ in the second-order extension of T (that is, the comprehension axioms governing each are inter-translatable).

In particular, the fusion extension of Peano arithmetic, here denoted PAF, interprets full second-order arithmetic, Z_2. Given the significantly higher arithmetic strength of Z_2 over PA, it follows that one can prove in PAF a great many *arithmetic* claims not provable in PA itself. For example, PAF proves a statement corresponding to Con(PA), which PA itself does *not* prove.

This then suggests that adding a theory of fusions of some original entities is perhaps not as "*metaphysically innocent*" as some have previously made out. We do not wish to enter deeply into this philosophical debate in this essentially technical article. However, there is a doctrine, articulated by many, and, for example, by David Lewis, that "… if you are already committed to some things, you incur no further commitment when you affirm the existence of their fusion. The new commitment is redundant, given the old one" (Lewis 1991: 81-82). The results in this paper may cast doubt on this doctrine of "redundancy".

2. Second-Order Comprehension

We suppose L is a first-order language with identity. Let L_2 be the result of extending L by adding monadic second-order variables X_i, with atomic

[3] Linnebo 2012 notes: "For instance PFO [plural first-order quantification] is equiinterpretable with atomic extensional mereology" (Linnebo 2012, Sc. 2.1.). But we have not found detailed presentation of these results.

formulas of the form $t \in X_i$, where t is an L-term.[4] For any L_2-formula ϕ in which X is not free, the (monadic) second-order comprehension axiom for ϕ is the formula:

$$\mathsf{Comp}_\phi\colon\ \exists X \forall x(x \in X \leftrightarrow \phi)$$

Thus, Comp_ϕ asserts the existence of $\{x \mid \phi(x)\}$.

Letting L be the first-order language of arithmetic with signature $\{0, s, +, \cdot\}$, Peano arithmetic PA is the L-theory with individual non-logical axioms:

$$\mathsf{PA}_1\colon\ \forall x(x = 0 \leftrightarrow \forall y(x \neq s(y))) \qquad \mathsf{PA}_2\colon\ \forall x, y(s(x) = s(y) \to x = y)$$
$$\mathsf{PA}_3\colon\ \forall x(x + 0 = x) \qquad \mathsf{PA}_4\colon\ \forall x, y(x + s(y) = s(x + y))$$
$$\mathsf{PA}_5\colon\ \forall x(x \cdot 0 = 0) \qquad \mathsf{PA}_6\colon\ \forall x, y(x \cdot s(y) = x \cdot y + x)$$

along with the axiom scheme of induction:[5]

$$(\phi^x_0 \wedge \forall x(\phi \to \phi^x_{s(x)})) \to \forall x \phi$$

where ϕ is any L-formula.

Second-order arithmetic Z_2 is the theory in L_2 whose axioms are the individual non-logical axioms PA_1-PA_6 of PA, along with:

$$\mathsf{Ind}\colon\quad (0 \in X \wedge \forall x \in X(s(x) \in X)) \to \forall x(x \in X)$$
$$\mathsf{Comp}_\phi\colon\ \exists X \forall x(x \in X \leftrightarrow \phi).$$

where ϕ is an L_2-formula in which X is not free. Here Ind is the second-order induction *axiom.*

Definition 1. *Weak second-order arithmetic,* denoted Z_2^-, is the theory in L_2 whose axioms are those of Z_2 except that Comp_ϕ is weakened to,

$$\mathsf{Comp}^-_\phi\colon\ \exists x\phi \to \exists X \forall x(x \in X \leftrightarrow \phi).$$

So, Z_2^- only asserts the existence of $\{x \mid \phi(x)\}$ on the assumption $\exists x\phi$. However, although Z_2^- is slightly weaker than Z_2, one can show that it interprets Z_2.

First, we give a little detail to the obvious fact that one can think of the numbers as "starting" with 1, instead of 0, by minor redefinitions of "addition" and "multiplication":[6]

[4] For the purposes of defining the interpretations below, it is convenient to assume that L has very few primitives: let us say L has $\neg$, $\to$ and $\forall$ as primitives, with $\wedge$, $\vee$, $\exists$, … given their usual classical definitions.

[5] The notation ϕ^x_t indicates the result of substituting the term t for all free occurrences of x in ϕ, relabelling bound variables in ϕ if necessary to avoid variable collisions.

[6] We do not mean to merely restrict attention to $\mathbb{N}^*$, thereby simply forgetting 0. Rather, we mean that the new structure, with its new operations, treats 1 as 0, etc.

Definition 2. Let $\mathbb{N}^* = \{1, 2, \ldots\}$. Let s^* be the restriction of s to $\mathbb{N}$. Define $+^*$ and $\times^*$ on $\mathbb{N}^*$ by

$$\begin{aligned} n +^* k &:= n + k - 1 \\ n \times^* k &:= n \times k - n - k + 2 \end{aligned}$$

For example, $3 +^* 1 = 3$. Note that

$$\begin{aligned} n +^* k &= (n - 1) + (k - 1) + 1. \\ n \times^* k &= (n - 1) \times (k - 1) + 1. \end{aligned}$$

Then:

Lemma 1. $s : (\mathbb{N}, 0, s, +, \times) \to (\mathbb{N}^*, 1, s^*, +^*, \times^*)$ is an isomorphism.

Proof. $s : \mathbb{N} \to \mathbb{N}^*$ is clearly a bijection. We need to show that, for all $n, k \in \mathbb{N}$, the usual homomorphism conditions hold:

$$\begin{aligned} s(s(n)) &= s^*(s(n)) \\ s(n + k) &= s(n) +^* s(k) \\ s(n \times k) &= s(n) \times^* s(k) \end{aligned}$$

The first is obvious. And $s(n) +^* s(k) = s(n) + s(k) - 1 = n + k + 1 = s(n + k)$, as required. And $s(n) \times^* s(k) = s(n) \times s(k) - s(n) - s(k) + 2 = n \times k + n + k + 1 - n - 1 - k - 1 + 2 = n \times k + 1 = s(n \times k)$, as required. □

To ease notation for the following results, let us assume some standard definitions are introduced over the language L: the usual connectives $\wedge, \vee, \ldots$ and quantifer $\exists$ are assumed to be given their standard definitions. 1 is defined as $s(0)$, 2 as $s(1)$; and the ordering $x < y$ and cut-off subtraction $x \dot{-} y$ are given standard definitions in the first-order language L of arithmetic.

Definition 3. We first define a translation $(.)^\dagger : L \to L$ by recursion on the build-up of formulas:

$$\begin{array}{llllll}
(x)^\dagger & := & x \text{ (with } x \text{ a variable)} & & & \\
(0)^\dagger & := & 1 & (s(t))^\dagger & := & s(t^\dagger) \\
(t + u)^\dagger & := & t^\dagger + u^\dagger \dot{-} 1 & (t \cdot u)^\dagger & := & t^\dagger \cdot u^\dagger \dot{-} t^\dagger \dot{-} u^\dagger + 2 \\
(t = u)^\dagger & := & t^\dagger = u^\dagger & & & \\
(\neg\phi)^\dagger & := & \neg\phi^\dagger & (\phi \to \theta)^\dagger & := & \phi^\dagger \to \theta^\dagger \\
(\forall x\phi)^\dagger & := & \forall x > 0 \phi^\dagger & & &
\end{array}$$

The intuitive idea is that the translation $(.)^\dagger$ "thinks" 0 is 1, etc. This accounts for the translations under $(.)^\dagger$ of terms of the form $t + u$ and $t \cdot u$ above. In fact, the translation will interpret theorems of PA as theorems of PA. For example, $\mathsf{PA} \vdash \exists x(x = 0)$. And $(\exists x(x = 0))^\dagger$ is the formula $\exists x > 0(x = 1)$, which is a theorem of PA.

