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A VARIANT OF CHURCH'S SET THEORY WITH A UNIVERSAL SET IN WHICH THE SINGLETON FUNCTION IS A SET*

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Abstract

A Platonistic set theory with a universal set, CUSι, in the spirit of Alonzo Church's "Set Theory with a Universal Set," is presented; this theory uses a different sequence of restricted equivalence relations from Church's, such that the singleton function is a 2-equivalence class and hence a set, but (like Emerson Mitchell's set theory, and unlike Church's), it lacks unrestricted axioms of sum and product set. The theory has an axiom of unrestricted pairwise union, however, and unrestricted complements. An interpretation of the axioms in a set theory similar to Zermelo-Fraenkel set theory with global choice and urelements (which play the rôle of new sets) is presented, and the interpretations of the axioms proved, which proves their relative consistency.

The verifications of the basic axioms are performed in considerably greater generality than necessary for the main result, to answer a query of Thomas Forster and Richard Kaye. The existence of the singleton function partially rebuts a conjecture of Church about the unification of his set theory with Quine's New Foundations, but the natural extension of the theory leads to a variant of the Russell paradox.

0. Introduction, Context, and Related Work

1. Philosophical Introduction and Motivation

> *Die Zeit ist nur ein psychologisches Erforderniss zum Zählen, hat aber mit dem Begriffe der Zahl nichts zu thun.*
>
> Time is only a psychological necessity for numbering, it has nothing to do with the concept of number.
>
> (Frege, *Die Grundlagen der Arithmetik* §40, tr. J.L. Austin)

* In this abridged version, all proofs and two bibliographic appendices (on the Church archives at Princeton, and works citing Church's paper) are omitted. The full proofs and appendices will be made available on the website of the Centre National de Recherches de Logique, http://www.logic-center.be/Publications/Bibliotheque.

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> ταῦτα δὲ πάντα μέρη χρόνου, καὶ τό τ᾽ ἦν τό τ᾽ ἔσται χρόνου γεγονότα εἴδη, ἃ δὴ φέροντες λανθάνομεν ἐπὶ τὴν ἀίδιον οὐσίαν οὐκ ὀρθῶς. λέγομεν γὰρ δὴ ὡς ἦν ἔστιν τε καὶ ἔσται, τῇ δὲ τὸ ἔστιν μόνον κατὰ τὸν ἀληθῆ λόγον προσήκει, τὸ δὲ ἦν τό τ᾽ ἔσται περὶ τὴν ἐν χρόνῳ γένεσιν ἰοῦσαν πρέπει λέγεσθαι—κινήσεις γάρ ἐστον…
>
> And these are all portions of Time; even as "Was" and "Shall be" are generated forms of Time, although we apply them wrongly, without noticing, to Eternal Being. For we say that it "is" or "was" or "will be," whereas, in truth of speech, "is" alone is appropriate to It; "was" and "will be," on the other hand, are properly said of the Becoming which proceeds in Time, since [both of] these are motions...
>
> (Plato, *Timæus* 37E, tr. R.G. Bury (Loeb), corrected slightly)

This paper is part of an effort towards a whole-heartedly Platonistic[1] set theory which avoids the set-theoretic paradoxes,[2] but still contains such Fregean sets as the universal set and Frege-Russell cardinals. The standard method of avoiding the set-theoretic paradoxes, Zermelo-Fraenkel set theory, has been pragmatically successful, but suffers from what I have called elsewhere[3] half-hearted Platonism: It relies for its philosophical justification on a metaphor of constructing sets in time (or something like it),[4] which violates a crucial tenet of mathematical Platonism: that mathematical objects are independent of time.[5]

The main philosophical advance in Church's theory, I would claim (Church is silent on his philosophical motivation[6]), is the rejection of a

[1] The relevant aspect of Platonism for the current discussion will simply be that mathematical objects are not temporal; concurrence with any of Plato's ideas about non-mathematical objects is not necessary. See further below.

[2] Particularly the Russell Paradox as it affected Frege's foundational program ([Frege 1903], afterword), but also the Burali-Forti Paradox of the set of all ordinals, and the Mirimanoff Paradox [Mirimanoff 1917b] of the set of all well-founded sets.

[3] [Sheridan 1982, 1989]; summarized in [Forster 1995] pp. 141-2.

[4] E.g., [Parsons 1977], [Gödel 1964] footnote 12, and [Almog 2008] pp. 550-1, 570-1. Even if this temporal metaphor is accepted, the theory would seem to violate it, by allowing sets to be constructed at an earlier level via quantification over sets constructed at a later level; but that is an internal matter for those who accept the metaphor. Unbeknownst to me, Church had noted, with perhaps a hint of scepticism, this impredicativity of ZFC in the notes for his Coble Memorial Lectures [Princeton University Church Archives, box 15, folder 10, typescript "Outline and Background Material, Arthur B. Coble Memorial Lectures"/"Sets of the Model Transfinitely Generated" page numbered 2, 39th page in folder, also mimeograph page 4]. (See below for the need for unwieldy citations to the archives.) See also [Holmes 2001] for an argument that the theory justified by the iterative conception is actually Zermelo Set Theory with Σ_2 replacement. According to Professor Holmes, "this contain[s] an error, which Kanamori pointed out to me and which I know how to fix."

[5] Plato *Timæus*, 37E; [Frege 1884] §40.

[6] The following remark, in a paper on which Church was working at around the same time, might be indicative of Church's state of mind, but this is speculative: "To avoid impredicativity the essential restriction is that quantification over any domain (type) must not be allowed to add new members to the domain, as it is held that adding new members changes

general comprehension axiom schema; such axioms seem also to use the suspect metaphor of temporal construction of sets.[7] Instead Church's theory posits the existence of Fregean sets denied by ZF, such as the universal set and Frege-Russell cardinals, via atemporal operations such as symmetric difference and equivalence classes as sets. Church's (restricted) axioms of generalized Frege cardinals shed, I hope, some light on later work by neo-Fregeans,[8] and may help to rescue Frege's definition by abstraction of cardinal numbers. This is still, I believe, the most natural definition, and Frege's insistence that the definitions of numbers reflect their application remains the best available, albeit partial, explanation of the unreasonable effectiveness of mathematics.[9]

Church's main pragmatic advance is a double use of standard Zermelo-Fraenkel set theory with global choice, both within his theory as axioms restricted to well-founded sets, and metatheoretically as the basis for his (apparently uncompleted) relative consistency proof. The inclusion of the axioms of ZFGC (as the well-founded subtheory, i.e., restricted to well-founded sets), while it makes the use of the theory as a foundation for mathematics easier for the working mathematician, does lay the theory

the meaning of quantification over the domain in such a way that a vicious circle results." "Comparison of Russell's Resolution of the Semantical Antinomies with that of Tarski," The *Journal of Symbolic Logic* volume 41, Number 4, Dec. 1976.

[Anderson 1998] p. 136 suggests that Church was "usually seen as a quite traditional Platonic Realist," at least in his mature period, with a caveat about Church eschewing the label because of its association with the thesis that only universals are real, which is not necessary for the present variety of Platonism. Church, at least in his early period, was not enough of a Platonist not to show some skepticism about the Axiom of Choice [Enderton 2008] p. 8, though in "Set Theory with a Universal Set" he uses an extremely strong form of it. (Church does, however, use Hailperin's finite axiomatization of Quine's New Foundations, rather than Quine's original comprehension schema, in his later theories; this is probably for technical reasons, though Hailperin's axioms might be seen as less Platonistically offensive.) Church was apparently working on a paper entitled "Frege on the Philosophy of Time" before he started work on his set theory [box 15, Folder 8, April 17, 1969]; I have not yet been able to obtain the manuscript from the Church Archives.

Church speculates [Church 1974a], pp. 298-9 about axiomatic possibilities for blaming the antinomies on *intermediate* sets, i.e., sets which are not *low* (i.e., equinumerous to a well-founded set), and whose complements are also not low; more specifically, on sets which are "balanced on the hazardous edge between low sets and intermediate sets." I am not aware of any progress on this approach. While it might apply to the Burali-Forti and Mirimanoff Paradoxes, it does not seem to apply to the Russell Class, which would contain, for instance, all normal singletons.

[7] See also [Sheridan 2005] and [Forster & Libert 2011] for an argument that a first-order comprehension axiom amounts to a claim of implausible fixed points in simple set theoretic operations such as adjunction.

[8] See especially [Burgess 2005]. Much of the neo-Fregean program was beginning while I was doing my initial work—some of it at Oxford while I was writing the initial version of this paper—but I was largely unaware of its achievements until I resumed work on this paper more than a decade later.

[9] See [Heck 2013], p. 41 ff, p. 222 ff.

open to the charge of philosophical hypocrisy. The coupling between the details of ZF, and Church's and my theories, is relatively weak, however; the ZF-like axioms are a clearly-distinguished subset. ([Sheridan 1985] shows that Church's theory can be viewed as a conservative extension of ZFGC, and a similar result seems clear for the current theory.) I have tried to keep my uses of the main one, Well-Founded Replacement, relatively isolated, though I have not succeeded as much as I had hoped.

Both of these features distinguish Church's theory from Quine's New Foundations [Quine 1937a], which in some other respects his theory resembles. Church ends his initial article with speculation that his theory might be unified with New Foundations. I found this philosophically objectionable, since NF has a comprehension axiom and lacks a clear philosophical motivation.[10] A later, unpublished, theory by Church attempts to converge with New Foundations, but he seems to have abandoned it; see the historical introduction below.

My main contribution is to distinguish theories of Church's sort further from New Foundations, by providing a variant in which the singleton function is a set. This is completely impossible in New Foundations.[11] I also provide a fully-worked out relative consistency proof. (Details of my proof are omitted in this abridged version, but will be made available on the web.) Church never published a full consistency proof, and the version in his archives, which refers to two earlier attempts, seems to have been abandoned as well. His notes written in 1989 suggest the need for "a new approach." My consistency proof also avoids the use of compactness needed by Church; I provide an interpretation for the full sequence of restricted equivalence relations, rather than an arbitrary finite subsequence.

How successful my endeavor was, however, is unclear: A natural extension of my theory is subject to a variant of the Russell Paradox, involving the set of all non-self-membered sets equinumerous to the universe,[12] and my equivalence classes (unlike Church's) are not closed under sum set and product set, though Mitchell's theory suffers a similar limitation. (My theory does, however, satisfy unrestricted pairwise union.)

The paradox, though it is not directly relevant to the consistency proof here, suggests that my equivalence classes ran afoul of what has since been called the Bad Company problem.[13] Church's equivalence classes are not obviously subject to the same difficulty, and are, as noted, closed under sum

[10] [Forster 1995] pp. 26-7; [Holmes 1998] p. 12; [Maddy 2011] p. 136, citing [Fraenkel, Bar-Hillel, and Lévy 1973] p. 164. Holmes' chapter eight does attempt to provide a philosophical motivation for stratification, but not from an atemporal perspective.

[11] [Holmes 1998], p. 110, 131.

[12] See the discussion following the 1-Isomorphism Lemma (16.4), below. Cp. also Holmes' proof of the non-set-hood of the membership relation, [Holmes 1998] p. 43.

[13] [Burgess 2005], pp. 164 ff.; [Boolos 1990], pp. 249–251; [Dummett 1991], pp. 188-9.

and product set. Thus it could be argued that the limitations of my endeavor are an argument for Church's conjecture about unification with New Foundations,[14] or possibly Forster's "Naturam expellas furca" claim[15].

1.1. *Criticism and Alternatives*

Thomas Forster has criticized Church's type of construction as a Potemkin Village:[16] "*this technique is not a great deal of use for constructing models of a theory T unless T has an easy word problem.* Set theories with an easily solvable word problem are unlikely to be of interest."[17] I don't think anyone has disagreed with this criticism; both Church's unpublished follow-on work [Church Archives box 15, folder 11, untitled manuscript, page numbered 3, 3rd page in folder], and [Malitz 1976], written under Church's supervision, redefine equality recursively for the sake of more powerful axioms. Mitchell's concluding comment addresses this possibility as well [Mitchell 1976], pp. 30-1. My own contemplation of more powerful axioms extending my system led to a variant of the Russell Paradox, noted above and discussed below. Even before encountering this paradox, I had expressed scepticism that interesting extensions of these theories would allow models using the same simple techniques.

Oberschelp's theory [1973] also avoids my difficulties, which may indicate the wisdom of the limitations of his approach. I feel that Oberschelp's work deserves far more attention than it has received, but the presentation is difficult, and part of the consistency proof (p. 48) is a reference to another paper ([Oberschelp 1964a]) with a different formalism.

2. Historical Introduction

When the rough draft of this paper was substantially finished, I learned of papers in Church's archives at Princeton University on later set theories with a universal set. Those theories are out of scope for the technical sections of this paper, but the papers I have obtained from the archives, and its catalog, are the main source for this historical introduction. I have so far only received

[14] Especially the cumulative hierarchical aspects of his 1975 and later unpublished archive notes, e.g. Church Archives Box 15, folder 10, typescript "Outline and Background Material, Arthur B. Coble Memorial Lectures"/"Sets of the Model Transfinitely Generated" page numbered 2, 37th page.

[15] [Forster 2006] p. 240, presumably alluding to Horace's *Epistles*, I. x. 24, about the necessity of the cumulative hierarchy for avoiding the paradoxes.

[16] The term "Potemkin Village," which is probably unfair to its namesake (a Governor-General showing villages in the Crimea to Catherine the Great), is used for constructions placed *only* where an observer will be looking for them. I believe the term was first used by me as a summary of Forster's criticism at his Stanford lecture on Church's theory, 11 April 2005.

[17] [Forster 2001], p. 6.

portions of the relevant papers from the archives, pending Princeton's procurement of scanning equipment. The page numbers on these papers are often missing or incorrect, so I have erred on the side of explicitness below in citing them.

On June 24th 1971, Alonzo Church presented a paper entitled "Set Theory with a Universal Set" to the Tarski Symposium, at the University of California at Berkeley.

In a fifty-page manuscript dated July 1971, labelled "Notes as to Set Theory with a Universal Set" (photocopy in archives box 47, Folder 10), Church states that "As even the amended model of April 1971, … is not yet satisfactory, we make a new start using the outline of June 1971." This seems to be an eventually-abandoned attempt at another consistency proof for the full CUS; the mathematics does not seem final, and the photocopy, if not the manuscript, ends abruptly.

In 1974, a version of the paper was printed in the conference proceedings, with a "minor but essential modification" to the definition of "the equivalences characterizing the model" [Church 1974a], p. 307, footnote 11.

The published paper presents three main features:

1. A sequence of equivalence relations generalizing equinumerosity,
2. A set of axioms for a set theory with a universal set and some equivalence classes, including Frege-Russell cardinals, as sets,
3. A model (actually an interpretation [Shoenfield 1967], though Church does not use the term) of the axioms, restricted to a finite subsequence of the sequence of equivalence relations, with length a fixed arbitrary natural number m.

The paper presents no proofs; Church states (p. 307) that the "details of the verification… are straightforward but (if $m > 0$) laborious." I believe most readers have found this an understatement.

I am not aware of any accounts of Church's talk (though Emerson Mitchell was present, see below; oddly, no correspondence with Mitchell is listed in the index to the Church archives). Given the length of time necessary to understand the published paper, it seems unlikely that Church had time to present much more than the definitions of the sequence of equivalence relations, the axioms, and a high-level sketch of the interpretation.

Church later issued an undated two-page correction to footnote 4 of the paper, covering a tangential remark on standard von Neumann and Bernays set theories, which was not relevant to his new theory.