Lemma 2. $(.)^\dagger$ yields a relative interpretation of PA into PA. I.e.:

$$\text{if } \mathsf{PA} \vdash \phi, \text{ then } \mathsf{PA} \vdash \phi^\dagger.$$

Proof. Since $(.)^\dagger$ preserves structure, translations of logical (i.e., propositional, quantificational, identity) axioms are derivable in PA. (In order to verify universal instantiation and Leibniz' law one needs to prove that $(\phi^x_t)^\dagger = (\phi^\dagger)^x_t$.)

If ϕ is one of the six individual axioms of PA, then its translation can be seen to be a theorem of PA as follows:[7]

Axiom	Translation under $(.)^\dagger$
$\forall x(x = 0 \leftrightarrow \forall y(x \neq s(y)))$	$\forall x > 0(x = 1 \leftrightarrow \forall y > 0(x \neq s(y)))$.
$\forall x, y(s(x) = s(y) \rightarrow x = y)$	$\forall x, y > 0(s(x) = s(y) \rightarrow x = y)$.
$\forall x(x + 0 = x)$	$\forall x > 0(x + 1 \dot{-} 1 = x)$.
$\forall x, y(x + s(y) = s(x + y))$	$\forall x, y > 0(x + s(y) \dot{-} 1 = s(x + y \dot{-} 1))$.
$\forall x(x \cdot 0 = 0)$	$\forall x > 0(x \cdot 1 \dot{-} x \dot{-} 1 + 2 = 1)$.
$\forall x, y(x \cdot s(y) = x \cdot y + x)$	$\forall x, y > 0(x \cdot s(y) \dot{-} x \dot{-} s(y) + 2 = x \cdot y \dot{-} x \dot{-} y + 2 + x - 1)$.

Instances of induction translate into the statement, provable in PA, that if $\phi^\dagger$ holds of 1 and, whenever $\phi^\dagger$ holds for a number > 0 then it holds for its successor, then $\phi^\dagger$ holds for all numbers greater than 0. $\square$

We next extend this translation $(.)^\dagger$ to a translation $(.)^\ddagger$ from L_2 to L_2 which interprets Z_2 into Z_2^-.

Definition 4. We define $(.)^\ddagger : L_2 \rightarrow L_2$ by:

$$\begin{array}{lll@{\qquad}lll}
(x)^\ddagger & := & x & & & \\
(0)^\ddagger & := & 1 & (s(t))^\ddagger & := & s(t^\ddagger) \\
(t + u)^\ddagger & := & t^\ddagger + u^\ddagger \dot{-} 1 & (t \cdot u)^\ddagger & := & t^\ddagger \cdot u^\ddagger \dot{-} t^\ddagger \dot{-} u^\ddagger + 2 \\
(t = u)^\ddagger & := & t^\ddagger = u^\ddagger & (t \in X)^\ddagger & := & t^\ddagger \in X \\
(\neg\phi)^\ddagger & := & \neg\phi^\ddagger & (\phi \rightarrow \theta)^\ddagger & := & \phi^\ddagger \rightarrow \theta^\ddagger \\
(\forall x\phi)^\ddagger & := & \forall x > 0\phi^\ddagger & (\forall X\phi)^\ddagger & := & \forall X\phi^\ddagger
\end{array}$$

The only significant extension is that the translation $(.)^\ddagger$ takes us from, e.g.,

$$0 \in X$$

to

$$1 \in X$$

etc. So, that when Z_2 says of some particular number n that $n \in X$, then Z_2^- interprets this to mean that $n + 1 \in X$.

[7] We are a bit sloppy about brackets in writing the translation, assuming associativity where needed for ease of notation.

Theorem 1. Z_2 is relatively interpretable in Z_2^-.

Proof. We wish to establish that, for any L_2-sentence ϕ,

$$\text{if } \mathsf{Z}_2 \vdash \phi, \text{ then } \mathsf{Z}_2^- \vdash \phi^\ddagger.$$

Again, $(.)^\ddagger$ preserves structure, and so translations of logical axioms are derivable in Z_2^-. As before, if ϕ is an individual axiom of PA, then its translation $\phi^\ddagger$ can be seen to be a theorem of Z_2^-.

Induction: induction in Z_2 is the second-order axiom Ind:

$$\forall X[(0 \in X \wedge \forall x(x \in X \rightarrow s(x) \in X)) \rightarrow \forall x(x \in X)].$$

The translation $(\mathsf{Ind})^\ddagger$ is:

$$\forall X[(1 \in X \wedge \forall x > 0(x \in X \rightarrow s(x) \in X)) \rightarrow \forall x > 0(x \in X)].$$

The translation $(\mathsf{Ind})^\ddagger$ is a theorem of Z_2^-, because the induction axiom Ind is an axiom of Z_2^- and Ind implies $(\mathsf{Ind})^\ddagger$. For suppose $1 \in X$ and $\forall x > 0(x \in X \rightarrow s(x) \in X)$. We aim to show $\forall x > 0(x \in X)$. Let $Y = X \cup \{0\}$. Then $0 \in Y$. And since $1 \in Y$, it also follows that $\forall x(x \in Y \rightarrow s(x) \in Y)$. So, by induction, $\forall x(x \in Y)$. And thus, $\forall x > 0(x \in X)$, as required.

Comprehension. Let $\phi(x)$ be given. The corresponding comprehension axiom is:

$$\exists X \forall x(x \in X \leftrightarrow \phi).$$

Its translation is:

$$\exists X \forall x > 0(x \in X \leftrightarrow \phi^\ddagger).$$

Clearly, PA proves $\exists x(\phi^\ddagger \vee x = 0)$. So

$$\mathsf{Z}_2^- \vdash \exists X \forall x(x \in X \leftrightarrow (\phi^\ddagger \vee x = 0)).$$

So,

$$\mathsf{Z}_2^- \vdash \exists X \forall x > 0(x \in X \leftrightarrow \phi^\ddagger).$$

as required. □

3. Classical Mereology

Suppose L is a first-order language with identity. Let $L_\preceq$ be the result of extending L with the primitive $x \preceq y$, intended to express the concept "x is a part of y". In addition, we define two new $L_\preceq$-formulas xOy and Ax by:

$$D_O \quad xOy \leftrightarrow \exists w(w \preceq x \wedge w \preceq y).$$
$$D_A \quad Ax \leftrightarrow \forall z(z \preceq x \rightarrow z = x).$$

Intuitively, xOy means "x overlaps y" and Ax means "x is an atom". For, on our intuitive understanding of the concepts, x "overlaps" y just if x and y have a common part; and x is an "(mereological) atom" just if the only part of x is itself.