In 1973, Arnold Oberschelp published an article, apparently independently, with a technique similar to Church's, but using urelements rather than displaced sequences for the construction, and with a richer model which included the singleton function as a set [Oberschelp 1973]. Note that part of the consistency proof in both [Oberschelp 1973] (p. 48) and [Friedrichsdorf 1979] (p. 382) is merely a reference to [Oberschelp 1964a], which uses a significantly different formalism. Neither Church nor Oberschelp seems to have been aware of the other's work at the time, and the underlying similarity between the two techniques is not necessarily obvious. (Indeed, though I cited [Oberschelp 1973] in [Sheridan 1989], I did not realize its significance until [Sheridan 1990].) See the comments below on my definition of j-isomorphism, and the limitations in proving its absoluteness in the interpretation.

In 1974, according to Forster [2001], Urs Oswald independently rediscovered Church's method of permutation models, published in his ETH Zürich Ph.D. thesis of 1976, *Fragmente von "New Foundations" und Typentheorie*. Forster calls Oswald's discovery simultaneous with Church's, but Church's original paper was presented in 1971. Werner Depauli-Schimanovich also makes a claim for the priority of his 1971 doctoral thesis in his Arxiv web article [2008], which I have not evaluated.

In the fall of 1974, Church presented a lecture entitled "Notes on a Relative Consistency Proof of Axioms A–K of Church's Set Theory with a Universal Set" [Church 1974b]. Church mailed me a copy in 1984, with a handwritten notation (in a different handwriting, presumably Church's): "Student notes of 1974 lectures by Alonzo Church." The notes are eleven pages of quite dense mathematics, but only cover the case m = 0, i.e., omitting the equivalence class axioms L_j, whose verification Church implied was laborious but not straightforward. (I do not believe that this is an overstatement.) Church's comment on these notes, in his archive, is apparently "Probably not of much value—but possibly worth some reflection" [box 47, Folder 5]

In 1984 I requested that Church send me any further relevant work, but never received a response; my attempts to see him in Los Angeles in 1985 and 1987 failed due to his absence in the Bahamas. Until the publishing of the catalog of his archives on the web [http://arks.princeton.edu/ark:/88435/fx719m49m], I had assumed that he did no further work on his set theory, but the catalog lists some further lectures which sound relevant, which I have so far been unable to obtain from the archives. I cannot tell whether Church's disinclination to send me his later work represents a repudiation of it.

On September 23-25, 1975, at the University of Illinois at Urbana-Champaign, Church delivered the Arthur B. Coble Memorial Lectures, entitled

"Set theory with a universal set."[18] Despite having the same title, this set theory was far more complicated than his 1974 paper; one of its main goals was to model Hailperin's finite axiomatization of Quine's New Foundations, though Church's notes (summarized below) indicates that he fails to model axiom P6. This topic deserves more investigation than it has received, but is outside the scope of this paper, and I have so far only been able to acquire a portion of the archives on this topic.

Princeton Archives Box 15, folder 10 consists of three versions of what I will call the Coble Theory. Two are hand-written, with the first apparently a corrected version of the second; the third is apparently a typewritten version of the corrected handwritten manuscript, with three hand-written corrections. The typewritten version also exists in a mimeographed copy in the possession of the late Professor Herbert Enderton of the University of California at Los Angeles (which he graciously allowed me to scan), and apparently in a version at the University of California at Berkeley, which I have not seen. The UCLA copy lacks the handwritten corrections, but contains an extra typewritten sheet, specifying (1) two of the corrections made to the Princeton copy, and (2) a more sweeping correction: "the probability there must be substantive changes at various places, especially on the last two pages of either manuscript of the notes."

Princeton Archives Box 15, folder 11 contains hand-written notes for a somewhat different, incompletely-developed theory, which I will call the Folder 11 Theory. I will refer to the published 1974 theory (the main basis for my theory) as the 1974 Theory.

Some striking features of the Coble construction are:

1. "a transfinite array of relations, one for each ordinal m, that are left unspecified, the intention being that different set theories result by different choices of the invariance relations inv^m." [Church Archives, Box 15, folder 10, 5th page, numbered 5.] The independent construction in the middle part of the current paper could be similarly described, though the details are quite different.
2. Construction of a non-trivial identity relation for the model, which may avoid Forster's Potemkin Village criticism, discussed above. [Church Archives ibid.], [Forster 2001], p. 6.
3. The Coble set theory may have influenced the 1974 theory, e.g., the Axiom of Substitutivity in the earlier theory seems superfluous, but is necessary in the later theory, since the relation i (corresponding to identity)

[18] http://www.math.uiuc.edu/Colloquia/coble_history.html, History section. Church's Princeton archives list the title of this and the lecture below as "Set Theory on a Universal Set"; this substitution of "on" for "with" seems to be a transcriber's mistake rather than a deliberate change—the University of Illinois web page lists the usual title, and no subsequent source ever seems to have used "on" rather than "with."

is constructed, and differs from the base equality. Keeping Substitutivity in the Basic Axioms allows Church to keep the axiom letters the same between the two theories.

4. The unnumbered last (19th) page of Professor Enderton's mimeographed copy of the Coble Lecture notes mentions "the probability there must be substantive changes at various places, especially on the last two pages of either manuscript of the notes." This correction is missing from the Princeton Archives copy.
5. "Given any set, there exist its complement and the set of complements of its members. The set of low sets, the set of intermediate sets, the set of high sets exist." ["Outline and Background Material, Arthur B. Coble Memorial Lectures"/"Sets of the Model Transfinitely Generated," section "Set Existence" Box 15, folder 10, page numbered 12, 47th page in folder]

Some features of the Folder 11 construction are:

1. Church calls this "the model of the Quine set theory which we seek to set up." [Church Archives box 15, folder 11, untitled manuscript, page numbered 2, 2nd page in folder.]
2. An explicit classification of constructed sets as low, high, low intermediate, high intermediate, and fully intermediate. [Church Archives box 15, folder 11, untitled manuscript, page numbered 3, 3rd page in folder.] This differs, at least in presentation, from the Coble Theory.
3. Church abruptly abandons a definition after clause 60, because of "the (later) discovery that the model obtained does not satisfy Hailperin's P6." He says that "an informal and partly heuristic account follows." [Church Archives box 15, folder 11, untitled manuscript, page numbered 15, 15th page in folder.] Church's statement of P6 is "$(\exists u) \,.\, x \,\mathrm{e}\, u \equiv_x (y) \,.\, <y, \iota`x> \mathrm{e}\, v$". [Church Archives Box 15, folder 10, typescript "Outline and Background Material, Arthur B. Coble Memorial Lectures"/"Sets of the Model Transfinitely Generated," section "Set Existence" unnumbered page, 48th page in folder] Hailperin's original formulation [Hailperin 1944], p. 10, is "$(\alpha)\ (E\beta)\ (x)\ [x \in \beta \equiv (u)(<u, \iota x> \in \alpha)]$." The following page begins with the observation that "The foregoing definition by recursion is a first draft." Church goes through various stages of believing that he can or cannot prove P6 or Hailperin's other axioms, and it is not clear what the final resolution was for this article:
   Two unnumbered pages note that the "proof of P6 requires modification of the above" [33rd page; different wording on the 34th page]. The unnumbered 36th page is headed "Analysis Directed Towards Proof of P6," but the page numbered 40 states "it is not immediately clear that this amendment will successfully result in a model in which all nine of the Hailperin axioms hold..." The following page, numbered 42, is headed "Proof of

the Main Lemma (for P6) from Lemmas 1-5," but the manuscript ends abruptly after the next page, five lines into the proof of case 1b.

Later items in the Church Archives catalog, which I have not yet been able to obtain, have titles mentioning P6 as well—I do not yet know if the later work overcomes this difficulty.

Church seems to have continued working on this approach until at least "box 47, Folder 2: Notebook: Recursion clauses for inv^m as revised Sept. 1980." I speculate that he had abandoned it by 1984, when he sent me [1974b] without mentioning the later theory. The undated addendum to the Coble lecture typed mimeograph, noting that substantive changes were probably needed, may have seemed at the time to be only a temporary setback.

4. Church notes "other divergences from the Quine set theory... the set of all sets a of pairs… evidently does not exist in the model" [Church Archives box 15, folder 11, untitled manuscript, p. 40, 40th page in folder], which demonstrates that the Folder 11 Theory is inferior in this respect to his 1974 theory as well as New Foundations.

In the Abo Akademi in Turku, Finland, on March 22, 1976, Church presented a shortened version of the Urbana, Illinois lectures [box 49, Folder 7].

In 1976, Emerson C. Mitchell was granted a Ph.D. from the University of Wisconsin at Madison for "A model of set theory with a universal set," which cites Church and builds on his technique to provide a set theory with unrestricted power set, but which lacks some of Church's other axioms [Mitchell 1976].[19] Mitchell was present at Church's 1971 lecture.[20] Mitchell's only bibliographic reference is to [Church 1974a]; the complexity of Mitchell's construction is reminiscent of Church's later unpublished work, but this is not proof that Church showed it to him. Cp. the resemblance between Church's unspecified sequence of equivalence relations in his later theories and my own.

In 1979 Ulf Friedrichsdorf published an article [Friedrichsdorf 1979] building on [Oberschelp 1973].

The Church archives list a number of notebooks on set theory from 1975 to 1983; some of these at least (e.g., box 15, Folder 10) seem to be on combining his set theory with New Foundations via Hailperin's finite axiomatization [Hailperin 1944].

In 1989 Church wrote about some of his notes, apparently the uncompleted consistency proof, "These notes are old [1971] but might be reconsidered for the sake of some truth in it, which might guide a new approach"

[19] Note that the spelling of Mitchell's first name on his thesis is an error.

[20] Abstract of [Mitchell 1976] in The Journal of Symbolic Logic, March 1977, Vol. 42, No. 1, p. 148.

[box 47, Folder 10]. This dissatisfied comment might be applied to all of the original work done in this area. For a survey of the field, the reader is referred to the articles and book by Forster in the bibliography, whose perspective is decidedly different.

3. Organization

The overall goal is to prove the equiconsistency of my theory, CUSι, with ZF, by defining an interpretation of CUSι in a base theory equiconsistent with ZF, and proving the interpretation of each axiom from the base theory. This will establish the relative consistency of CUSι: Any proof of an inconsistency from the axioms of CUSι can be translated into a proof of an inconsistency in the base theory.

The central section of this paper (Part II, which was originally written separately, for Professor Church's cancelled ninetieth birthday festschrift, and is omitted except for a few key results in this abridged version) defines an interpretation of a partially-specified ill-founded set theory, and proves the interpretation of the Axiom of Extensionality. (The proofs of the other Basic Axioms—the first group of axioms, below, which are restricted versions of axioms of ZF—which constitute Part I, are simpler.) Unlike Church's and Mitchell's interpretations, but like Oberschelp's, my interpretation uses urelements in the rôle of the new, ill-founded, sets; this avoids having to rearrange the old sets to make room for the new. To answer a query of Forster and Kaye when a much earlier attempt at this result was presented as a doctoral thesis, and also to keep open the possibility of iterating this type of construction, or to do it with other set theories as a base theory, Part II was done in much greater generality than is needed for the main result, and with limited use of Choice and Foundation. (It is not clear at this point that either endeavor was worth the effort.) In particular, rather than using the specific sequence of restricted equivalence relations $\Bumpeq^{j}$ defined in Part III, it was done with an arbitrary sequence $\backsim^{j}$ satisfying certain properties.

After the interpretation is defined and the interpretation of the Axiom of Extensionality proven for the partially-defined interpretation defined in terms of $\backsim^{j}$ in Part II, in Part III the required properties of $\backsim^{j}$ are shown to hold for $\Bumpeq^{j}$. This establishes the interpretation of Extensionality for the specific theory under investigation, CUSι, with the specific equivalence relations $\Bumpeq^{j}$. The interpretations of the new axioms of interest can then finally be established, along with their consequences of interest, such as the existence of Frege-Russell cardinals and complements. In Part III both Global Choice and Foundation are assumed for the base theory, which reduces the generality but simplifies the derivations.

The verification of the rest of the Basic Axioms, which constitutes Part I, is also done in considerably more generality than necessary, but with a different and weaker set of requirements: primarily on the form of the definition of the new membership relation, plus a requirement that new sets be ill-founded, and some sanity requirements on well-foundedness in terms of the new membership relation.

One of the consequences of the requirements on the partially-specified equivalence relations, needed for Extensionality in Part II (which was written first), is, in effect, that the new sets are too large to be low. The verification of the other Basic Axioms (which were proven later) are simpler in Part I, and rely largely upon this fact; this may make verification of these axioms for other possible theories easier. The key to this result in the specific case in Part III is the Replacing at Level*j Construction, which for any non-degenerate $\Bumpeq^j$ equivalence class, takes an arbitrary object and embeds it into the transitive closure of an object in the given equivalence class. Given that result, the verification that the results of Part I apply to the interpretation in Part III is far simpler than the application of Part II.

Note that I generally follow Church in referring to both $\backsim^j$ and $\Bumpeq^j$ informally as equivalence relations, though they are actually only provably equivalence relations on the well-founded sets. (Conveniently, this will not matter in the base theory, and hence in the consistency proof; it only affects discussion of results within the theory of interest.) Indeed, it is not even clear that symmetry or reflexivity hold for objects equivalent to the Universal Set, for even the first of Church's relations in his interpretation, nor for my $\Bumpeq^1$ in mine. Church did not address this point in his published writings, though it is presumably the motivation for the restriction in his axiom of generalized Frege cardinals, and would have needed to be addressed in his full consistency proof. (The point does not arise in his surviving lecture notes for case $m=0$, and I was not able to find mention of it in his abandoned consistency proof in the archives.) It is possible that he was aware of subtleties which eluded me, since the obvious extension of my theory, asserting that these relations are unrestricted equivalence relations (plus some natural assumptions about the existence of mappings), runs into a variant of the Russell Paradox which his equivalence relations apparently avoid. It is even possible that he was aware of this when suggesting a unification of his theory with Quine's New Foundations, but there is no evidence of this.

The proofs in Parts I and II could be applied to Church's original theory; the generality in Part II was crafted to include Church's j-equivalence relations as well as my own. The Replacing at Level*j Construction in Part III would require substantial modification, but would be substantially easier for Church's relations. Such a proof would be significant, given Church's apparent abandonment of the consistency proof for his full system. It would not be complete, however, since Church's theory has unrestricted axioms of sum and product set, which my theory does not; they depend on the details of Church's equivalence relations, which do not seem to have been addressed in the abandoned consistency proof in the archives. The replacing result seems to be true for Church's theory, as noted in [Sheridan 1989], but Church does not seem to have addressed it in his surviving writings, though his presentation suggests that it may have been part of his motivation for the definition of his equivalence relations.

4. Discussion of the Axioms

The base theory in which results will be proven will be the Basic Axioms (below), plus a bookkeeping axiom about urelements. Use of Choice and Foundation will be avoided in Parts I and II, except for some explicitly-mentioned uses of a consequence of Foundation near the end of Part II, but will be needed extensively in Part III. The Basic Axioms are equivalent to their usual counterparts in the presence of Foundation. The relative consistency of the form of global choice used, and of the book-keeping axiom, are unproblematic: global choice by the well-know result of Gödel, and the book-keeping axiom by a trivial use of the technique of [Church 1974a], or in the simpler form presented in [Forster 2001].

Note that since the base theory must allow urelements, Extensionality is restricted to nonempty objects. The theory of interest is largely neutral about the existence of urelements, though the Axiom of Generalized Frege Cardinals implies that the collection of all empty objects is a set; Well-Founded Replacement forbids this object to be the size of the Universe. Hence the interpretation presented below excludes urelements.

CUSι, the theory of interest, includes the Basic Axioms, but necessarily excludes Foundation. It also avoids dependence on Choice, though it is consistent to add it in the strong form used here, as it is incidentally true in the interpretation presented. (The global well-ordering used does not mention set membership, so it is unaffected by the reinterpretation of the membership relation.) The theory also includes the Restricted Axiom of Generalized Frege Cardinals, asserting that for the sequence of equivalence relations $\Bumpeq^{j}$ (for $j \in \omega$), any well-founded set has a set of all sets to which it is $\Bumpeq^{j}$, for each $j \in \omega$. The restriction to well-founded sets is for the purpose of the consistency proof; neither Church's technique nor Oberschelp's seems sufficient to provide unrestricted axioms of generalized Frege cardinals. Note that the equivalence classes themselves are not restricted to well-founded sets. For example, the set of all singletons, (which is the union of at most two $\Bumpeq^{1}$ equivalence classes) contains the singleton containing the Universal Set, and hence is ill-founded. For simplicity, $\Bumpeq^{0}$ is the universal relation, so the (unique) 0-equivalence class is the universal set.

Note that in the absence of Foundation, some of the restrictions on the Basic Axioms become significant. Sum Set, for instance, does not apply to the new sets, again because of the limitations in technique. Church's proof of his theory's consistency would have needed to demonstrate that his combinations of equivalence classes were closed under unrestricted sum set, but this seems to depend on the details of his equivalence relations, and does not apply to my modifications.

Part I

Language, Definitions, Basic Axioms, and $\in\dagger$-Interpretations

5. Language

The primitive symbols of the language, in addition to the usual first-order logical apparatus, are "=", "$\in$", "Ø", "Υ", and (for use with a strong form of Global Choice) "$\mathcal{G}$". Two special symbols are needed for use in book-keeping axioms, "Υ" and "Ø"; an explicit symbol for Ø is needed for use in distinguishing it from urelements, and "Υ" will denote an assumed injection of the sets into the urelements.

Several symbols will be often used rather like primitive symbols, but are in fact defined terms: "$\not\equiv$" denotes exclusive disjunction, i.e., $P \not\equiv Q \equiv_{df} (P \vee Q) \& \neg (P \& Q)$. $\in_1$, $\in_2$, and $\in_3$ will be the ill-founded set membership relations of interest in Parts I, II, and III respectively; $\in_1$ and $\in_2$ will be partially specified (in somewhat different ways) in Parts I and II, to show general results about broad classes of interpretations; $\in_3$ will be the specific membership relation used in Part III to show the relative consistency of CUSι. Since Church uses "$\in$" without a subscript to denote the new membership relation, rather than the old one (or for purposes of emphasis), I will often use "$\in_0$" as a synonym for "$\in$". Once I have defined "$\in_1$", a formula followed by a subscript "$_1$" will abbreviate that formula with "$\in_1$" substituted for all (including implicit) occurrences of "$\in$." Similarly for "$_2$" and "$_3$."

Limited use is made of class terms as a syntactic convenience without ontological commitment, as in [Quine 1969] and [Levy 1979] §3.1. Definition schemas are explicitly marked as such with "$\equiv_{dfs}$" and distinguished from single definitions marked with "$\equiv_{df}$". Following Church's modification of the Peano/Russell practice, dots and double dots are sometimes used informally as substitutes for brackets.

6. Definitions

These definitions are not intended to be surprising (where possible they are simply from [Levy 1979]), but the weakness of the base theory requires more elaboration than usual, and the proofs require tedious attention to primitive notation. The only surprising point is the definition of well-foundedness, not directly, but in terms of ill-foundedness and unending chains, which is important in the absence of Dependent Choices. The reader should feel free to skip these definitions (and the development of addition, below) on the assumption that the definitions do indeed mean what they are supposed to mean.

6.1. *Logic*

"$\Rightarrow$" and "$\Longleftrightarrow$" indicate implication and equivalence with least close possible binding. "$\equiv_{\mathrm{df}}$" of course, has looser binding still. "$\equiv_{\mathrm{dfs}}$" indicates a definition schema.

"$\not\equiv$" indicates exclusive disjunction, inequality for truth values. Informally, exclusive disjunction is associative: $(P \not\equiv Q) \not\equiv R$ ... iff an odd number of P, Q, R ... are true iff $P \not\equiv (Q \not\equiv R \ldots)$.

"$\exists! x.\, \Phi(x)$" abbreviates "$\exists x.\, \Phi(x)\ \&:\ \forall x'.\, \Phi(x') \rightarrow x' = x$", where Φ is a predicate.

"$\iota x.\, \Phi(x)$", if $\exists! x.\, \Phi(x)$, denotes that x, and otherwise is undefined, where Φ is a predicate with one free variable.

6.2. *Sets and Membership*

nonempty(x) $\equiv_{\mathrm{df}} \exists y \in x$; $z \notin x \equiv_{\mathrm{df}} \neg z \in x$.

set(x) $\equiv_{\mathrm{df}}$ $x = \emptyset \vee$ nonempty(x); **urelement**(x) $\equiv_{\mathrm{df}}$ $\neg$set(x).

SET[Φ] $\equiv_{\mathrm{dfs}}$ $\exists x.$ set(x) & $\forall z.\ z \in x \equiv \Phi(z)$, where Φ is a predicate with one free variable. (Note that the definition does not require that this x be unique, though extensionality would imply this.)

$\{x, y\} =_{\mathrm{df}} \iota p.\ \forall w.\ w \in p \equiv (w = a \vee w = b)$; $\langle x, y\rangle =_{\mathrm{df}} \{\{x\}, \{x, y\}\}$, i.e., the Kuratowski ordered pair.

$x \simeq y \equiv_{\mathrm{df}} \forall z.\ z \in x \equiv z \in y$. (Read "x is **coextensive** with y.") (Thus $x \simeq_0 y \Longleftrightarrow \forall z.\ z \in_0 x \equiv z \in_0 y$, and [once I have defined "$\in_1$"] $x \simeq_1 y \Longleftrightarrow \forall z.\ z \in_1 x \equiv z \in 1\ y$.)

Unique(y) $\equiv_{\mathrm{df}} \forall x.\ x \simeq y \rightarrow x = y$; x and y are **disparate** $\equiv_{\mathrm{df}} \neg\, x \simeq y$.

$x \subseteq y \equiv_{\mathrm{df}}$ set(x) $\&:\ \forall z.\ z \in x \rightarrow z \in y$.

$x \subset y \equiv_{\mathrm{df}} x \subseteq y\ \&\ \exists z \in y.\ z \notin x$.

6.3. *Mapping*

maps(f, a, b) $\equiv_{\mathrm{df}} \forall p \in f\ \exists x \in a\ \exists y \in b.\, p = \langle x, y\rangle\ \&$
$\forall x \in a\ \exists! y \in b\ \exists p \in f.\ p = \langle x, y\rangle\ \&$
$\forall y \in b\ \exists x \in a\ \exists p \in f.\ p = \langle x, y\rangle$.
I.e., the function f maps a onto b, not necessarily one-to-one.

function(f) $\equiv_{\mathrm{df}} \exists a, b.$ maps(f, a, b).

domain(f) $=_{\mathrm{df}} \iota a.$ set(a) & $\exists b.$ maps(f, a, b).

range(f) $=_{\mathrm{df}}$ ιb. set(b) & ∃a. maps(f, a, b).

Note that, unlike $\mathrm{maps}_{\mathrm{formula}}$, below, if maps(f, a, b), then f's domain is a and range is b.

$\mathbf{maps}_{1\text{-}1}(f, a, b) \equiv_{\mathrm{df}} \forall p \in f\ \exists x \in a\ \exists y \in b.\ p = \langle x, y\rangle\ \&$
$\forall x \in a\ \exists! y \in b\ \exists p \in f.\ p = \langle x, y\rangle\ \&$
$\forall y \in b\ \exists! x \in a\ \exists p \in f.\ p = \langle x, y\rangle.$

Note that $\mathrm{maps}_{1\text{-}1}(f, a, b)$ implies maps(f, a, b).

$a \approx b \equiv_{\mathrm{df}} \exists f.\ \mathrm{maps}_{1\text{-}1}(f, a, b)$. (Read "a is **equinumerous** to b.")

FUNCTION(Φ, a) $\equiv_{\mathrm{dfs}}$ $\forall x \in a.\ \exists! y.\ \Phi(x, y)$.[21]

$\mathbf{maps}_{\mathbf{formula}}(\Phi, a, b) \equiv_{\mathrm{dfs}} \mathrm{FUNCTION}(\Phi, a)\ \&\ \forall x \in a.\ \exists y \in b.\ \Phi(x, y)\ \&$
$\forall y \in b.\ \exists x \in a.\ \Phi(x, y).$

6.4. *Well-Foundedness*

unending-chain(c) $\equiv_{\mathrm{df}}$ nonempty(c) & $\forall x \in c\ \exists y \in c.\ y \in x$.

ill-founded(w) $\equiv_{\mathrm{df}}$ $\exists c.\ w \in c$ & unending-chain(c).

wf(w) $\equiv_{\mathrm{df}}$ ¬ ill-founded(w). (Read "**well-founded**.")

low(x) $\equiv_{\mathrm{df}}$ $\exists f \exists w$. wf(w) & maps(f, w, x).

We will not need the notion of a low class, since the restricted Axiom of Replacement will imply that any such would correspond to a set. Informally, say that there are **many** P's if the class of P's does not correspond to a low set. Church [1974a], page 298 defines a set as low if it is equinumerous to a well-founded set; it is not hard to show the two definitions equivalent in the presence of a global well-ordering. Unbeknownst to me, Church's abandoned consistency proof (Box 47, Folder 10) has a predicate "retrogressive," which is similar to my unending chain: $x\ \varepsilon\ r \rightarrow_x (Ey)\ .\ y\ \varepsilon\ r\ .\ y\ \varepsilon\ x.$

transitive(a) $\equiv_{\mathrm{df}}$ $\forall y \forall z.\ z \in y\ \&\ y \in a. \rightarrow z \in a.$

6.5. *Infinity and Ordering*

Dedekind-infinite(x) $\equiv_{\mathrm{df}}$ $\exists y.\ y \subset x\ \&\ x \approx y.$

Dedekind-finite(x) $\equiv_{\mathrm{df}}$ ¬Dedekind-infinite(x).

totally-linearly-orders(R, a) $\equiv_{\mathrm{dfs}}$

$\forall x \in a.\ \neg xRx\ \&$

$\forall x \in a\ \forall y \in a\ \forall z \in a.\ xRy\ \&\ yRz \rightarrow xRz\ \&$

$\forall x \in a\ \forall y \in a.\ xRy \vee x = y \vee yRx.$

[21] I will systematically confuse function symbols with relation symbols which I have proved functional.

well-orders(R, a) $\equiv_{dfs}$
totally-linearly-orders(R, a) &
$\forall z \subseteq a$. nonempty(z) $\rightarrow \exists m \in z\ \forall w \in z.\ \neg wRm$.

ordinal(x) $\equiv_{df}$ transitive(x) & wf(x) & well-orders($\in$, x) & set(x) & $\forall z \in$ x. set(z).

$\omega =_{df}$ ıw. $\forall x.\ x \in w \equiv$. Dedekind-finite(x) & ordinal(x).

If α and β are ordinals, define $\alpha < \beta$ iff$_{df}$ a $\in \beta$; $\alpha \leq \beta$ iff$_{df}$ a $\in \beta \vee \alpha = \beta$.

It may seem redundant to require that an ordinal be both well-founded and well-ordered by $\in$; but the obvious proof of my version of well-foundedness from well-ordering requires the existence of the given set's intersection with an unending chain, which in turn apparently requires well-foundedness.

I will use small Greek letters as variables for ordinals. As the ordinals less than some fixed ordinal μ perform the same function here as do the natural numbers less than or equal to some fixed natural number m in [Church 1974a], and Church uses j in such contexts, I will here use i, j, k, and n extensively as variables for ordinals less than or equal to μ.

6.6. *Class Abstracts*

"$\{x \mid \Phi(x)\}$", or, for emphasis "$\{x \mid \Phi(x)\}_0$" indicates ıs. set$_0$(s) & $\forall x$. $x \in_0 s \equiv \Phi(x)$, if that exists, otherwise merely the virtual class (i.e., predicate) $\Phi(x)$. Analogously "$\{x \mid \Phi(x)\}_1$". The latter notion is of little interest if Φ was defined in terms of "$\in_0$" rather than "$\in_1$". Note that a class abstract$_0$ is never an urelement$_0$.

"$\{x \in y \mid \Phi(x)\}$" abbreviates "$\{x \mid x \in y\ \&\ \Phi(x)\}$."

For $\tau(y)$ a term, $\{\tau(y) \mid \Phi(y)\} =_{dfs} \{x \mid \exists y.\ \Phi(y)\ \&\ x = \tau(y)\}$.

"Δ" normally means **symmetric difference**. For typographic convenience, "Δ" will be used in Part II for symmetric difference in the sense of $\in_2$; "δ" will mean symmetric difference in the sense of $\in_0$. I.e., $x\ \delta\ y =_{df} \{z \mid z \in_0 x \not\equiv z \in_0 y\}_0$, and $x\ \Delta\ y =_{df} \{z \mid z \in_2 x \not\equiv z \in_2 y\}_2$. ("$\partial$", defined in a later section, will also be distinct.)

$\Sigma x =_{df} \{z \mid \exists y.\ z \in y\ \&\ y \in x\}$; $x \cup y =_{df} \{z \mid z \in x \vee z \in y\}$.

$\cap x =_{df} \{z \mid \forall y \in x.\ z \in y\}$; $x \cap y =_{df} \{z \mid z \in x\ \&\ z \in y\}$.

a — b $=_{df} \{x \in a \mid x \notin b\}$.

POW(a)$=_{df} \{x \mid x \subseteq a\}$.

Define for a term τ and ordinal α, $\bigcup_{\alpha \leq j < \mu} \tau(j) =_{dfs} \{x \mid \exists j\ \exists y.\ \alpha \leq j < \mu\ \&\ x \in y \in \tau(j)\}$. Similarly $\bigcup_{j \leq \mu} \tau(j) =_{dfs} \{x \mid \exists j\ \exists y.\ j \leq \mu\ \&\ x \in y \in \tau(j)\}$.

Define f'x $=_{df}$ ıy. $\langle x, y\rangle \in f$; $\Phi^{\leftarrow}y =_{df}$ ıx. $\Phi(x) = y$. (Read "Φ **inverse** of y.")

Φ"a $=_{dfs} \{\Phi(x) \mid x \in a\}$, for a a set; for a an urelement, Φ"a $=_{dfs}$ a. (Read "the **image** of a under Φ.") More explicitly, and partially following

[Levy 1979], p. 27 for the non-empty case, Φ“a $=_{\mathrm{dfs}}$ a, if empty(a), else $\{y \mid \exists x \in a \,.\, y = \Phi(x)\}$. Note that the obvious simpler definition in terms of class abstracts would have meant that the value of “ for any formula and any urelement would have been the empty set, but it will be important for results about j-isomorphism that it instead be the urelement itself.

$\Phi^{\Leftarrow}(h) =_{\mathrm{df}} \{x \mid \Phi(x) = h\}$.

Define $f|a =_{\mathrm{df}} \{\langle x, y\rangle \in f \mid x \in a\}$. (Read “f **restricted** to a.”)