The notion of a *fusion* of objects x, y is less easily pinned down intuitively. There seem to be two distinct but intimately related explications. One might say that z is a fusion of x and y just if the overlappers of z are exactly those thing that overlap either x or y. Or one might say that z is a fusion of x and y just if x and y are parts of z, and any part of z overlaps either x or y. We express these notions in $L_{\preceq}$ by the formulas $\mathfrak{f}(x, y, z)$ and $\mathfrak{F}(x, y, z)$, as follows:

$$D_{\mathfrak{f}} \quad \mathfrak{f}(x, y, z) \leftrightarrow \forall w(wOz \leftrightarrow (wOx \vee wOy)).$$
$$D_{\mathfrak{F}} \quad \mathfrak{F}(x, y, z) \leftrightarrow x \preceq z \wedge y \preceq z \wedge \forall w \preceq z(wOx \vee wOy).$$

Clearly the formulas $\mathfrak{f}(x, y, z)$ and $\mathfrak{F}(x, y, z)$ are both symmetric in x, y. However, they are not logically equivalent. Showing the equivalence $\mathfrak{f}(x, y, z) \leftrightarrow \mathfrak{F}(x, y, z)$ is suprisingly non-trivial. It is briefly mentioned below. The second notion $\mathfrak{F}(x, y, z)$, expressing that z is the fusion of the elements of the set $\{x, y\}$, has a fairly obvious schematic generalization, as follows:

Definition 5. Let ϕ be an $L_{\preceq}$-formula in which z is not free. We define the schematic "fusion" $L_{\preceq}$-formula $\mathfrak{F}_{\phi}(z)$ as follows:[8]

$$\mathfrak{F}_{\phi}(z) : \forall x(\phi \to x \preceq z) \wedge \forall y \preceq z \exists x(\phi \wedge yOx).$$

So, informally, $\mathfrak{F}_{\phi}(z)$ can be read:[9]

$$z \text{ is the "fusion" of the elements of } \{x \mid \phi\}.$$

In particular, $\mathfrak{F}_{x=u \vee x=v}(z)$ expresses that z is the fusion of u and v. As expected, since this is the generalization, we have:

Lemma 3. $\vdash \mathfrak{F}_{x=u \vee x=v}(z) \leftrightarrow \mathfrak{F}(u, v, z)$.

Definition 6. The theory CEM in $L_{\preceq}$, known as *Classical Extensional Mereology*, can be axiomatized as follows:

$$\text{Tran} \quad (x \preceq y \wedge y \preceq z) \to x \preceq z.$$
$$\text{UF} \quad \exists x\phi \to \exists! z \mathfrak{F}_{\phi}(z).$$

[8] Note that x becomes bound in the formula $\mathfrak{F}_{\phi}(z)$.

[9] See Hovda 2009 for further explanation of the intuitive meaning of $\mathfrak{F}_{\phi}(z)$, which Hovda expresses as $Fu_2(z, [x \mid \phi])$, as its relation to the notions of minimal and least upper bounds in order theory. Hovda also provides a second schematic notion, denoted $Fu_1(z, [x \mid \phi])$, and generalizing our $\mathfrak{f}(x, y, z)$.

The non-logical axioms are Tran, stating that $\preceq$ is transitive, and the axiom scheme UF, for *Unrestricted Fusion*, which expresses that if ϕ defines a non-empty set X, there is a unique z such that (i) all elements of X are parts of z and (ii) every part of z overlaps some element of X.[10]

Lemma 4. The following are theorems of CEM:[11]

Reflexivity of $\preceq$ (Ref)	$x \preceq x$.
Anti-symmetry of $\preceq$ (Anti)	$(x \preceq y \wedge y \preceq x) \to x = y$.
Part Extensionality (PE)	$\forall w(w \preceq x \leftrightarrow w \preceq y) \to x = y$.
Overlap Extensionality (OE)	$\forall w(wOx \leftrightarrow wOy) \to x = y$.
Supplementation (S)	$(x \neq y \wedge x \preceq y) \to \exists w \preceq y \neg wOx$.
Strong Supplementation (SS)	$\forall z \preceq x(zOy) \to x \preceq y$.
Complementation (C)	$x \not\preceq y \to \exists z \forall w(w \preceq z \leftrightarrow (w \preceq x \wedge \neg wOy))$.
Equivalence of $\mathfrak{f}$ and $\mathfrak{F}$ fusion	$\mathfrak{f}(x, y, z) \leftrightarrow \mathfrak{F}(x, y, z)$.
Existence of $\mathfrak{f}$ fusion	$\exists z \mathfrak{f}(x, y, z)$.
Uniqueness of $\mathfrak{f}$ fusion	$(\mathfrak{f}(x, y, z_1) \wedge \mathfrak{f}(x, y, z_2)) \to z_1 = z_2$.

There are multiple dependencies. For example:

Lemma 5. Assuming reflexivity of $\preceq$, Complementation (C) implies Strong Supplementation (SS). Assuming anti-symmetry of $\preceq$, Strong Supplementation implies Supplementation (SS). Reflexivity and Strong Supplementation deliver the left-to-right direction of the equivalence $\mathfrak{f}(x, y, z) \leftrightarrow \mathfrak{F}(x, y, z)$:

(i) Ref, C $\vdash$ SS.
(ii) Anti, SS $\vdash$ S.
(iii) Ref, SS $\vdash \forall x \forall y \forall z \mathfrak{f}(x, y, z) \to \mathfrak{F}(x, y, z))$.

Proof. For (i), suppose C holds: $x \not\preceq y \to \exists z \forall w(w \preceq z \leftrightarrow (w \preceq x \wedge \neg wOy))$. Now suppose $x \not\preceq y$. So, relabelling variables, we have u such that $\forall w(w \preceq u \leftrightarrow (w \preceq x \wedge \neg wOy))$. So, $u \preceq u \leftrightarrow (u \preceq x \wedge \neg uOy))$. By reflexivity, $u \preceq x$ and $\neg uOy$. Hence, it is not that case that for any $u \preceq x$, uOy. By contraposition, if for any $u \preceq x$, uOy, then $x \preceq y$. This is Strong Supplementation (SS).

For (ii), first let $x \neq y$ and $x \preceq y$. By anti-symmetry, $\neg(y \preceq x)$. Next suppose SS (variables relabelled) holds: $\forall u \preceq y(uOx) \to y \preceq x$. Hence, there is some $u \preceq y$ with $\neg uOx$, which is Supplementation.

[10] For further details of the many possible axiomatizations of mereology and some of their model-theoretic properties, see Hovda 2009. As Hovda points out, this axiomatization is due, effectively, to Tarski 1935.

[11] For proofs of some of these, see Hovda 2009.

For (iii), let $\mathfrak{f}(x, y, z)$. So, $\forall w(wOz \leftrightarrow (wOx \vee wOy))$. Suppose $w \preceq z$. By Ref, $w \preceq w$. So, wOz. And so, $wOx \vee wOy$. Thus,

$$\mathfrak{f}(x, y, z) \rightarrow \forall w \preceq z(wOx \vee wOy). \tag{1}$$

Next, for a contradiction, suppose $\neg(x \preceq z)$. By Strong Supplementation, $\forall u \preceq x(uOz) \rightarrow x \preceq z$. So, there is some $u \preceq x$ such that $\neg uOz$. And, by Ref, we have $u \preceq u$ and thus uOx and therefore $uOx \vee uOy$. Since $\mathfrak{f}(x, y, z)$, we have $uOz \leftrightarrow (uOx \vee uOy)$. Hence, uOz, which is a contradiction. We reason similarly to show $y \preceq z$. Thus,

$$\mathfrak{f}(x, y, z) \rightarrow (x \preceq z \wedge y \preceq z). \tag{2}$$

For (1) and (2), we used Ref and SS. From these two, (iii) follows. □

In light of the existence and uniqueness of fusions, we may extend $L_{\preceq}$ to $L_{\preceq,\oplus}$ by introducing a new binary function symbol $\oplus$, with $z = x \oplus y$ intuitively meaning "z is the fusion of x and y".

Definition 7. CEM$^+$ is the theory in $L_{\preceq,\oplus}$ obtained by adding to CEM the following definition:

$$D_{\oplus} \qquad z = x \oplus y \leftrightarrow \mathfrak{F}(x, y, z),$$

where $\mathfrak{F}(x, y, z)$ is the formula $x \preceq z \wedge y \preceq z \wedge \forall w \preceq z(wOx \vee wOy)$.

4. "Fusion Theory"

A somewhat different, but definitionally equivalent, formalization of classical mereology is possible. Let L be a first-order language again and let $L_{\oplus}$ be the result of extending L with a binary function symbol $\oplus$, with $x \oplus y$ intuitively meaning "the fusion of x and y". We then extend $L_{\oplus}$ to $L_{\oplus,\preceq}$ by introducing the binary predicate $\preceq$.[12] Then the formulas xOy, Ax, $\mathfrak{f}(x, y, z)$, $\mathfrak{F}(x, y, z)$, $\mathfrak{F}_{\phi}(z)$ are defined as above.

An algebraic formulation F of classical extensional mereology may then be given as follows.[13]

Definition 8. The theory F in $L_{\oplus,\preceq}$, *Fusion Theory*, has the following axioms:

Ass	$x \oplus (y \oplus z) = (x \oplus y) \oplus z.$
UF	$\exists x\phi \rightarrow \exists! z\mathfrak{F}_{\phi}(z).$
$D_{\preceq}$	$x \preceq y \leftrightarrow y = x \oplus y.$

[12] The languages $L_{\preceq,\oplus}$ and $L_{\oplus,\preceq}$ are identical.

[13] Hovda discusses similar algebraic formulations and their connection with Boolean algebra in Part Four, "Strong complements and Boolean algebra", of Hovda 2009.

The axiom Ass states that binary fusion $\oplus$ is associative. The axiom scheme UF is formulated exactly as before. However, note that $\preceq$ is explicitly defined in F. In principle F can be formulated in the language $L_{\oplus}$, within which $\oplus$ is the mereological primitive. We shall call this theory F^{-}.

Lemma 6. $\mathsf{Ass}, D_{\preceq} \vdash \mathsf{Tran}$.

Proof. Suppose $x \preceq y$ and $y \preceq z$. Using $D_{\preceq}$, we have $y = x \oplus y$ and $z = y \oplus z$. Hence, $z = (x \oplus y) \oplus z$. By Ass, $z = x \oplus (y \oplus z)$. Hence, $z = x \oplus z$. Hence, $x \preceq z$. □

As noted, the symbol $\preceq$ is explicitly defined in F and therefore F can be regarded as being formulated entirely in $L_{\oplus}$. On the other hand, CEM is formulated in $L_{\preceq}$. We next show that, despite being formulated in different languages, the theories CEM and F are *definitionally equivalent.*

First, we show that CEM is a subtheory of F:

Lemma 7. For all $\phi \in L_{\preceq}$, if $\mathsf{CEM} \vdash \phi$, then $\mathsf{F} \vdash \phi$.

Proof. It is sufficient to prove that this holds when ϕ is an *axiom* of CEM. For suppose ϕ is a theorem of CEM. Then ϕ is derived from axioms $\theta_1, \ldots, \theta_n$ of CEM. But by assumption, each θ_i is a theorem of F. Hence, ϕ is a theorem of F too. Now, by Lemma 6, $\mathsf{F} \vdash \mathsf{Tran}$. The other axiom of CEM is the scheme UF, already an axiom scheme of F. □

Lemma 8. The idempotence and commutativity of $\oplus$ follow from the other axioms of F:[14]

$$\mathsf{F} \vdash \forall x(x \oplus x = x).$$
$$\mathsf{F} \vdash \forall x \forall y(x \oplus y = y \oplus x).$$

Proof. The reflexivity and anti-symmetry of $\preceq$ are theorems of CEM and hence of F too. These imply, respectively, the idempotence and commutativity of $\oplus$. □

Lemma 9. $\mathsf{F} \vdash \forall x \forall y \forall w(wOx \to wO(x \oplus y))$.

Proof. Working in F, let wOx. We want to show $wO(x \oplus y)$. Now w and x have a common part, say, $z \preceq w$ and $z \preceq x$. So, $x = x \oplus z$. So, $y \oplus x = y \oplus (x \oplus z)$. So, reasoning using Lemma 8, $x \oplus y = z \oplus (x \oplus y)$. So, $z \preceq (x \oplus y)$. So, z is a common part of w and $x \oplus y$. So, $wO(x \oplus y)$, as required. □

We next show that $\oplus$ is explicitly definable in F.

[14] We are grateful to a referee for emphasizing this point.

Lemma 10. $\mathsf{F} \vdash \forall x \forall y \forall z(\mathfrak{F}(x, y, z) \to z = x \oplus y)$.

Proof. Working in F, assume that $\mathfrak{F}(x, y, z)$, i.e.

$$x \preceq z \wedge y \preceq z \wedge \forall u \preceq z(uOx \vee uOy) \tag{3}$$

Then $x \oplus z = z = y \oplus z$ and so $z = x \oplus (y \oplus z) = (x \oplus y) \oplus z$ and hence $x \oplus y \preceq z$. We now want to show that $z \preceq x \oplus y$ in order to conclude $z = x \oplus y$ (by Part Extensionality, PE). Assume, for the sake of contradiction, that $\neg(z \preceq x \oplus y)$. Then by Complementation (C) there is some u such that

$$\forall w(w \preceq u \leftrightarrow (w \preceq z \wedge \neg(wO(x \oplus y)))).$$

Instantiate the quantifer $\forall w$ to u. Since $u \preceq u$ we get

$$u \preceq z \wedge \neg(uO(x \oplus y)) \tag{4}$$

But $u \preceq z$ implies with (3) that u overlaps either x or y. But then u overlaps also $x \oplus y$, by Lemma 9. This contradicts (4). Thus $z \preceq x \oplus y$. □

Lemma 11. $\mathsf{F} \vdash \forall x \forall y \forall z(z = x \oplus y \to \mathfrak{F}(x, y, z))$.