7. The Axioms

7.1. *The Basic Axioms*

Extensionality: $\forall a \forall b.\ \mathrm{nonempty}(a)\ \&\ \forall z : z \in a \equiv z \in b. \Rightarrow a = b$

Null Set: $\forall x.\ x \notin \emptyset$

Pair: $\forall x \forall y \exists p \forall w.\ w \in p \equiv (w = x \vee w = y)$

Well-Founded Sum Set: $\forall z.\ \mathrm{wf}(z) \Rightarrow \exists u \forall x.\ x \in u \equiv .\ \exists y.\ x \in y\ \&\ y \in z$

Well-Founded Power Set: $\forall x.\ \mathrm{wf}(x) \Rightarrow \exists p \forall z.\ z \in p \equiv z \subseteq x$

Infinity: $\exists w \forall x.\ x \in w \equiv .\ \text{Dedekind-finite}(x)\ \&\ \mathrm{ordinal}(x)$

Well-Founded Replacement: a schema, one instance for each two-place predicate φ:

$\forall a.\ \mathrm{wf}(a)\ \&\ \mathrm{FUNCTION}(\varphi, a) \Rightarrow \exists b.\ \mathrm{maps}_{\mathrm{formula}}(\varphi, a, b)$

7.2. *Global Choice*

A global well-ordering, as in [Church 1974a]:

Axiom Schema of Global Well-Ordering:

$\forall x.\ \neg x\mathscr{G}x\ \&$

$\forall x \forall y.\ x\mathscr{G}y\ \&\ y\mathscr{G}z \rightarrow x\mathscr{G}z\ \&$

$\forall x.\ \varphi(x) \Rightarrow \exists y.\ \Phi(y)\ \&\ \forall z.\ \Phi(z) \rightarrow y\mathscr{G}z \vee y{=}z$

For convenience below, we will use a slight rearrangement of the global well-ordering, in which $\emptyset$ is the first element. I.e., define $x\mathscr{G}'y$ iff $(x{=}\emptyset\ \&\ y \neq \emptyset) \vee (x \neq \emptyset\ \&\ y \neq \emptyset\ \&\ x\mathscr{G}y)$. By abuse of notation, I will use $\mathscr{G}$ for $\mathscr{G}'$.

7.3. *Foundation*

Axiom of Foundation: $\forall x.\ \mathrm{wf}(x)$

7.4. *Base Theory*

RZFU (the base theory) is the Basic Axioms plus the following axiom:

Urelement Bijection Axiom:

$$\forall x.\ \mathrm{set}(x) \to \exists! u.\ u = \Upsilon(x)\ \&$$
$$\forall x \forall u.\ u = \Upsilon(x) \Rightarrow \mathrm{urelement}(u)\ \ \&\ \ \mathrm{set}(x)\ \&$$
$$\forall x \forall y \forall u.\ u = \Upsilon(x)\ \ \&\ \ u = \Upsilon(y) \Rightarrow x = y\ \&$$
$$\forall u.\ \mathrm{urelement}(u) \Rightarrow \exists x.\ u = \Upsilon(x)$$

The last clause is not used in the proof of the Basic Axioms Theorem; it is only needed for the final construction, hence "Injection" rather than "Bijection" in the name of the axiom in the 1993 version of this paper.

This axiom will be used, via a Cantor-Schroeder-Bernstein-Dedekind construction, to show a bijection from the class of urelements to the class of indexes, defined below, which will be used to keep track of the new, ill-founded, sets.

7.5. *CUSι*

CUSι will consist of the Basic Axioms plus the following axioms, where $\Bumpeq^j$ (j-isomorphism) will be defined below (III.15.2).

Restricted Axiom of Generalized Frege Cardinals:

$$\forall j \in \omega\ \forall b.\ \mathrm{wf}(b) \Rightarrow \exists F\, \forall x.\ x \in F \equiv b \Bumpeq^j x$$

Note that, while b is restricted to well-founded sets, x is not. Thus, given a reasonable amount of transitivity (which will be non-trivial), sets j-isomorphic to a well-founded set may also have generalized Frege cardinals.

Unrestricted Axiom of Symmetric Difference:

$$\forall x \forall y \exists z \forall w.\ w \in z \equiv (w \in x \not\equiv w \in y)$$

Note that, since the 0-cardinal of anything is the universal set, symmetric difference also gives us unrestricted complementation. Trivially this gives us unrestricted union of disjoint sets, and hence (the non-trivial case for) adjunction (i.e., the existence of $x \cup \{y\}$.) It does not seem to give us general pairwise union, however, so the following axiom is also necessary. Pairwise union together with complement will give pairwise intersection, of course, by the usual identity: $a \cap b = \sim(\sim a \cup \sim b)$.

Unrestricted Axiom of Pairwise Union:

$$\forall x \forall y \exists z \forall w.\ w \in z \equiv (w \in x \vee w \in y)$$

8. Elementary Lemmata

Uniqueness of Pairs

The set required by the Pair Axiom will be unique by Extensionality, since the required set is nonempty. (Note that the name is slightly inaccurate, since the case x=y implies the existence of singletons as well.) This uniqueness is not necessarily preserved in an arbitrary interpretation, though it will be for any interpretation of interest.

Well-Founded Pairwise Union

Observe that the Well-Founded Sum Set Axiom gives us pairwise union for well-founded sets in the Base Theory:

Lemma 8.1 (Pairwise Union for Well-Founded Sets)**.** $\forall x \forall y\ \mathrm{wf}(x)\ \&\ \mathrm{wf}(y) \rightarrow \exists z \forall w.\ w \in z \equiv (w \in x \vee w \in y)$

The proof is straightforward; it and later proofs are omitted in this abridged version. Thus it is unnecessary to add an axiom for well-founded pairwise union to the base theory. The unrestricted version for CUSι will depend on the details of the sequence of equivalence relations; it is not necessarily true for an arbitrary $\in\dagger$-interpretation, defined below.

9. $\in\dagger$-Interpretations and Proof of the Basic Axioms

I define a type of interpretation, an $\in\dagger$-interpretation, and show that any such interpretation over the base theory automatically satisfies the Basic Axioms except for Extensionality. The proofs (omitted in this abridged version) are straightforward, since the axioms are restricted to well-founded sets, whose rôles do not change in the interpretation. The use of urelements avoids much of the tedium of [Church 1974b] and, to a lesser extent, [Mitchell 1976] and [Forster 2001]. It may also make similar interpretations of different ill-founded set theories more convenient, since it eliminates the initial need to verify the uninteresting old axioms and allows immediate attention to Extensionality and the new axioms.

An $\in\dagger$-interpretation will be a relation (called $\in_1$) defined in the form below, together with two ancillary two-place predicates Φ and Υ', satisfying the additional requirements below. $\in_1$ will differ from the base membership relation only in that urelements (in the old sense) become ill-founded sets in the new sense. Φ and Υ' are partly-specified but otherwise arbitrary in Part I. It may be easier, for now, for the reader to think of Υ' as the injection Υ of sets to urelements required by the Urelement Bijection Axiom, though in Part III a rearrangement will be necessary to avoid too many unused urelements. (In Part II, either Υ or Υ' would suffice, so for simplicity Υ will be used.)

9.1. $\in\dagger$*-Interpretations*

Let $\in_1$, Υ', and Φ be two-place formulæ defined in the language of the base theory. Let "$\in_0$" denote the usual membership relation; the "$_0$" will merely emphasize the distinction from the newly-defined membership relation, and avoids confusion with Church's notation, which adopts the opposite convention. (Similarly, subscripts 0 and 1 will be used to distinguish other formulæ defined in terms of the old and new membership relations. Where these formulæ already include subscripts, a comma will be used to separate the 0 or 1.) Abbreviate **unaltered**(x) $\equiv_{df} \forall y.\ y{\in_0}x \equiv y{\in_1}x$, and define **altered**(x) $\equiv_{df}$ ¬unaltered(x). (In Parts II and III, which treat $\in_2$ and $\in_3$ respectively, these terms will be redefined for convenience to suit the context.)

Definition Schema: $\in$1, Υ', and Φ constitute an **$\in$†-interpretation** iff_{dfs}

$\in_1$ Definition

$x \in_1 y \equiv$

(a) urelement(y) & $\exists L.\ y = \Upsilon'(L)$ & $\Phi(L, x)$

$\vee$

(b) $x \in_0 y$.

Υ' Injection Requirement

$\forall x \forall u.\ u = \Upsilon'(x) \Rightarrow \text{urelement}_0(u)$ & $\text{set}_0(x)$ &

$\forall x \forall y \forall u.\ u = \Upsilon'(x)$ & $u = \Upsilon'(y) \Rightarrow x = y$

Ill-Foundedness Requirements

(1) $\forall x.$ altered(x) $\rightarrow$ $\text{ill-founded}_1(x)$

(2) $\forall x \forall y.\ \text{ill-founded}_1(x)$ & $x \subseteq_1 y \rightarrow \text{ill-founded}_1(y)$

(3) $\forall x \forall y.\ \text{ill-founded}_1(x)$ & $x \in_1 y \rightarrow \text{ill-founded}_1(y)$

Discussion. I will prove, in the Base Theory, for an arbitrary $\in$†-interpretation, the interpretation of each of the Basic Axioms except Extensionality. The current goal of this result is a relative consistency proof for the special case of an $\in$†-interpretation which is my interpretation of CUSι in the Base Theory, but the result might also be useful for other ill-founded set theories.

The domain of the interpretation is the same as that of the ground model. At this level of generality, however, without Foundation in the base theory or Extensionality in the interpretation, the sets of the ground model need not be definable within the interpretation. This will be different for the relation $\in_3$ in Part III.

The altered objects are urelements_0, whose membership is decided by clause (a) of the definition above. Informally, the altered objects will sometimes be called the new sets, where "set" is used in the sense of the new membership relation, since they are urelements in the sense of the old.

Ill-Foundedness Requirements (2) and (3) are not as trivial as they seem, since we don′t have pairwise union in general for the new sets. Unrestricted pairwise union will be true in the interpretation of CUSι, but is not necessarily true in general for $\in$†-interpretations.

Informally, the most obvious ways to prove (2) and (3) fail. If we have an ill-founded$_1$ set w, and an unending-chain$_1$(c), and wish to show that a superset$_1$ s of w, (or a set x containing$_1$ w) is ill-founded$_1$, we could show the existence of $c' = c \cup \{s\}$ (respectively $c \cup \{x\}$.) There are two obvious approaches: First, we could try to show the existence of such a c′ in the new theory, but this presumably would require an unrestricted axiom of pairwise union in the new theory. Second, we could try to show the existence of a suitable unending chain in the base theory; but the given c might not even be a set in the base theory.

As an alternative, we could try to find a low$_1$ subset$_1$ of c which contains$_1$ x and is still an unending-chain$_1$; Replacement in the base theory might then give us the required union of the new chain and $\{w\}$. The Axiom of Dependent Choices is the obvious candidate for constructing such a sub-chain. This was the motivation for the even weaker Low Chain Axiom in some of my previous work: $\forall a \forall c.$ unending-chain(c) & $a \in c \Rightarrow \exists d.$ low(d) & unending-chain(d) & $a \in d$, which is a consequence of either Dependent Choices or Foundation. A still weaker alternative would be the Chain Adjunction Axiom: $\forall a \forall c.$ unending-chain(c) & $a \in c$ & $s \in a \Rightarrow \exists d.$ unending-chain(d) & $s \in d$. This is normally a consequence of the Low Chain Axiom (given low pairwise union in the interpretation), or of unrestricted pairwise union, or even merely unrestricted adjunction: take $c \cup \{s\}$ as d. At the current level of generality, demonstrating the interpretation of these axioms would be inconvenient at this stage of the proof, so I adopt the Ill-Foundedness Requirements instead.

9.2. *Basic Axioms Theorem*

Theorem 9.1 (Basic Axioms Theorem)**.** For an arbitrary $\in\dagger$-interpretation $\in_1$, the interpretations in terms of $\in_1$ of the Basic Axioms except Extensionality are provable from the Base Theory.

The proofs (largely omitted in this abridged version) for each of the Basic Axioms except Extensionality will take the remainder of Part I, but I begin with a simple lemma. (Henceforward I will use heavily the convention noted above, about complex expressions using subscript zero or one to distinguish notions defined in terms of the new membership relation from those defined in terms of the old.)

Lemma 9.2 (Well-Foundedness Lemma)**.** $\mathrm{wf}_1(x) \rightarrow \mathrm{wf}_0(x)$.

9.2.1. Proofs of the Interpretations of the Basic Axioms except Extensionality in an Arbitrary $\in\dagger$-Interpretation

The proofs of the first four axioms are straightforward:

Null Set: $\forall x.\ x \notin_1 \emptyset$

Pair: $\forall x \forall y \exists p \forall w.\ w \in_1 p \equiv (w = x \vee w = y)$
Well-Founded Sum Set: $\forall z.\ wf_1(z) \Rightarrow \exists u \forall x.\ x \in_1 u \equiv .\ \exists y.\ x \in_1 y \ \&\ y \in_1 z$
Well-Founded Power Set: $\forall x.\ wf_1(x) \Rightarrow \exists p \forall z.\ z \in_1 p \equiv z \subseteq_1 x$

Infinity: $\exists w \forall x.\ x \in_1 w \equiv .\ \text{Dedekind-finite}_1(x) \ \&\ \text{ordinal}_1(x)$

The proof will require three results, below. Let ω denote the set required by the uninterpreted axiom; it will suffice to show that this set also has the properties required by the interpretation of the axiom. Since ω is non-empty$_0$ and hence unaltered, it will suffice to show that $\forall x.\ \text{Dedekindfinite}_1(x) \ \&\ \text{ordinal}_1(x) \equiv \text{Dedekind-finite}_0(x) \ \&\ \text{ordinal}_0(x)$.

The interpretation of the axiom will follow from three results, the first of them trivial: the Set Lemma, the Ordinal Absoluteness Theorem, and the Dedekind Infinite Absoluteness Lemma.

Lemma 9.3 (Set Lemma)**.** $\forall z.\ \text{set}_0(z) \rightarrow \text{set}_1(z)$.

Theorem 9.4 (Ordinal Absoluteness Theorem)**.** $\forall \alpha.\ \text{ordinal}_0(\alpha) \equiv \text{ordinal}_1(\alpha)$.

The property of being an ordinal is absolute, i.e., is true of an object in the sense of $\in_1$ iff it is true of that object in the sense of $\in_0$. This will permit omitting subscripts 0 and 1 when saying that something is an ordinal.

Lemma 9.5 (Dedekind Infinite Absoluteness Lemma)**.** For any ordinal α, $\text{Dedekind-infinite}_0(\alpha) \equiv \text{Dedekind-infinite}_1(\alpha)$.

Corollary 9.6. $\forall x.\ \text{Dedekind-finite}_1(x) \ \&\ \text{ordinal}_1(x) \Leftrightarrow \text{Dedekind-finite}_0(x) \ \&\ \text{ordinal}_0(x)$.

Thus ω is also the set of all Dedekind-finite ordinals in the sense of $\in_1$, as required, which completes the proof of the interpretation of the Axiom of Infinity.

The proof of the interpretation of the last axiom is also straightforward:

Well-Founded Replacement: a schema, one instance for each two-place predicate φ: $\forall a.\ wf_1(a) \ \&\ \text{FUNCTION}_1(\varphi, a) \Rightarrow \exists b.\ \text{maps}_{\text{formula},1}(\varphi, a, b)$.
This establishes the Basic Axioms Theorem.