Proof. Assume $z = x \oplus y$. Then it is easily computed (using Lemma 8) that $x, y \preceq x \oplus y$, so by identity $x, y \preceq z$. Now let w with

$$w \preceq z \tag{5}$$

be given. We have to show that w overlaps either x or y.

Let $\phi(v)$ be the formula $v = x \vee v = y$. Since $\exists v \phi(v)$, UF yields (with some variable relabelling) that there is a unique z' such that

$$\forall v(\phi(v) \to v \preceq z') \wedge \forall u \preceq z' \exists u'(\phi(u') \wedge uOu') \tag{6}$$

or in short, $\mathfrak{F}_\phi(z')$. Now by Lemma 3, $\mathfrak{F}(x, y, z') \leftrightarrow \mathfrak{F}_\phi(z')$. And so $\mathfrak{F}(x, y, z')$. From Lemma 10,

$$\mathfrak{F}(x, y, z') \to z' = x \oplus y$$

So, $z' = x \oplus y$. So by identity $z = z'$. Since by (5) $w \preceq z$ also $w \preceq z'$, the second conjunct of (6) implies that $\exists u'(\phi(u') \wedge wOu')$. But clearly, by definition of ϕ, u' is either x or y. So $wOx \vee wOy$, as desired. □

Recall that $D_\oplus$ is the following explicit definition of $\oplus$:

$$z = x \oplus y \leftrightarrow \mathfrak{F}(x, y, z).$$

Consequently, the previous two lemmas imply the definability of $\oplus$ in F:

Lemma 12. $\mathsf{F} \vdash D_\oplus$.

This now allows us to conclude that CEM^+ is a subtheory of F:

Lemma 13. For any $\phi \in L_{\preceq,\oplus}$, if $\mathsf{CEM}^+ \vdash \phi$ then $\mathsf{F} \vdash \phi$.

Proof. It is sufficient to show that this holds for each axiom of CEM^+. By Lemma 7, we have that CEM is a subtheory of F, so each axiom of CEM is a theorem of F. Furthermore, $D_\oplus$ is also theorem of F, by Lemma 12. So each axiom of CEM^+ is a theorem of F, as required. □

We next show the converse, that CEM^+ proves all the theorems of F. First,

Lemma 14. $\mathsf{CEM}^+ \vdash \mathsf{Ass}$.

Proof. Because CEM already proves the equivalence $\mathfrak{F}(x, y, z) \leftrightarrow \mathfrak{f}(x, y, z)$, CEM^+ proves

$$z = x \oplus y \leftrightarrow \mathfrak{f}(x, y, z) \tag{7}$$

Working in CEM^+ we want to show

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z,$$

which is logically equivalent to,

$$(a = y \oplus z \wedge b = x \oplus a \wedge c = x \oplus y \wedge d = c \oplus z) \rightarrow b = d$$

And, by (7), this is equivalent to:

$$(\mathfrak{f}(y, z, a) \wedge \mathfrak{f}(x, a, b) \wedge \mathfrak{f}(x, y, c) \wedge \mathfrak{f}(c, z, d)) \rightarrow b = d$$

To show this, for a contradiction, assume we have objects x, y, z, a, b, c, d such that $\mathfrak{f}(y, z, a)$, $\mathfrak{f}(x, a, b)$, $\mathfrak{f}(x, y, c)$ and $\mathfrak{f}(c, z, d)$ all hold but that $b \neq d$, which by Overlap Extensionality, implies there is some u such that uOb and $\neg uOd$. Now, $uOd \leftrightarrow (uOc \vee uOz)$, and so $\neg uOc$ and $\neg uOz$. But $uOc \leftrightarrow (uOx \vee uOy)$ and so $\neg uOx$ and $\neg uOy$. And $uOb \leftrightarrow (uOx \vee uOa)$, and so $uOx \vee uOa$. So, uOa. Now $uOa \leftrightarrow (uOy \vee uOz)$, and so $uOy \vee uOz$. And so uOz. Contradiction. □

Lemma 15. $\mathsf{CEM}^+ \vdash D_\preceq$.

Proof. In CEM^+, we have the theorem $z = x \oplus y \leftrightarrow \mathfrak{f}(x, y, z)$ and hence,

$$y = x \oplus y \leftrightarrow \mathfrak{f}(x, y, y). \tag{8}$$

It is straightforward, using the definition of the formula $\mathfrak{f}(x, y, z)$, to show,

$$\mathfrak{f}(x, y, y) \leftrightarrow \forall w(wOx \rightarrow wOy). \tag{9}$$

We wish to prove $D_\preceq$, the formula,

$$x \preceq y \leftrightarrow y = x \oplus y. \tag{10}$$

First, let $x \preceq y$. We want to prove $y = x \oplus y$, which by (8) is equivalent to $\mathfrak{f}(x, y, y)$. But $\mathfrak{f}(x, y, y)$ is equivalent to $\forall w(wOx \rightarrow wOy)$ by (9). Suppose wOx. Then we have some $z \preceq w$ such that $z \preceq x$. By Tran, $z \preceq y$. And therefore, wOy. Since w was arbitrary, we have $\forall w(wOx \rightarrow wOy)$ as required.

Next, let $y = x \oplus y$. We want to prove $x \preceq y$. By (8), we have $\mathfrak{f}(y, x, y)$. By (9) we have $\forall w(wOx \rightarrow wOy)$. Now let $z \preceq x$. Then zOx. So, zOy. Since z was arbitrary, we have $\forall z \preceq x(zOy)$. By Strong Supplementation, $\forall z \preceq x(zOy) \rightarrow x \preceq y$. And therefore $x \preceq y$, as required. □

The preceding lemmas give us:

Lemma 16. For any $\phi \in L_{\oplus,\preceq}$, if $\mathsf{F} \vdash \phi$, then $\mathsf{CEM}^+ \vdash \phi$.

Proof. It is sufficient to show that this holds whenever ϕ is an axiom of F. So, ϕ is either Ass, $D_\preceq$ or an instance of UF. All instances of UF are already axioms, and therefore theorems of CEM^+. By Lemma 14, CEM^+ proves Ass. By Lemma 15, CEM^+ proves $D_\preceq$. □

Recall that F is a definitional extension of F^-. Lemmas 13 and 16 tell us that $\mathsf{F} = \mathsf{CEM}^+$: these theories have exactly the same theorems, and so CEM^+ is a definitional extension of F^-. But clearly, CEM^+ is a definitional extension of CEM. So, F^- and CEM have a common definitional extension, namely CEM^+. This yields the second main theorem:

Theorem 2. CEM and F (or F^-) are definitionally equivalent.

5. A Conservation Theorem

Recall that, given a (first-order) base language L, then $L_{\oplus,\preceq}$ is the extended language obtained by adding the new symbols $\oplus$ and $\preceq$. We then defined $L_{\oplus,\preceq}$-formulas xOy, Ax and $\mathfrak{f}(x, y, z)$ and gave the axioms for the algebraically-formulated fusion theory F in $L_{\oplus,\preceq}$.