Part II

Extensionality and Arbitrary Restricted Equivalence Relations

In Part II, I introduce a somewhat different partially-specified membership relation, $\in_2$, and show that it satisfies Extensionality. This membership relation is defined in terms of an arbitrary series of relations satisfying the $\backsim^j$ Requirements, below.

The use of Choice is avoided in the base theory, as is Foundation, except, near the end of this Part, for explicitly-mentioned uses of the **$\mathbf{Lowness_0}$ Assumption:** $\forall$s. low(s). I append an "s" to the names of theorems which assume this. In Part III I will show that a specific series of restricted equivalence relations satisfies the requirements in this part, which will establish that my interpretation satisfies Extensionality. The proof of Extensionality in this part is quite involved and special-purpose; it is omitted in the abridged version, except for a few key definitions and results; but the full proof will be made available on the web.

10. Weak Arithmetic

To bypass a long uninteresting proof, I avoid induction on the natural numbers. This also keeps open the possibility of application to Quine′s *New Foundations*, in which full induction fails even for the natural numbers even with the addition of the Axiom of Counting [Forster 1992], p. 30. Natural proofs that the ordinals are linearly ordered seem to require some form of induction [Forster 1992], p. 44. The behavior of ordinals in Oberschelp is even more obscure. In lieu of induction on ω, I will need only a few simple arithmetic facts. (With the stronger assumptions in Part III, arithmetic will become much easier.)

First note that, even with my unusual definitions, an ordinal is well-founded. By definition, of course, the ordinals less than some ordinal are linearly ordered.

10.0.2. *Ordinal Addition*

All we require ordinal addition for is the elementary properties below, primarily of oddness and evenness. I do not even need to show that every natural number is either odd or even but not both; I simply will use only such natural numbers. If we assumed definition by recursion on ordinals, the ordinary definition of addition would suffice. Without definition by recursion, we could still define "+0," "+1," and "+2" everywhere, and define addition on the finite ordinals via (Cantor) cardinal addition; see [Levy 1979], §III.3. To spare the reader′s patience, and for greater applicability of my construction, I will instead omit the development of the definition, and present only the elementary properties of ordinal addition which I actually need.

Define $\mathbf{0} =_{df} \varnothing$. $\mathbf{1} =_{df} \{\varnothing\}$. $\mathbf{2} =_{df} \{\varnothing, \{\varnothing\}\}$.

Define **odd**(a) iff$_{df}$ $\exists$n, k $\in$ ω. n = k + k + 1 & n $\approx$ a; **even**(a)iff$_{df}$ $\exists$n, k $\in$ ω. n = k + k & n $\approx$ a. **Odd-or-even**(a) iff$_{df}$ odd(a) $\not\equiv$ even(a). The **parity** of x is odd (even) iff$_{df}$ x is odd (even). (N.b., these predicates may apply to sets, not just to natural numbers.)

10.1. *Required Properties of* +

The Required Properties of +, a two-place function on the ordinals, are:

(i) $\alpha+\mathbf{0}=\alpha$; $\alpha+\mathbf{1}=\alpha \cup \{\alpha\}$; $\alpha+\mathbf{2} = \mathbf{(\alpha + 1) + 1.}$

(ii) $\forall x.\ \neg\mathrm{odd}(x) \vee \neg\mathrm{even}(x)$.

(iii) Parity Property: If odd-or-even(a) and odd-or-even(b) then $\{\mathrm{odd}(a\ \delta\ b) \Longleftrightarrow [\mathrm{odd}(a) \not\equiv \mathrm{odd}(b)]\}$ and $\{\mathrm{even}(a\ \delta\ b) \Longleftrightarrow [\mathrm{odd}(a) \equiv \mathrm{odd}(b)]\}$.

(iv) $\forall \alpha, \beta.\ \mathrm{ordinal}(\alpha)\ \&\ \mathrm{ordinal}(\beta)\ \&\ \alpha < \beta \Rightarrow \alpha + 1 \leq \beta$.

11. $\sim$j Requirements

Let μ be an ordinal.[22] Let **j-rep**(ξ) be a two-place function, $\xi \backsim^j \zeta$ a three-place predicate, and **rank** a one-place function satisfying the following conditions:

(α) $\forall j, k \leq \mu\ \forall x, y.\, j \leq k\ \&\ x \sim^k y \Rightarrow x \sim^j y$,

(β) $\forall x, y.\, x \sim^0 y$,

(γ) $\forall x, y.\, x \sim^\mu y \equiv x = y$,

(δ) $\forall j \leq \mu\ \forall b\ \exists r.\ r = \text{j-rep}(b)$,

(ε) For $0 \leq j \leq \mu$, $x \sim^j y$ iff $\text{j-rep}(x) = \text{j-rep}(y)$,

(ζ) $\forall h.\ \mathrm{rank}(h) \leq \mu$, and $\forall g.\ \mathrm{rank}(g) = j \Rightarrow \exists x.\ g = \text{j-rep}(x)$,

(η) $\mathrm{rank}(\text{0-rep}(\emptyset)) = 0$, and $\neg\exists s: \mathrm{low}(s)\ \&\ \forall d.\ d \in s \leftrightarrow \exists x.\ \text{1-rep}(x) = d$.

In prose, say "g is a **j-rep**" iff$_{\mathrm{df}}$ $\exists x.\ g = \text{j-rep}(x)$. A j-rep g is **rankable** iff$_{\mathrm{df}}$ it is in the domain of rank.

The main requirement on the given sequence of equivalence relations is (α), increasing strictness; $\sim^0$ and $\sim^\mu$ can be appended to any sequence satisfying it. Requirements (δ), (ε), and (ζ) are for the existence of representative functions, and can be satisfied for arbitrary equivalence relations in the presence of either Global Choice or Foundation.

Define **daughter**(h, g) iff$_{\mathrm{df}}$ $\exists j < \mu\ \exists x.\ j = \mathrm{rank}(g)\ \&\ \text{j-rep}(x) = g\ \&\ \text{j+1-rep}(x) = h$. (Read "h is a daughter of g.") Informally, a daughter of g is a member of j + 1-rep"j-rep$^{\Leftarrow}$g, where j = rank(g).

Define **j-prolific**(g) iff$_{\mathrm{df}}$ $\mathrm{rank}(g) = j\ \&\ \neg\exists s: \mathrm{low}(s)\ \&\ \forall d.\ d \in s \leftrightarrow \mathrm{daughter}(d,g)$. Informally, something is j-prolific iff its rank is j and it has many daughters. The unique 0-rep is 0-prolific, since by (η) there are many 1-reps, all of which are daughters of the unique 0-rep.

[22] If μ has a predecessor, it corresponds to the arbitrary natural number m in [Church 1974a].

12. μ+1-tuples and Sprigs

A set of ordered pairs L is a **μ+1-tuple** iff$_{df}$ $\exists$r. maps(L, μ + 1, r). Abbreviate L'j to L^j, and call it L's **j-component**. L will usually be denoted "(L^0 L^1 … L^n … L^μ)"; I follow Church in using "()" rather than "$\langle\rangle$" for this sort of tuple, and omit commas. A μ+1-tuple may have components which are urelements, but attention below will be restricted to μ+1-tuples whose components are all sets.

Informally, the intent is for new sets to be represented by urelements, tagged with a sequence of length μ+1, conventionally represented (L^0 … L^j … L^μ), L for short. The idea is that x is a member of a new set (old urelement, tagged by L) if there are an *odd* number of j′s such that j-rep(x) is in L^j.

Thus the universal set will be the urelement with tag ({0-rep(Ø)} Ø … Ø), since everything has the same 0-rep, and 1 is odd. The set of all pairs will, in Part III, be tagged by (Ø {1-rep(2)} Ø … Ø), and the singleton function by (Ø Ø {2-rep(<Ø,{Ø}>)} Ø … Ø). The complement of ω will be tagged by ({0-rep(Ø))} Ø … μ-rep"ω). Machinery will be developed below, first to formalize the notion of an odd number of j's, and then to restrict the new sets to those needed for the interpretation.

More formally, the *sprig* of a μ + 1-tuple (L^0 L^1 … L^n … L^μ) *for* an object x will be a partially-defined sequence from 0 to μ, with its value for j, j-rep(x) if j-rep(x) $\in L^j$, and otherwise undefined. Define

sprig((L^0 L^1 … L^n … L^μ)) $=_{df}$ $\{\langle$j, j-rep(x)$\rangle \mid j \leq \mu$ & j-rep(x) $\in L^j\}$.

13. Indices and Urelements

Define **INDEX**(L) $\equiv_{df}$

a. μ + 1-tuple(L) & $\forall j \leq \mu$. set(L^j),
b. low($\bigcup_{j\leq\mu} L^j$),
c. $\exists j < \mu\ \exists x.\ x \in L^j$,
d. $\forall j < \mu\ \forall a \in L^j$. rank(a) = j & j-prolific(a),
e. $\forall a \in L^\mu\ \exists x.\ a = \mu$-rep(x),
f. $\forall$x. odd-or-even(sprig(L, x)).

Note the prohibition in clause (a) of urelements as components L^j. The formalism is neutral on whether urelements are members of these components, but their primary rôle will be through the membership of their μ-reps in L^μ.

Routine verificationshows that *INDEX* has three additional properties:

Proposition 13.1 (Degeneracy/Diversity Properties)**.**

g. INDEX(L) & INDEX(M) & diverse(L, M) $\Rightarrow$ INDEX(L ∂ M),

h. $\mathrm{INDEX}(L)$ & $\mathrm{low}(a) \Rightarrow \mathrm{INDEX}((L^0\ L^1 \ldots L^n \ldots [L^\mu\ \delta\ \mu\text{-rep``}a]))$,
i. $\mathrm{INDEX}((\{0\text{-rep}(\emptyset)\}\ \emptyset \ldots \emptyset))$.

For greater generality, we could take the above, not as a definition of INDEX, but as a minimum requirement, provided we add the Degeneracy/Diversity Properties as additional conditions on INDEX(L); in this Part the only use I make of the "$\Leftarrow$" part of the definition of INDEX is in the proof of these three properties. This will allow us to replace the condition "j-prolific(a)" in clause (d) by a stronger predicate, when applying the results of this section to the specific consistency proof for CUSι.

13.1. *Urelements and* *

By the Urelement Bijection Axiom, we have a function $\Upsilon(x)$ injecting the sets into the urelements. (We could also use the partially-specified function Υ' from Part I, but for this Part that level of generality is not necessary.) I will define the function "*" as a restriction of Υ, and will abbreviate *(x) to $*x$. Define $*x =_{df} \Upsilon(x)$, provided INDEX(x); undefined otherwise. By this definition and the Urelement Bijection Axiom, we have the following:

Lemma 13.2 (* Properties Lemma)**.**

a. $\forall x.\ \mathrm{INDEX}(x) \rightarrow \exists! u.\ u = *x$,
b. $\forall x, u.\ u = *x \Rightarrow \mathrm{urelement}(u)\ \&\ \mathrm{INDEX}(x)$,
c. $\forall x, y, u.\ u = *x\ \&\ u = *y \Rightarrow x = y$.

In prose, read "INDEX(L)" as "L is **an index.**" Let the function *index* be the inverse of the function *. I.e., define **index**(u) $=_{df}$ $\iota x.\ u = *x$. Thus $\forall x.\ \mathrm{INDEX}(x) \rightarrow \mathrm{index}(*x) = x$. By (a), $\exists! u.\ u = *x$, which establishes the existence part of the definition of index($*x$); (c) establishes uniqueness. In prose call x **the index of** $*x$, and call $*x$, x′s **urelement.**

14. $\in_2$

Define $x \in_2 y$ iff$_{df}$

(a) $\exists L.\ y = {}^*L\ \&\ \mathrm{INDEX}(L)\ \&\ \mathrm{odd}(\mathrm{sprig}(L, x)) \vee$
(b) $x \in_0 y$.

Note that since $y = {}^*L \Rightarrow \mathrm{urelement}_0(y)$, the two clauses are mutually exclusive, and $\mathrm{set}_0(y) \Rightarrow x \in_2 y \equiv x \in_0 y$. Redefine (analogously to the definition in Part I) **unaltered**(x) iff$_{df}$ $\forall z.\ z \in_0 x \equiv z \in_2 x$; **altered**(x) iff$_{df}$ $\neg$unaltered(x). Thus

Lemma 14.1. $\mathrm{set}_0(y) \rightarrow \mathrm{unaltered}(y)$, and $\mathrm{altered}(y) \rightarrow \exists L.\ y = {}^*L$.

The two cases in the definition of $\in_2$ correspond to the six cases of Church's definition [1974a], page 306. Considerable simplification is achieved by the use of urelements (in place of Church's *i-analogue* function) and the definition of sprig, though at the cost of the Urelement Bijection Axiom and the non-primitive notations "*", "INDEX", "sprig", and "odd" in the definition.

Note that Church's use of Compactness ([1974a], p. 307) is here unnecessary, since this construction uses the full sequence of partially-defined restricted equivalence relations, rather than Church's initial segment of $\backsim^j$'s, for $j \leq m$, with unspecified length m.

Observe that the definition of $\in_2$ immediately gives us many $\backsim^j$-equivalence classes as sets$_2$ with $\backsim_0^j$ in, regrettably, the sense of the old membership relation:

Observation 14.2 (Equivalence Class Observation)**.** Let a be an object, with $j < \mu$. If L is an index with $L^j = \{\text{j-rep}(a)\}$, and $L^k = \emptyset$ for $j \neq k$, then $\forall x.\ x \in_2 {}^*L \equiv x \backsim_0^j a$.

Note that what we actually want is this result with "$\backsim^j$" replaced by its interpretation. Say that $\backsim^j$ is absolute iff$_{\text{dfs}}$ $\forall x, y.\ x \backsim_0^j y \equiv x \backsim_2^j y$. Consideration of this requirement leads naturally to Oberschelp's comprehension schema; see [Sheridan 1990]. Trivially, though, the Equivalence Class Observation gives us:

Corollary 14.3. For any $j < \mu$ and any a, if $\backsim^j a$ is absolute and $(\emptyset \ldots \{\text{j-rep}(a)\} \ldots \emptyset)$ is an index, then $\forall x.\ x \in_2 {}^*L \equiv x \backsim_2^j a$.
I.e., $^*L$ is a's Frege j-cardinal in the sense of the new membership relation.

Lemma 14.4 (Universal Set Lemma)**.** $\forall y.\ y \in_2 {}^*(\{\text{0-rep}(\emptyset)\}\ \emptyset \ldots \emptyset)$.

Theorem 14.5 (Symmetric Difference$_2$ Theorem (s))**.** $\forall a \forall b \exists z \forall w.\ w \in_2 z \Longleftrightarrow (w \in_2 a \not\equiv w \in_2 b)$.

Lemma 14.6 (Nonemptiness$_2$ Lemma)**.** $\text{INDEX}(L) \Rightarrow \exists x.\ x \in_2 {}^*L$.

Theorem 14.7 (Interpretation of the Axiom of Extensionality for Sets (s))**.** $\forall a \forall b.\ \text{nonempty}_2(a)\ \&\ \forall z.\ z \in_2 a \equiv z \in_2 b.\ \Rightarrow a = b$.

Part III

j-Isomorphism, Foundation, Choice, the Interpretation, and Proof of the Axioms of CUSι

15. j-Isomorphism

In Part III, I define a specific sequence of restricted equivalence relations, $\approx^j$ (read "j-isomorphic"), and prove its two key properties: that the singleton function is the union of a small finite number (six in general, one in the current context) of 2-isomorphism classes (17.14), and that any nondegenerate j-isomorphism class is non-low (16.16).