Definition 9. If ϕ is an L-formula, ϕ^A is the $L_{\oplus,\preceq}$-formula obtained by relativizing all quantifiers in ϕ to A. If T is a theory in L, then T^A is $\{\phi^A \mid \phi \text{ is an } L\text{-sentence and } T \vdash \phi\}$.

We introduce this relativization to ensure that when we *extend* L to $L_{\oplus,\preceq}$, then for any claim ϕ in L, the part of its content that is only "about" atoms, namely ϕ^A, can be isolated.

There is a conservation theorem for F, obtained by showing how to transform any L-structure M into an $L_{\oplus,\preceq}$-structure $M_{\oplus,\preceq}$ which satisfies F.

Theorem 3. Let T be an L-theory and ϕ an L-sentence. Then

$$\text{if } T^A \cup \mathsf{F} \vdash \phi^A \text{ then } T \vdash \phi.$$

Proof sketch. Let an L-structure M be given. We wish to define an $L_{\oplus,\preceq}$-structure $M_{\oplus,\preceq}$ with certain properties. Let

$$dom(M_{\oplus,\preceq}) := \mathcal{P}(dom(M)) \setminus \{\varnothing\}.$$

So, the domain of $M_{\oplus,\preceq}$ is the power set of $dom(M)$, with the empty set removed. This yields a model of classical mereology (cf., Tarski 1935) in which the original elements of M have become singletons in $M_{\oplus,\preceq}$, and play the role of atoms in $M_{\oplus,\preceq}$. In particular, it follows that $M_{\oplus,\preceq} \vDash \mathsf{F}$. The original relations on the elements of M can be redefined onto the atoms in $M_{\oplus,\preceq}$ by the singleton mapping $m \mapsto \{m\}$. This yields an embedding of M into $M_{\oplus,\preceq} | L$, and consequently, for any L-sentence ϕ,

$$M \vDash \phi \Leftrightarrow M_{\oplus,\preceq} \vDash \phi^A.$$

From this it follows that any model $M \vDash T$ can be converted to a model $M_{\oplus,\preceq} \vDash T^A \cup \mathsf{F}$. Theorem 3 then follows by the Completeness Theorem. $\square$

This tells us that extending T with the fusion axioms leads to a conservative extension, at least so long as any axiom scheme in T is restricted to L-formulas. One might then argue in favour of a kind of *mereological fictionalism* – that is, taking mereological atoms as the sole real entities, regard (arbitrary) fusions of atoms as "useful fictions" – based on this. This would be analogous to Hartry Field's argument for fictionalism about numbers and sets based on a similar conservation result obtained when one extends a nominalistic theory with set existence axioms (Field 1980, Ch. 1).

A natural question is then to ask what happens if *axiom schemes* in T are extended to the full fusion language $L_{\oplus,\preceq}$? We show below that the unrestricted fusion scheme UF interprets second-order comprehension. This implies that the extension of T with F amounts to a form of *second-order* extension. In particular, if fusion theory is added to PA with the axiom scheme of induction extended to all $L_{\oplus,\preceq}$-formulas, the result is equivalent in strength to full second-order arithmetic Z_2, and therefore a highly non-conservative extension in its arithmetic content.[15]

[15] The situation seems to be entirely analogous to what happens in the Field-style case. Extending a nominalistic theory N with set existence axioms is conservative if axioms schemes of N are not extended to the enriched language, containing $\in$. However, the full extension may be highly non-conservative. Again, the situation is analogous to extending PA with set existence axioms. The extension ACA_0, asserting only the existence of arithmetic sets, and with set induction, is conservative. However, extending induction to all second-order formulas introduces a non-conservative system, ACA.

6. Arithmetic With Fusions: PAF

We introduce the theory obtained by adding fusion theory to Peano arithmetic, in a way that recovers the numbers as atoms.

Definition 10. The axioms of the theory PAF in $L_{\oplus,\preceq}$ are the relativized PA axioms:

$$\begin{array}{ll}
(\mathsf{PA}_1)^A & \forall x(Ax \to (x = 0 \leftrightarrow \forall y(Ay \to x \neq s(y)))) \\
(\mathsf{PA}_2)^A & \forall x, y(Ax \wedge Ay \to (s(x) = s(y) \to x = y)) \\
(\mathsf{PA}_3)^A & \forall x(Ax \to (x + 0 = x)) \\
(\mathsf{PA}_4)^A & \forall x, y(Ax \wedge Ay \to (x + s(y) = s(x + y))) \\
(\mathsf{PA}_5)^A & \forall x(Ax \to (x \cdot 0 = 0)) \\
(\mathsf{PA}_6)^A & \forall x, y(Ax \wedge Ay \to (x \cdot s(y) = x \cdot y + x))
\end{array}$$

and the relativized axiom scheme of induction (ϕ any $L_{\oplus,\preceq}$-formula):

$$[(\phi^x_0 \wedge \forall x(\phi \to \phi^x_{s(x)})) \to \forall x\phi]^A$$

and the axioms of F,

$$\begin{array}{ll}
\mathsf{Ass} & x \oplus (y \oplus z) = (x \oplus y) \oplus z \\
\mathsf{UF} & \exists x\phi \to \exists! z\mathfrak{F}\phi(z) \\
D_{\preceq} & x \preceq y \leftrightarrow y = x \oplus y
\end{array}$$

along with the following "closure axioms":

$$\begin{array}{ll}
A0 & Ax \to As(x) \\
Ax \wedge Ay \to A(x + y) & Ax \wedge Ay \to A(x \cdot y)
\end{array}$$

From now on, we shall pretend that the language of PAF is $L_{\oplus}$, since clearly $\preceq$ is explicitly defined and may be eliminated. The theory PAF asserts the existence of arbitrary "fusions" of numbers, such as $0 \oplus 1 \oplus 2$, $0 \oplus 2 \oplus 4 \oplus \ldots$, and so on. In principle, we can also apply the usual algebraic arithmetic operations of successor, additions and multiplication, to these fusions, obtaining terms such as $s(0 \oplus 1 \oplus 2)$, $(0 \oplus 2 \oplus 4 \oplus \ldots) \cdot (0 \oplus 2)$, and so on. But the relativization prevents anything non-trivial being provable about these entities.

Now consider the fusion $0 \oplus 2 \oplus 4$. Suppose that x is an atom and x is part of $0 \oplus 2 \oplus 4$. It follows that $x = 0$ or $x = 2$ or $x = 4$. In other words,

$$\mathsf{PAF} \vdash \forall x(Ax \to ((x \preceq 0 \oplus 2 \oplus 4) \leftrightarrow (x = 0 \vee x = 2 \vee x = 4)))$$

As this indicates, we can reconstruct the "elements" of a given fusion as its *atomic parts*.