After defining j-isomorphism, rather than proving the properties of a partially-specified membership relation (such as $\in_1$ or $\in_2$ in Parts I and II), I will instead define a specific relation $\in_3$; and I will assume for the base theory, in addition to the Basic Axioms, the Axioms of Foundation and Global Well-Ordering. Some of the uses of these axioms might be eliminable with sufficient care to relativization and the use of Scott's Trick [1955], but substantial use of Foundation seems necessary for the Replacing at Level*j construction, section 16.4 below.

15.1. *Ordinals and Avoidance of Advanced Recursion*

For the following subsection I will continue to avoid development of recursion on the finite ordinals beyond that used above. This may facilitate use of these techniques in other contexts, though whether this justifies the additional effort is by no means clear.

For $1 \leq j < \omega$, define $y \in^j a \equiv_{df} \exists f.\ \exists c.\ \text{maps}(f, j{+}1, c)\ \&\ f`0 = a\ \&\ f`j = y\ \&\ \forall k{\in}j.\ f`k{+}1 \in f`k$. Read "y is a member at **level j** of a"

For convenience, define $y \in^0 a \equiv_{df} y = a$; this differs from Church's usage, but is convenient for usage with the j^{th} cumulative union, defined below. Repeated application of the Axiom of Pairs trivially shows that $y \in^1 a \equiv y \in a$.

Define $y \in^{<j} a \equiv_{df} \exists k.\ 0 \leq k < j.\ y \in^k a$. (Note that this means $y \in^{<j} y$, for $j \geq 1$.)

Define $y \in^{\leq j} a \equiv_{df} \exists k.\ 0 \leq k \leq j.\ y \in^k a$.

Define $y \in^{*j} a \equiv_{df} y \in^j a\ \&\ \neg\exists\ i < j.\ y \in^j a$. Read "y is a member at level*j *proper* of a"; **level*j** of a is the class of all members at level*j of a. Thus level*0 of a is $\{a\}$.

Define $\Xi^j a =_{df} \{y \mid y \in^{\leq j} a\}$, for $0 \leq j < \omega$. Read "the j^{th} *cumulative union* of a." This is a class abstract; it will have to be proved to be a set before making use of it. (Recall that a class abstract is never an urelement, so it is only necessary to eliminate the possibility that it is an ultimate class.) Note that $\Xi^0 a = \{a\}$, and that $\Xi^j a$ contains a for any j.

Define **TC**(a) $=_{df} \{y \mid \exists\ j \in \omega.\ y \in^j a\}$. Note that, because of my definition of $\in^0$, this differs slightly from the standard **transitive closure** of a, in that TC(a) also includes a. Informally, call a member of the transitive closure of x, a **constituent** of x.

15.2. *Definition of j-Isomorphism*

Define, for $j \leq 1 < \omega$, $a \rightleftharpoons^j b \equiv_{df} \exists F:$

(1) SET($\Xi^j a$) & SET($\Xi^j b$) &

(2) $\text{maps}_{1\text{-}1}(F, \Xi^j a, \Xi^j b)$ &
(3) F‘a = b &
(4) $\forall y \in^{<j} a$. F‘y = F“y

Read “a is **j-isomorphic** to b.” The first clause will be superfluous (by the Cumulative Union Lemma (16.1), below) in the presence of Foundation; it is only needed for ill-founded objects in the interpretation. When F is known, I will also write “F: a $\dualmap^j$ b.” For convenience, define $\dualmap^0$ as the universal relation which holds between any two objects, and $\dualmap^\omega$ as equality.

15.2.1. *j-Isomorphism Notes*

(1) Note that since $\in^0$ is equality, clauses (3) and (4) imply that b = F“a, for $j \geq 1$.
(2) By my definition of “, which maps empty objects to themselves, an empty object can be j-isomorphic only to itself, for $j \geq 1$.
(3) Conversely, for any empty a, the mapping {<a, a>} is a j-isomorphism for any $j \leq \omega$. So any empty object is j-isomorphic to itself, and, for $j \geq 1$, only to itself.
(4) Since b = F“a, if F|a (i.e., F restricted to a) is a set (which it will be in the presence of Foundation), then a is equinumerous to b.
(5) The intent is that the $\dualmap^j$ are of increasing strictness, but proving this will require Foundation, which I assume below.
(6) Note that, despite my informal terminology, I have not yet proved that these are equivalence relations, nor even that they are reflexive. They won’t necessarily be either for ill-founded sets, since the obvious proofs require Replacement.
(7) The state of j-isomorphism in the interpretation will be inelegant, especially the existence of set mappings witnessing j-isomorphism, rendering them merely restricted equivalence relations. (A similar difficulty arises with Church’s theory, though he did not need to address it in his surviving writings.) It will be simple to show that j-isomorphism is an equivalence relation in the presence of Foundation, and j-isomorphism will be absolute for sets which are unaltered down to level j in the interpretation (see the j-Pure j-Isomorphism Absoluteness Theorem (17.10), below). This is somewhat short of showing that j-isomorphism will be a restricted equivalence relation in the interpretation, since some new set might be a mapping which witnesses a j-isomorphism for a new set, with no obvious guarantee of the existence of other mappings required for an equivalence relation.
(8) There will be further shortcomings of these equivalence relations in the interpretation. They will only provably be absolute for well-founded

sets and those j-isomorphic to them, the j-pure sets (defined formally below); contemplating the requirements for absoluteness of such relations leads naturally to Oberschelp's existence criterion [Oberschelp 1973], [Sheridan 1990], which may loosely be described as mandating the set-hood of any predicate whose definition is absolute. The situation is inelegant even for $j = 1$. The intent is for two sets to be 1-isomorphic if they are equinumerous, and either both or neither are self-membered. No two urelements are 1-isomorphic in the base theory, but in the interpretation an old urelement might contain itself and be externally equinumerous to the universe (e.g., the urelement which represents the universal set itself), or not contain itself and be externally equinumerous to the universe (e.g., the set of all non-self-membered singletons). See also the discussion of the Bad Company problem in the philosophical introduction for further difficulties with j-isomorphism.

The crucial difference between my j-isomorphism and Church's j-equivalence (abbr: $\approx_j$) is that, in my definition, while the first two clauses deal with membership at level $\leq$j, the last deals with membership at level $<$ j, in order to enable the set-hood of the singleton function. A lesser differences is that I have a single mapping required to be one-one across all levels, while his sequence of mappings are only required to be individually one-one.

j-isomorphism classes do not seem to be closed under sum set, which is why my theory (unlike Church's) does not have an unrestricted axiom of sum set. The 2-isomorphism class of $\{\{\emptyset, \{\emptyset\}\}, \{\emptyset\}\}$ will be a set in my theory, but its sum set does not seem to be. This union should be the set of all singletons plus the set of all pairs of the form $\{a, \{a\}\}$, but the latter does not seem to be a j-isomorphism class, nor a manageable combination thereof.

Church's equivalence relations have the property that if a class is roughly (i.e., modulo a well-founded set) closed under j-equivalence, its sum set is roughly closed under j-1-equivalence [Sheridan 1989], p. 75, 84; this would have been crucial in Church's consistency proof.

16. Foundation, Choice, j-isomorphism, and Less Generality

16.1. *Foundation and Global Well Ordering*

For the remainder of this work, I will drastically reduce the generality in which I have been working. I will work in a base theory which includes, in addition to RZFU, the Axiom of Foundation (sometimes merely three of its consequences—see below) and the Axiom Schema of Global Well-Ordering. This renders the Basic Axioms, some of them restricted to well-founded sets, equivalent to their standard counterparts in, for example, [Levy 1979].

(Since we are now assuming Choice, the usual proof will also go through that ω—here defined as the set of all Dedekind finite ordinals—is itself an ordinal.) This will allow the use of the standard results of ZFC, e.g., definition by recursion and Separation, and hence requires far less formality. It will also render unproblematic use of the (Cantor) cardinality of any set, with the standard definition as the least ordinal equinumerous to the given set.

As a further specialization, for the arbitrary ordinal μ, I will substitute ω; for the arbitrary sequence of relations $\backsim^j$ $(j \leq \mu)$ I substitute $\Bumpeq^j$ $(j \leq \omega)$, with $\Bumpeq^\omega$ being equality and $\Bumpeq^0$ being the universal relation. I will also substitute for the partially-specified relations $\in_1$ and $\in_2$, and predicate INDEX, a specific relation $\in_3$ and predicate INDEX3, defined below. For the partially-specified function "+" on the odd-or-even ordinals, I substitute the usual addition function on the finite ordinals. For convenience, I will reuse subsidiary terminology (e.g., j-rep, rank, *, and sprig) without explicitly distinguishing it, though the notations should henceforth be understood as defined in terms of $\Bumpeq^j$ rather than $\backsim^j$.

Some of the uses below of Foundation in the base theory are essential; the most extreme case is the Replacing at Level*j Construction, which is defined by recursion on the Cumulative Hierarchy. Some of the uses, however, are needed merely for three unrestricted consequences of the normal ZF axioms: Separation, pairwise union, and unrestricted sum set. Where appropriate, I will mark results which require only these consequences of Foundation.

The uses of unrestricted Separation are largely of one type, that a subclass of a set function (or of its domain or range) is also a set; I will call this the Function Subset Assumption: In set theories like Church's, this seems little, if any, weaker than full Separation, which needs to be restricted to well-founded sets. (Consider the identity function, which could plausibly be a set, and its subclass, the identity function restricted to non-self-membered sets, which is likely to lead to a paradox.) But the assumption recurs frequently enough in what follows that it seems worth calling attention to, for possible use of this construction in other theories; e.g., [Aczel 1988], which has self-membered sets but unrestricted Replacement.

The following three results are straightforward.

Lemma 16.1 (Cumulative Union Lemma)**.** For $0 \leq j < \omega$, $\forall a.\ \mathrm{SET}(\Xi^j a)$.

Lemma 16.2 (Transitive Closure Lemma)**.** $\forall a.\ \mathrm{SET}(\mathrm{TC}(a))$.

Lemma 16.3 (j-Isomorphism/Level j Lemma)**.** If $F\colon a \Bumpeq^j b$ and $i \leq j$, then $\forall x.\ x \in^i a \equiv F`x \in^i b$.

The following result characterizes the first non-trivial j-isomorphism relation.

Lemma 16.4 (1-Isomorphism Lemma (Function Subset Assumption, Pairwise Union))**.** $\forall$a,b. non-empty(a) $\Rightarrow$ a $\Bumpeq^1$ b $\equiv$. a $\approx$ b & (a $\in$ a $\equiv$ b $\in$ b).

(Recall that two empty objects are $\Bumpeq^j$ iff they are equal, for j $\geq$ 1.) Note that the final conjunct is significant only in the absence of full Foundation; the main interest of this result is for CUSι and possible extensions, not the Base Theory, but such applications are beyond the scope of this paper.

16.2. *1-Isomorphism and Paradox*

This result, though it aids the consistency proof in this paper, would have disturbing consequences for the goal of extending my theory, CUSι, though apparently not Church's original theory: Natural extensions of CUSι (with unrestricted axioms of generalized Frege-Russell cardinals and some natural mappings as sets) lead to a variant of the Russell Paradox. (I would have hoped that some expansion of my theory could be useful for working mathematicians, for example, in category theory [Feferman 2006], but this seems to rule that out.)

Call a set *blasphemous* iff the universe is equinumerous to it via a set mapping; the formal definition is below. (This is a pun on the name Church and his conjecture about high sets ([1974a], p. 299), plus Cantor's notion of absolute infinity as presented in [Hallett 1984].) A sometimes helpful informal notion is being *weakly blasphemous,* via a class mapping rather than a set mapping. More formally, this is a definition schema: b is **weakly blasphemous via** φ iff$_{\mathrm{dfs}}$ FUNCTION(φ) & $\forall$x. $\exists$!y $\in$ b. φ(x,y). Often I will elide the formula in informal exposition; if I were to do so formally, there would be a risk of hidden quantification over virtual classes.

Informal Motivation: An easier, but not quite sufficient, version of this paradox is the 1-isomorphism class of the universe. Since the universe is a member of itself, this will be the set of all self-membered blasphemous sets. Does this set contain itself? If it does, then it does; if it doesn't, it doesn't. This isn't a paradox, but suggests a problem with such equivalence-class axioms, that in this case they say too little.

This does evoke a familiar route to a genuine paradox: Take a blasphemous set that isn't self-membered; the set of all singletons is a convenient one. The idea, which I work out in detail below, is that its 1-isomorphism class contains itself iff it doesn't.

Informally assume we are working in some partially-specified stronger theory than CUSι (call it CUSι#), which I will show inconsistent, with the *unrestricted* existence of 1-isomorphism classes, plus three additional properties, formally stated following the definitions.

(II), below, will mean simply that the 1-Isomorphism Lemma is still true in CUSι#, even for the new 1-isomorphism classes of ill-founded sets.

(III) is that being equinumerous to the set of all singletons (abbreviated $\mathbb{1}$) is equivalent to being equinumerous to the universe. To motivate this, note that the singleton function maps the universe one-one onto the set of all singletons. Thus any set equinumerous to $\mathbb{1}$ is at least weakly blasphemous, via the obvious composition map. Actually proving (III) in general would seem to require a fair amount of compositionality, which Church's technique does not seem to provide.

(IV) is that there is a set mapping from the universe one-one onto the 1-isomorphism class (abbreviated $\mathscr{J}$) of the set of all singletons. (Defined formally below; $\mathscr{J}$ will be a set by the unrestricted Axiom of Generalized Frege Cardinals.)

To motivate this, I will exhibit a class mapping which is one-one and might reasonably be hoped to map the universe into $\mathscr{J}$. (For readers worried about implicit quantification over ultimate classes, I stress that this motivational section is purely motivational: I am arguing that the desired properties of the hypothetical theory CUSι#, which turn out to lead to paradox, would have been reasonable to desire in the absence of paradox.)

Let $\mathbb{2}$ be the set of all pairs, which exists by a similar argument to that for $\mathbb{1}$. By the Axiom of Pairwise Union, $\mathbb{2} \cup \{\{x\}\}$, for arbitrary x, exists. It's non-self-membered, since it has more than two members, unlike any of its members. It's at least weakly blasphemous: Consider the mapping $z \to \langle z, z \rangle$, which maps the universe one-one into $\mathbb{2}$, hence also into $\mathbb{2} \cup \{\{x\}\}$. This does not suffice to show that $\mathbb{2} \cup \{\{x\}\}$ is blasphemous, but does (I hope) make that seem a reasonable desideratum for CUSι#. If $\mathbb{2} \cup \{\{x\}\}$ is blasphemous and non-self-membered, it's a member of $\mathscr{J}$, the 1-isomorphism class of the set of all singletons.

Consider the class mapping $x \to \mathbb{2} \cup \{\{x\}\}$. It is obviously one-one. By the preceding, if $\mathbb{2} \cup \{\{x\}\}$ is blasphemous for each x, this mapping would inject the universe (abbr: $\mathbb{U}$) into $\mathscr{J}$. So $\mathscr{J}$ would also be weakly blasphemous, so it seems a reasonable desideratum that $\mathbb{U} \approx \mathscr{J}$.

More formally, define $\mathbb{U} =_{\mathrm{df}}$ ℩u $.\forall x.\ x \in u$. This will exist by the Unrestricted Axiom of Symmetric Difference.

Define **blasphemous**(b) iff$_{\mathrm{df}}$ $\mathbb{U} \approx b$, i.e. $\exists f.\ \text{maps}_{1\text{-}1}(f, \mathbb{U}, b)$.

Let $\mathbb{1}$ be the set of all singletons; this exists by the Unrestricted Axiom of Generalized Frege 1-Cardinals, as the 1-isomorphism class of $\{\emptyset\}$, unioned with the 1-isomorphism class of any self-membered singleton, if such exists. It has more than one member, hence does not contain itself.

$\mathscr{J}$ will be the class of things to which $\mathbb{1}$ is 1-isomorphic: $\mathscr{J} =_{\mathrm{df}} \{x \mid \mathbb{1} \Bumpeq^{1} x\}$. This will be a set by the Unrestricted Axiom of Generalized Frege 1-Cardinals, (I) below.

Assumptions on CUSι#:

(I) Unrestricted Axiom of Generalized Frege 1-Cardinals: $\forall b.\ \exists F\ \forall x.\ x \in F \equiv b \Bumpeq^{1} x$

(II) The 1-Isomorphism Lemma still holds in CUSι#:
$\forall$a,b. non-empty(a) $\Rightarrow$ a $\Bumpeq^1$ b $\equiv$. a $\approx$ b & (a $\in$ a $\equiv$ b $\in$ b)

(III) $\forall$x. $\mathbb{1} \approx$ x iff $\mathbb{U} \approx$ x

(IV) $\mathbb{U} \approx \mathscr{I}$

Thus, by (II) and the definition of $\mathscr{I}$, since $\mathbb{1} \notin \mathbb{1}$, we have $\forall$x. x $\in \mathscr{I} \equiv \mathbb{1} \approx$ x & x $\notin$ x.

By (III), $\forall$x. x $\in \mathscr{I} \equiv \mathbb{U} \approx$ x & x $\notin$ x.

Substituting $\mathscr{I}$ in the preceding, we have $\mathscr{I} \in \mathscr{I} \equiv \mathbb{U} \approx \mathscr{I}$ & $\mathscr{I} \notin \mathscr{I}$. But $\mathbb{U} \approx \mathscr{I}$ by (IV), so $\mathscr{I} \in \mathscr{I} \equiv \mathscr{I} \notin \mathscr{I}$, contradiction.

This could be interpreted as an example of the Bad Company Argument against equivalence sets ([Dummett 1991] pp. 188-9, [Boolos 1990], p. 214) or the Embarrassment of Riches Argument [Weir 2003], p. 28, or perhaps a confirmation of Forster's "Naturam expellas furca" argument [Forster 2006], p. 240. Cp. also Holmes' proof of the non-set-hood of the membership relation [Holmes 1998] p. 43, and Remark 7.7 on cardinalities and paradox in [Forster & Libert 2011].

I do not believe this is a counterexample to Heck's observation that "there are no set-theoretic paradoxes specifically concerning cardinal numbers" ([Heck 2013], p. 224), nor even evidence against Frege-Russell cardinals for ill-founded sets, but merely a hazard of a relation which can code enough information about membership to emulate the Russell Paradox.

This may also mean that extensions of Oberschelp's theory (which like CUSι, has the Singleton Function as a set, and which I believe also proves the existence of j-isomorphism classes for wellfounded sets) cannot prove the set-hood of unrestricted generalized Frege cardinals and/or some of the preceding natural mappings, on pain of inconsistency.

16.3. *Well-Founded Equivalence Relations*

Theorem 16.5 (Well-Founded Equivalence Relation Theorem). $\forall$j $\in \omega$, $\Bumpeq^{j}$ is an equivalence relation on the well-founded sets.

The result is actually slightly stronger; only one of the sets need be assumed well-founded. Note that we are now assuming Foundation, so both the assumption and the title of the theorem are redundant; but for possible use over other base theories, and to emphasize the nature of the result, I will limit my direct use of Foundation. (Explicitly calling out the indirect assumptions necessary for this theorem would be non-trivial, however, because of the use of recursion and the Cumulative Union Lemma.)

Theorem 16.6 (Singleton Function/2-Isomorphism Theorem (Foundation for Finite Sets)). $\forall$b. $\langle \emptyset, \{\emptyset\} \rangle \Bumpeq^2$ b $\equiv \exists$d. b = $\langle$d, {d}$\rangle$.

I.e., the singleton function is a 2-isomorphism equivalence class. The use of Foundation is only for the second part of the proof, is only needed for sets with three or fewer members, and could be avoided by explicitly cataloging the six possible failures of Foundation, as in [Sheridan 1989]. Note that $<\emptyset, \{\emptyset\}>$ expands, by the definition of Kuratowski ordered pair, to $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.

This will help to show below that, in CUSι, the singleton function is a set; but there will be some non-obvious additional effort required, e.g., to show that an object which is 2-isomorphic to $<\emptyset, \{\emptyset\}>$ in the base theory remains so in the interpretation, and to verify that there are no new objects which are of the form $<x, \{x\}>$ but are not 2-isomorphic to $<\emptyset, \{\emptyset\}>$. If, for example, there were a Q such that $Q = \{Q\}$, then $<Q, \{Q\}> = \{\{Q\}, \{Q, \{Q\}\}\} = \{Q, \{Q\}\} = \{Q\} = Q$. This has only a single member, hence would not be 2-isomorphic to $<\emptyset, \{\emptyset\}>$. In the interpretation, all the new sets will be non-low, so this is not an issue. In a more general context, there are only a finite number of ways that this can go wrong, so it would not be hard to construct the singleton function as the union of a finite number of 2-isomorphism classes.

Lemma 16.7 (Level <j Equinumerosity Lemma (Function Subset Assumption)). If $F: x \Bumpeq^j y$ and $z \in^{<j} x$, then $F`z \approx z$.

Lemma 16.8 (Increasing Strictness Lemma (Function Subset Assumption)). $\forall j, k \leq \omega\ \forall x, y.\ j \leq k\ \&\ x \Bumpeq^k y \Rightarrow x \Bumpeq^j y$. (Cp. $\backsim^j$ Requirements (α), below.)

16.4. *Replacing at Level*j Construction*

Given a well-founded set, a, not empty at level*j (for $j \in \omega$), an arbitrary object z, and an arbitrary infinite Cantor cardinal χ larger than the transitive closure of a, I will construct below a set b(z) such that $a \Bumpeq^j b(z)$ and $z \in^{j+2} b(z)$. Additionally, b(z) will have a member at level j of cardinality χ.

The function b(z) is one-one, so the universe can be injected into a's j-isomorphism class. In the interpretation below, the value of b( ) with a's j-isomorphism class (which will be a set_3) as argument, gives a membership_3 loop of length j+2; so every non-degenerate j-isomorphism class will be ill-founded.

Gandy's and my conjecture in [Sheridan 1989] that the following construction could be done by reverse recursion on membership depth seems to be false. A counterexample to the natural construction seems to be $\{\{0\}, \{\{\emptyset\}\}\}$, replacing 0 with $\aleph_1$ at level 2. The natural construction by reverse recursion on depth would leave $\{\{\emptyset\}\}$ at level 1 unchanged, since $\{0\}$ at level 2 would also be unchanged. But this would fail to preserve the level 1 graph edge from $\{\{\emptyset\}\}$ to $\{0\}$, since $\{\{\emptyset\}\}$ would be unchanged, but $\{0\}$ would map to $\{\aleph_1\}$. (Part of the difficulty is that $\{0\}$ is a member at level 1, hence not at level*2. Using maximal rather than minimal depth would not work, since, for instance, 0 is a member of ω at all finite levels.)

16.4.1. Preliminary Definitions

a is not empty at level*j, so it has a member at level*j, d. Choose an arbitrary object z; take χ as an arbitrary infinite (Cantor) cardinal larger than TC(a). Construct F and b(z), which will be a with d replaced at level*j by χ(z) [defined below], with F: a $\Bumpeq^{j}$ b(z). F is constructed by transfinite recursion on a variant of the Cumulative Hierarchy, modified for urelements; thus this construction is essentially dependent on the Axiom of Foundation. (I had hoped that the consistency of the Axiom of Generalized Frege Cardinals would not be dependent on its restriction to well-founded sets, even though the available consistency proof is. But the above paradox of the set of all non-self-membered blasphemous sets mandates caution.) F will be constructed in stages, F^{α} for each ordinal $\alpha \leq \rho(a)$, where ρ is the usual Cumulative Hierarchy rank function ([Levy 1979], §6.6); R(α) will be the usual α^{th} stage of the Cumulative Hierarchy, modified for the inclusion of the relevant urelements in R(0), as follows.

Let R(0) be the set of all urelements in TC(a); define, similarly to the usual cumulative hierarchy, $R(\alpha) = \bigcup_{\xi<\alpha} P(R(\xi))$, where P is power set. (Since TC(a) is a set, so will be the R(α)'s, and hence the F^{α}'s defined on them below.) Let R*(α) be the collection of objects first appearing in stage R(α), i.e., $R(\alpha) — \bigcup_{\xi<\alpha} R(\xi)$. (So R*($\alpha$) will be empty if α is a limit ordinal. R*(0) will be equal to R(0).) The union of the F^{α}'s will be a mapping on the transitive closure of a; the desired F will be the restriction of the union of the F^{α}'s to Ξ^{j}a. The desired b(z) will be F'a.

χ was taken above as an arbitrary infinite (Cantor) cardinal larger than TC(a); define $\chi(x) = \chi — \{\{\emptyset\}\} \cup \{\{x\}\}$. Observe that χ(x) is one-one, and χ(x) contains x at level 2.

Let β be the first ordinal such that $d \in R(\beta)$, i.e., β is unique such that $d \in R^*(\beta)$.

16.4.2. The Construction

Define F^{α} on R*(α), for ordinals α, as follows; the recursion will end at the first stage, γ, containing a (i.e., R*(ρ(a)+1). γ must be a successor ordinal, so γ - 1 exists. Observe that at each stage $\alpha \leq \beta$, F^{α} will be obviously one-one, since χ(z) is distinct from—because it is larger than—any member of TC(a). Showing that later functions, and their union, are one-one will be more difficult.

If d is not an urelement, then F^0 will be the identity function on R*(0) (the urelements in TC(a)). Otherwise F^0 maps d to χ(z), and is the identity on the rest of R*(0); and β is 0. Formally,

$F^0 =$

$\{<d, \chi(z)>\} \cup \{<x,x> \mid x \in R^*(0) \ \& \ x \neq d\}$, if urelement(d)

$\{<x, x> \mid x \in R^*(0)\}$, otherwise.

F^1, if d = Ø (and hence β = 1), maps Ø to χ(z), and is the identity on the rest of R*(1); otherwise F^1 is just the identity function on R*(1). Formally,

F^1 =

$\{<\emptyset, \chi(z)>\} \cup \{<x,x> \mid x \in R^*(1) \;\&\; x \neq \emptyset\}$, if d = Ø,
$\{<x, x> \mid x \in R^*(1)\}$, otherwise.

For stages α between 1 and β (if any; if β is 0 or 1, this clause is vacuous, and the following clause coincides with clause 0 or 1), F^α is the identity on members of TC(a) in R*(α). I.e.,

$$F^\alpha = \{<x,x> \mid x \in R^*(\alpha) \cap TC(a)\}.$$

At stage β (where d ∈ R*(β)), F^β maps d to χ(z), and is otherwise the identity on members of TC(a) in R*(β). I.e.,

$$F^\beta = \{<d, \chi(z)>\} \cup \{<x,x> \mid x \in R^*(\beta) \;\&\; x \in TC(a) \;\&\; x \neq d\}.$$

Define $F^{\leq\alpha} = \bigcup_{\delta\leq\alpha} F^\delta$; this is a function, since the domains of the F^α are disjoint. (The continuation of the definition below maintains this disjointness; each Fα will be restricted to R*(α).)

For successor ordinals α+1 greater than β and less than γ, $F^{\alpha+1}$‘x is $F^{\leq\alpha}$“x, i.e.,

$$F^{\alpha+1} = \{<x, \{F^{\leq\alpha}‘w \mid w \in x\} > \mid x \in R^*(\alpha+1) \cap TC(a)\}.$$

Observe that each $F^{\leq\alpha}$‘w will be defined, since w ∈ x, and hence w is earlier in the Cumulative Hierarchy.

The limit ordinal case is trivial, since R*(α) is empty for α a limit ordinal.

F^γ is defined only for a:

$$F^\gamma = \{<a, \{F^{\leq\gamma\text{-}1}‘y \mid y \in a\}>\}.$$

Let b(z) be $\{F^{\leq\gamma\text{-}1}‘w \mid w \in a\}$, i.e., F^γ‘a. (For brevity, in the rest of this proof, since z is fixed, abbreviate b(z) to b.) Let

$F^+ = \bigcup_{\delta\leq\gamma} F^\delta$; let F be F^+ restricted to Ξ^ja, i.e.,
$F = \{<x, y> \mid <x,y> \in F^+ \;\&\; x \in \Xi^j a\}$.

Example: Let j = 2, a = 3 — {Ø} = {{Ø}, {{Ø}}} = {1, {1}}, d = 0, γ = 4, β = 1, χ = ω, χ(z) = ω — {{Ø}} ∪ {{z}}.

R*(0) = Ø (Since there are no urelements in TC(a).)
R*(1) = {Ø}
R*(2) = {{Ø}}
R*(3) = {{{Ø}} …}
R*(4) = {{{Ø}, {{Ø}}} …}

F^0 is the empty function.

$F^1 = F^\beta = \{<\emptyset, \chi(z)>\}$.

$F^2‘x = F^{\leq\beta}“x = \{<x, \{F^{\leq 1}‘w \mid w \in x\} > \mid x \in R^*(2) \cap TC(a)\} = \{<\{\emptyset\}, \{F^{\leq 1}‘\emptyset\} > \} = \{<\{\emptyset\}, \{\chi(z)\} > \}$.

$F^3‘x = F^{2+1}‘x = F^{\leq 2}“x = \{<x, \{F^{\leq 2}‘w \mid w \in x\} > \mid x \in R^*(3) \cap TC(a)\} = \{<\{\{\emptyset\}\}, \{F^{\leq 2}‘w \mid w \in \{\{\emptyset\}\}>\} = \{<\{\{\emptyset\}\}, \{F^{\leq 2}’\{\emptyset\}\}>\} = \{<\{\{\emptyset\}\}, \{\{\chi(z)\}\} > \}$.

$F^4 = \{<a, \{F^{\leq 3}‘y \mid y \in a\}>\} = \{<a, \{F^{\leq 3}‘\{\emptyset\}, F^{\leq 3}‘\{\{\emptyset\}\}\}>\} = \{<a, \{\{\chi(z)\}, \{\{\chi(z)\}\}\} > \}$.

So $b(z) = \{\{\chi(z)\}, \{\{\chi(z)\}\}\}$.

16.4.3. Properties of the Construction

Theorem 16.9 (Replacing Theorem)**.** F: a $\cong^j$ b.

Lemma 16.10 (Domain Lemma)**.** The domain of F is Ξ^ja.

Lemma 16.11 (Cardinality Lemma)**.** $\forall x \in$ domain(F). $F‘x \neq x$ iff $TC(F‘x) \geq \chi$.

(For this and the following lemma, "$>$" and "$\geq$" will denote the usual cardinality inequalities; given the presence of Foundation and Choice, this is unproblematic.)

Corollary 16.12 (Cardinality Corollary)**.** $\forall x \in \Xi^j b.\ x > TC(a) \leftrightarrow x = F‘d$.

Lemma 16.13 (One-One Lemma)**.** F is one-one.