Definition 11. Define the "standard" model $\mathbb{N}_\oplus$ for PAF as follows. First, fix the domain and interpretation of $\oplus$:

$$dom(\mathbb{N}_\oplus) := \mathcal{P}(\omega) \setminus \varnothing \qquad \oplus^{\mathbb{N}_\oplus} := \cup$$

For the other mereological notions explicitly defined in $L_\oplus$, let

$$\begin{aligned} \preceq^{\mathbb{N}_\oplus} &:= \subseteq \\ O^{\mathbb{N}_\oplus} &:= \{(X, Y) \in \mathcal{P}(\omega)^2 \mid X \cap Y \neq \varnothing\} \\ A^{\mathbb{N}_\oplus} &:= \{\{n\} \mid n \in \omega\} \\ (\mathfrak{f})^{\mathbb{N}_\oplus} &:= \cup \end{aligned}$$

Then, for arithmetic symbols, let:

$$\begin{array}{ll} 0^{\mathbb{N}_\oplus} := \{0\} & (s)^{\mathbb{N}_\oplus}(\{n\}) := \{s(n)\} \\ \{n\} +^{\mathbb{N}_\oplus} \{k\} := \{n+k\} & \{n\} \cdot^{\mathbb{N}_\oplus} \{k\} := \{n \cdot k\}. \end{array}$$

And when $X \subseteq \omega$ and X is not a singleton, then let

$$(s)^{\mathbb{N}_\oplus}(X) = \{0\}.$$

If X, $Y \subseteq \omega$ and either X or Y is not a singleton, then let

$$\begin{aligned} X \cdot^{\mathbb{N}_\oplus} Y &= \{0\} \\ X +^{\mathbb{N}_\oplus} Y &= \{0\}. \end{aligned}$$

So, the model $\mathbb{N}_\oplus$ is an atomic power-set mereology in Tarski's sense. The atoms inside $\mathbb{N}_\oplus$ are simply singletons of the form $\{n\}$, with $n \in \omega$. In this model, the part-of relation is simply subset $\subseteq$, and the fusion operation is simply set union $\cup$. To overlap is simply to have a non-empty intersection. The model $\mathbb{N}_\oplus$ treats the usual numerical symbols as referring to the *singletons* of what they originally referred to in $\mathbb{N}$, and the "don't care" cases all refer to $\{0\}$, as a matter of convenience.

With this definition in hand, then:

$$\mathbb{N}_\oplus \models \mathsf{PAF}.$$

7. PAF Interprets Z_2

We next show that how to translate any instance of comprehension,

there is a **set** X of things x such that $\phi(x)$.

to a provable instance of the fusion scheme, roughly,

there is a **fusion** z of all **atoms** x such that $\phi^\circ(x)$.

First, recall the definitions,

$$
\begin{array}{ll}
D_O & xOy \leftrightarrow \exists w(w \preceq x \wedge w \preceq y). \\
D_A & Ax \leftrightarrow \forall z(z \preceq x \rightarrow z = x).
\end{array}
$$

Lemma 17. $D_O, D_A \vdash \forall x \forall y(Ay \rightarrow (xOy \rightarrow y \preceq x))$.

Proof. Suppose Ay. So, for any z, if $z \preceq y$, then $z = y$. Suppose xOy. So, there is some z such that $z \preceq x$ and $z \preceq y$. So, $z = y$. So, $y \preceq x$. □

Next, consider the formula $\mathfrak{F}_{Ax \wedge \phi(x)}(z)$, which expresses that z is a fusion of all atoms x such that $\phi(x)$. We can show:

Lemma 18. $D_O, D_A \vdash \forall z(\mathfrak{F}_{Ax \wedge \phi(x)}(z) \rightarrow \forall x(Ax \rightarrow (x \preceq z \leftrightarrow \phi(x))))$.

Proof. Suppose $\mathfrak{F}_{Ax \wedge \phi(x)}(z)$. So,

$$
\begin{gathered}
\forall x((Ax \wedge \phi(x)) \rightarrow x \preceq z) \\
\forall y \preceq z \exists w(Aw \wedge \phi(w) \wedge yOw).
\end{gathered}
$$

Suppose Ax. We want to show:

$$x \preceq z \leftrightarrow \phi(x)$$

Clearly, if $\phi(x)$, we get $x \preceq z$.

Conversely, suppose that $x \preceq z$. Then there is some w such that $Aw \wedge \phi(w) \wedge xOw$. By Lemma 17, $w \preceq x$. But since Ax, we have $w = x$. So, $\phi(x)$, as required. □

We then have the following corollary:

Lemma 19. $D_O, D_A, \exists z \mathfrak{F}_{Ax \wedge \phi(x)}(z) \vdash \exists z \forall x(Ax \rightarrow (x \preceq z \leftrightarrow \phi(x))))$.

Definition 12. Partition the variables x_i of $L_\oplus$ into the y_i and the z_i, where $y_i = x_{2i}$ and $z_i = x_{2i+1}$. Define a translation $(.)^\circ: L_2 \rightarrow L_\oplus$ by:

$$
\begin{array}{lllllll}
(x_i)^\circ & := & y_i & \quad & (X_i)^\circ & := & z_i \\
(ft_1 \ldots t_n)^\circ & := & f(t_1)^\circ \ldots (t_n)^\circ & & (Pt_1 \ldots t_n)^\circ & := & P(t_1)^\circ \ldots (t_n)^\circ. \\
(t = u)^\circ & := & t^\circ = u^\circ & & (t \in X_i)^\circ & := & t^\circ \preceq z_i \\
(\neg\phi)^\circ & := & \neg\phi^\circ & & (\phi \rightarrow \theta)^\circ & := & \phi^\circ \rightarrow \theta^\circ \\
(\forall x_i \phi)^\circ & := & \forall y_i(Ay_i \rightarrow \phi^\circ) & & (\forall X_i \phi)^\circ & := & \forall z_i \phi^\circ
\end{array}
$$

For the next lemma, let $\phi(x)$ be an L_2-formula in which X is not free. Let $y = x^\circ$ and let $\phi^\circ(y) = (\phi(x))^\circ$. Then $\mathfrak{F}_{Ay \wedge \phi^\circ(y)}(z)$ expresses that z is a fusion of all atoms y such that $\phi^\circ(y)$.

Lemma 20. $D_O, D_A, \exists z \mathfrak{F}_{Ay \wedge \phi^\circ(y)}(z) \vdash (\mathsf{Comp}_{\phi(x)})^\circ$.

Proof. $(\mathsf{Comp}_{\phi(x)})^\circ$ is the formula $\exists z \forall y(Ay \to (y \preceq z \leftrightarrow \phi^\circ(y)))$. The required result is then immediate from Lemma 19. $\square$

Note that no special axioms of fusion theory are needed except the explicit definitions of O and A. Next, we show that Z_2^- is relatively interpretable in PAF.

Lemma 21. If ϕ is one of PA_1-PA_6, then $\mathsf{PAF} \vdash \phi^\circ$.

Proof. If ϕ is an individual axiom of PA, then ϕ° is ϕ^A and so $\mathsf{PAF} \vdash \phi^\circ$. $\square$

Lemma 22. $\mathsf{PAF} \vdash (\mathsf{Ind})^\circ$.

Proof. The formula $(\mathsf{Ind})^\circ$ is:

$$\forall z[(0 \preceq z \wedge \forall x(Ax \wedge x \preceq z \to (s(x) \preceq z)) \to \forall x(Ax \to x \preceq z)]$$

Suppose that $0 \preceq z$ and $\forall x(Ax \wedge x \preceq z \to (s(x) \preceq z))$. We want to prove $\forall x(Ax \to x \preceq z)$. Let $\phi(x)$ be $Ax \to x \preceq z$ (here z is a parameter). The corresponding induction axiom in PAF is,

$$[(A0 \to 0 \preceq z) \wedge \forall x((Ax \to x \preceq z) \to$$
$$(As(x) \to s(x) \preceq z))] \to \forall x(Ax \to x \preceq z).$$

So, we need to prove:

$$A0 \to 0 \preceq z$$
$$\forall x((Ax \to x \preceq z) \to (As(x) \to s(x) \preceq z)).$$

Since $0 \preceq z$, we have the first.

Suppose $Ax \to x \preceq z$. We already have that $\forall x(Ax \wedge x \preceq z \to (s(x) \preceq z))$. So, $s(x) \preceq z$. So, trivially, $As(x) \to s(x) \preceq z$, as required. So, we have the second. $\square$

Lemma 23. If θ is a comprehension instance of Z_2^- in L_2, then $\mathsf{PAF} \vdash \theta^\circ$.

Proof. If $\phi(x)$ is an L_2-formula with X is not free, the translation of the comprehension instance for $\phi(x)$ in Z_2^- is:

$$\exists y \psi(y) \to (\mathsf{Comp}_{\phi(x)})^\circ,$$

where $\psi(y)$ is $Ay \wedge \phi^\circ(y)$. By UF, we have

$$\mathsf{PAF} \vdash \exists y \psi(y) \to \exists! z \mathfrak{F}_{\psi(y)}(z).$$

By Lemma 20,

$$\mathsf{PAF} \vdash \exists z \mathfrak{F}_{\psi(y)}(z) \to (\mathsf{Comp}_{\phi(x)})^\circ.$$

So,

$$\mathsf{PAF} \vdash \exists y \psi(y) \to (\mathsf{Comp}_{\phi(x)})^{\circ},$$

as required. □

The previous three lemmas jointly establish:

Theorem 4. For any L_2-sentence ϕ, if $\mathsf{Z}_2^{-} \vdash \phi$, then $\mathsf{PAF} \vdash \phi^{\circ}$.

Finally,

Definition 13. Define the translation $(.)^{*} : L_2 \to L_{\oplus}$ by composition,

$$\phi^{*} := (\phi^{\ddagger})^{\circ}.$$

From Theorems 1 and 4, we conclude that, for any L_2-sentence ϕ,

$$\text{if } \mathsf{Z}_2 \vdash \phi, \text{ then } \mathsf{PAF} \vdash \phi^{*}.$$

This yields the main result of this paper:

Theorem 5. PAF relatively interprets Z_2.

It is perhaps worth adding two remarks. First, the closure axioms ($A0$, $Ax \to As(x)$, $Ax \wedge Ay \to A(x+y)$, $Ax \wedge Ay \to A(x \cdot y)$), stating that 0 is an atom, and that the atoms are closed under arithmetic operations, are not needed for this result, and so need not be included in PAF for this to hold. Second, there is nothing particularly special here about the role of PA. For example, if one extends ZFC with the fusion axioms, with separation and replacement schemes extended to the full language, one obtains a theory which interprets the second-order set theory.

8. Discussion

As noted in Section 5, we have a kind of conservation result for mereology:

$$\text{if } T^{A} \cup \mathsf{F} \vdash \phi^{A}, \text{ then } T \vdash \phi.$$

In a sense, adding the theory of fusions to a theory T in L leads to a conservative extension, *if* axiom schemes in T are not extended to the full language $L\oplus$. In particular,

$$\text{if } \mathsf{PA}^{A} \cup \mathsf{F} \vdash \phi^{A}, \text{ then } \mathsf{PA} \vdash \phi.$$

However, if axiom schemes appear amongst T's axioms, and are extended, then the result may be non-conservative. In particular, if we begin with PA, the fusion extension PAF is non-conservative under the relativization. This follows from Theorem 5,

$$\text{if } \mathsf{Z}_2 \vdash \phi, \text{ then } \mathsf{PAF} \vdash \phi^{*}.$$

To illustrate, let ϕ be some arithmetic sentence provable in Z_2 but unprovable in PA. For example, ϕ might be some standard consistency assertion $\mathsf{Con}(\mathsf{PA})$ for PA, such that $\mathsf{PA} \nvdash \mathsf{Con}(\mathsf{PA})$. Or ϕ might be a more natural mathematical assertion, such as Goodstein's Theorem. Then ϕ^* is the formula ϕ^A, the result of relativizing all quantifiers to Ax. Now $\mathsf{Z}_2 \vdash \phi$, and so, by Theorem 4, we have $\mathsf{PAF} \vdash \phi^A$ even though $\mathsf{PA} \nvdash \phi$. Consequently, under the relativization to Ax, PAF is a non-conservative extension of PA, having the arithmetic strength of full impredicative Z_2.[16]

References

[1] Boolos, G. (1984). "To Be is to Be A Value of a Variable (or Some Values of Some Variables)". *Journal of Philosophy* 81: 430–50.
[2] Hovda, P. (2008). "What is Classical Mereology?" *Journal of Philosophical Logic* 38: 55-82.
[3] Field, H. (1980). *Science Without Numbers*. Oxford: Blackwell.
[4] Leśniewski, S. (1916). *Podstawy oglnej teoryi mnogosci. I*. Moskow: Prace Polskiego Kola Naukowego w Moskwie, Sekcya matematycznoprzyrodnicza (Eng. trans. by D. I. Barnett: "Foundations of the General Theory of Sets. I", in S. Leśniewski, *Collected Works*, ed. S. J. Surma, J. Srzednicki, D. I. Barnett, and F. V. Rickey, Dordrecht: Kluwer, 1992, Vol. 1, pp. 129-173).
[5] Leivant, D. (1994). "Higher-Order Logic". In D.M. Gabbay, C.J. Hogger, J.A. Robinson and J. Siekmann, eds., 1994: *Handbook of Logic in Artificial Intelligence and Logic Programming*, Volume 2: 229-321. Oxford: Oxford University Press.
[6] Lewis, D. (1991). *Parts of Classes*. Cambridge: Wiley-Blackwell.
[7] Linnebo, Ø. (2012). "Plural Quantification". Online *Stanford Encyclopedia of Philosophy*.
[8] Simpson, S. (1998). *Subsystems of Second-Order Arithmetic*. Berlin: Springer-Verlag.
[9] Tarski, A. (1935). "Zur Grundlegung der Booleschen Algebra I." *Fundamenta Mathematicae*, 24, 177-98. English translation, "On The Foundations of Boolean Algebra I", in A. Tarski, 1983, *Logic, Semantics and Metamathematics*: *Papers from 1923-1938 by Alfred Tarski*, second edition (ed. J. Corcoran), Indianapolis: Hackett.
[10] Varzi, A. (2011). "Mereology". Online *Stanford Encyclopedia of Philosophy*.

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