Observations: $F‘d = \chi(z)$: By the definition of F^0, if d is an urelement; by the definition of F^1, if $d = \emptyset$; otherwise by the definition of F^β.

$\chi(z) \in^j b$, since $\chi(z) = F‘d$, and $d \in^j a$.

z is a member at level j+2 of b. (Recall that $\chi(z)$ contains z at level 2.)

Observation 16.14 (Cardinality Replacing Observation)**.** The above construction (considered now as a function of χ) provides an injection of the infinite cardinals larger than the transitive closure of a into a's j-isomorphism class.

16.5. *Definition of j-rep(x)*

For arbitrary x, and j in ω, let r be the first object in the global well-ordering such that x $\cong^j$ r (if any, otherwise let r be x itself[23]); define **j-rep**(x) $=_{df}$

[23] This case does not arise in the base theory. The situation will be far more complicated in the interpretation, but the impact on the present consistency proof is limited.

<j, r>. Say that <j, r> is a **j-rep** iff$_{df}$ there is an object x such that j-rep(x) is <j, r>.

Since ω-isomorphism is equality, define ω-**rep**(x)$=_{df}$<ω, x>. Since Ø is the first object in the global well-ordering, 0-rep(x) = <0, Ø>. j-isomorphism on urelements will play little part in what follows, since they will be used for the new sets, but (as no two empty objects are j-isomorphic for j > 0) for 1 < j < ω and u an empty object, j-rep(u) will be <j, u>.

Define **rank**(h) = j, for j ≤ ω, if ∃s. h = j-rep(s); undefined otherwise. Since any j-rep is an ordered pair with first component j, this will be single-valued.

16.6. *Proof of* $\backsim^{j}$ *Requirements*

Lemma 16.15 ($\Bumpeq^{j}$ Requirements Lemma)**.** $\Bumpeq^{j}$, ω, rank, and j-rep satisfy the $\backsim^{j}$ Requirements from Part II.11.

Substituting $\Bumpeq^{j}$ for $\backsim^{j}$, ω for μ, and using the specific definitions of j-rep and rank, the $\backsim^{j}$ Requirements are:

(α). ∀j, k ≤ ω ∀x, y. j ≤ k & x $\Bumpeq^{k}$ y ⇒ x $\Bumpeq^{j}$ y,
(β). ∀x, y. x $\Bumpeq^{0}$ y,
(γ). ∀x, y. x $\Bumpeq^{\omega}$ y ≡ x = y,
(δ). ∀j ≤ ω ∀b ∃r. r = j-rep(b),
(ε). For 0 ≤ j ≤ ω, x $\Bumpeq^{j}$ y iff j-rep(x) = j-rep(y),
(ζ). ∀h. rank(h) ≤ ω and ∀g. rank(g) = j ⇒ ∃x. g = j-rep(x),
(η). rank(0-rep(Ø)) = 0 and ¬∃s: low(s) & ∀d. d ∈ s ↔ ∃x. 1-rep(x) = d.

Lemma 16.16 (Non-Emptiness and Prolificity Lemma)**.** If <j, a> is a j-rep and a is not empty at level*j, then <j, a> is j-prolific. (This is a generalization of $\backsim^{j}$ Requirement (η.2).)

Corollary 16.17 ($\backsim^{j}$ Requirements (η))**.**

Lemma 16.18 (j-Empty j-Isomorphism Lemma)**.** If a is empty at level j, then the only thing which is j-isomorphic to a is a itself.

Note that this is not true in a theory which violates Finsler Strong Extensionality [Aczel 1988].

Define **j-pure**(x) iff x has no members at less than level j which are urelements; more formally: **j-pure**(x) iff$_{df}$ ∀y. y$\in^{<j}$x → ¬urelement(y). (For j = 1, 1-pure(y) reduces to ¬urelement(y); 0-pure is vacuously true and will not be used.) (Note that this is simpler than the definition in [Sheridan 1993].) The intent is to exclude altered objects from being relevant to j-isomorphism, since this would make j-isomorphism different in the interpretation. This is

manifested in the j-Isomorphism j-Purity Lemma, below. If I were doing this construction without using all urelements in the base theory as new sets, which would allow for urelements in the interpretation, it would make sense to define j-pure in terms of altered objects rather than urelements.

17. The Interpretation $\in_3$

17.1. *Definition of INDEX3*

As noted above in the introductory remarks for Foundation, Choice, j-Isomorphism, and Less Generality (III.16), the definition of INDEX3 will be similar to the earlier definition of INDEX, with the following differences:

- ω is substituted for the arbitrary ordinal μ.
- For the arbitrary sequence of relations $\backsim^j$ ($j \leq \mu$) I substitute $\Bumpeq^j$ ($j \leq \omega$), with $\Bumpeq^\omega$ being equality and $\Bumpeq^0$ being the universal relation.
- One of the clauses (bracketed below) in the definition of INDEX is now redundant, given the assumption of Foundation in the Base Theory.
- Stricter conjuncts are substituted in clause (d), as noted in the original definition of INDEX (II.13).

Informally, an INDEX3 will be an ω+1-tuple, with at least one of its components (other than ω) non-empty, in which each L^j contains only j-reps of j-pure objects not empty at level*j.

Define **INDEX3**(L) $\equiv_{df}$

(a) ω+1-tuple(L) & $\forall j \leq \mu$. set(L_j),
[(b) low($\cup_{j\leq\omega} L^j$)],
(c) $\exists j < \omega\ \exists x.\ x \in L^j$,
(d) $\forall j < \omega\ \forall a \in L^j$. rank(a) = j & 2nd(a) is not empty at level*j & 2nd(a) is j-pure,
(e) $\forall a \in L^\omega\ \exists x.\ a = \omega$-rep(x),
(f) $\forall x$. odd-or-even(sprig(L, x)).

Note that, despite the specialization of the arbitrary family of relations $\backsim^j$ (for j less than an arbitrary ordinal μ), to the $\Bumpeq^j$ for finite j, (except for the trivial relation $\Bumpeq^0$), conjunct (f) is still significant. The ω+1-tuple ({0-rep(ω)} {1-rep(ω)} … {2-rep(ω)} … {j-rep(ω)} … {ω-rep(ω)}) is *not* an INDEX3, since its sprig for ω is neither odd nor even. The definition does not exclude all such unbounded ω+1-tuples, however: ({} {1-rep(1)} … {2-rep(2)} … {j-rep(j)} … {}) is an INDEX3, since by the Increasing Strictness Lemma, its sprig for any object is of length either zero or one.

17.2. *Excess Urelements*

Previous uses of the Urelement Bijection Axiom have ignored urelements not used as indexes for new sets. If the class of unused urelements were equinumerous to the universe, this would cause problems with the Axiom of Generalized Frege Cardinals. The Frege 1-cardinal of the empty set is a set containing the empty set plus all urelements, and is well-founded. If this set can be mapped onto the universe, Well-Founded Replacement would then require the existence of the Russell Set, leading to a contradiction.

So rather than the mapping Υ given by the Urelement Bijection Axiom, I will employ a mapping Υ'' based on Υ, but which is one-one from the class of INDEX3's onto the class of urelements. (Thus Υ'' will also satisfy the Υ' Injection Requirement.)

A Cantor-Schroeder-Bernstein-Dedekind construction will give a mapping Υ'' from the class INDEX3 one-one onto the class of urelements. Note that the required definition for Υ'' need merely be a particular definable formula; there is no need for a set mapping.

Since the class of all sets can be injected into the class INDEX3 (e.g., by the mapping from ξ to $(\{0\text{-rep}(\emptyset)\} \ldots \{\omega\text{-rep}(\xi)\})$), the mapping defined in the standard proof of the Cantor-Schroeder-Bernstein-Dedekind Theorem, e.g., [Levy 1979] p. 85, gives a class mapping from the class INDEX3 one-one onto the class of all sets. The composition of this with the original bijection Υ (from the sets one-one onto the urelements) gives the required bijection $\boldsymbol{\Upsilon}''$ from the class INDEX3 one-one onto the class of urelements.

Redefinition of *: Let * henceforth be an abbreviation for Υ''; it will normally be used with parentheses omitted, as before. As noted above, I am reusing this terminology (as well as j-rep and rank); it is now being used in a more specific sense than in the more general proofs. For convenience, Υ'' will be abbreviated to $\boldsymbol{\Upsilon}$, since the original Υ will not be used again.

17.3. *Definition of* $\in_3$ *& Interpretations of the Axioms of CUSι*

Define $x \in_3 y \equiv_{df}$

(a) $\exists L.\ y = {*L}\ \&\ \text{INDEX3}(L)\ \&\ \text{odd}(\text{sprig}(L, x))$
$\vee$
(b) $x \in_0 y$.

As with $\in_1$ and $\in_2$, I will adopt the convention that a formula with subscript "$_3$" represents the formula with $\in_3$ substituted for the base theory's membership relation. As before, for convenience, "altered" will be redefined in terms of $\in_3$.

Discussion. As in Part II, the domain of the interpretation is the same as that of the ground model. Church's use of Compactness is again unnecessary, since the entire sequence of relations $\Bumpeq^{j}$ is used, not merely a finite subsequence.

Since we are now assuming Foundation in the base theory, the $sets_0$ of the ground model will turn out to be definable as the low_3 $sets_3$ in the interpretation: after proving the Cardinal Injection Observation (17.1), the Ill-Foundedness Requirements, and the Unaltered Domain Lemma (17.3), below, it would not be hard to show, in the presence of a global well-ordering, that the altered objects are the non-low_3 $sets_3$. Observe that since the collection of old sets is definable in the interpretation, then so is the old membership relation, as the two relations differ only in that some old urelements are new sets.

It would be straightforward to alter this construction to use Church's j-equivalence (abbr: $\approx_j$) instead of $\Bumpeq^{j}$. Chapter 7 of [Sheridan 1989] sketches a proof that any Church j-equivalence class is the union of a low number of j-isomorphism classes. For any two constructions using such related relation sequences, there is a natural embedding from the model with the looser relation into the model with the stricter. The embedding moves only the altered objects; substitute for each j-rep (in the sense of the looser relation) in the j^{th} component of the associated ω+1-tuple, the low (by hypothesis) collection of j-reps (in the sense of the stricter relation) whose second components bear the looser relation to the original j-rep's second component.

Provided that both relation sequences are absolute, the image of the embedded model is definable in the model with the stricter relation sequence: It is the unaltered objects, plus the altered objects which correspond to the combination of looser equivalence classes defined by the corresponding looser ω+1-tuple. E.g., define, for this section only, $\approx$j-rep(x) as the representative of the $\approx^j$ equivalence class of x, and INDEX4 and $\in_4$ as the INDEX predicate and membership relation, defined analogously to j-rep(x), INDEX3, and $\in_3$, but in terms of Church's j-equivalence in place of my j-isomorphism. Then x is in the image of the embedding of the looser model ($\in_4$) in the stricter ($\in_3$), iff it is either low_3 or there is an ω+1-tuple N which satisfies the requirements for INDEX4, such that membership in x (in terms of $\in_3$) satisfies the requirements specified by N in terms of $\approx^j$.

Somewhat more formally, this predicate is: $low_3(x) \vee \exists N.\ INDEX4(N)\ \&\ \forall z.\ z \in_3 x \equiv odd_3(\{<j, \approx j\text{–}rep(z)> \mid j \leq \mu\ \&\ \approx j\text{–}rep(z) \in_3 N^j\}_3)$. Note that the meaningfulness of this predicate depends heavily on the absoluteness of, among others, $\approx$j–rep and membership in low_3 $sets_3$.

17.3.1. Organization of the Verification of the Interpretations of the Axioms of CUSι

Verifying that the interpretation $\in_3$ satisfies the axioms of CUSι, which constitutes the rest of the body of the paper, will be organized as follows:

(1) $\in\dagger$ Lemma: Verify that $\in_3$ satisfies the requirements for an $\in\dagger$-interpretation (§ I.9.1), i.e., the form of the definition, the Ill-Foundedness Requirements, and the ϒ' Injection Requirement. This, by the Basic Axioms Theorem (I.9.1), will establish the Basic Axioms except Extensionality.

(2) Verify that $\in_3$ satisfies the various assumptions of Part II. This will establish the Axioms of Extensionality (by Theorem II.14.7) and Symmetric Difference (by Theorem II.14.5). These assumptions are:

    (2.1) The definitions of INDEX3 and $\in_3$ are of the required form, with clause (d) of the former satisfying a strengthened requirement.

    (2.2) INDEX3 satisfies the Degeneracy/Diversity Properties (II.13.1).

    (2.3) Required Properties of + (II.10.1).

    (2.4) $\Bumpeq^j$ and j-rep satisfy the $\backsim^j$ Requirements from section II.11. This was established in the $\Bumpeq^j$ Requirements Lemma (III.16.15), above.

(3) Verify the interpretation of the Unrestricted Axiom of Pairwise Union.

(4) Prove the j-Pure j-Isomorphism Absoluteness Lemma.

(5) This lemma, plus the Equivalence Class Observation (II.14.2), yields the interpretation of the Axiom of Generalized Frege Cardinals.

17.4. $\in\dagger$ *Lemma*

I will show in the following that $\in_3$ (along with Υ'') satisfies the requirements for an $\in\dagger$-interpretation. An immediate corollary will be, by the Basic Axioms Theorem (I.9.1), that $\in_3$ satisfies the Basic Axioms except Extensionality. The requirements on the membership relation for the Basic Axioms Theorem are (a) that the relation be defined in a certain form, which is true by inspection, (b) the Υ' Injection Requirement, which is true for Υ'', as noted in its construction, and (c) Ill-Foundedness Requirements (1)–(3), the proofs of which are after the following two results.

Observation 17.1 (Cardinal Injection Observation)**.** Given an altered set x, we can inject the sufficiently large (i.e. infinite_0 and larger than the transitive closure_0 of the index of x) Cantor cardinals_0 into the members_3 of x.

Note that the sense of Cantor cardinality used here is that of the Base Theory; the Unaltered Domain Lemma (17.3), below, will lessen this difficulty. Note also that the injection constructed here is a formula in the base theory, not necessarily a function in the interpretation.

Corollary:

Observation 17.2 (Absolute Pairs Observation)**.** "x = {y, z}" and "x = <y, z>" (unordered and Kuratowski ordered pairs) are both absolute.

This result will be frequently used without comment.

Corollary:

Lemma 17.3 (Unaltered Domain Lemma)**.** (1) Any function_3 whose domain_3 is unaltered, is unaltered. (2) If a function_3 is unaltered, its domain_3 is unaltered. (3) If a function_3 is unaltered, its range_3 is unaltered.

17.4.1. Verification of the Ill-Foundedness Requirements for $\in_3$

Corollary:

Corollary 17.4 (Basic Axioms $\in_3$ Corollary)**.** $\in_3$ satisfies the Basic Axioms except Extensionality.

17.5. *Extensionality, Symmetric Difference, and the Application of Part II*

This section verifies (as specified in §III.17.3.1) that $\in_3$ satisfies the assumptions in Part II required for the Symmetric Difference_2 Theorem (II.14.5) and the Interpretation of the Axiom of Extensionality for Sets (II.14.7). The requirements for the applicability of these results are as follows:

(I) The definition of INDEX3 (section III.17.1) is of the form required by the definition of $\in_2$ (II.14), with clause (d) of the definition of INDEX3 satisfying an additional requirement, as noted after the Degeneracy/Diversity Properties (II.13.1).

(II) The definition of $\in_3$ (III.17.3) is of the required form (II.14).

(III) INDEX3 satisfies the Degeneracy/Diversity Properties (II.13.1).

(IV) Addition on the natural numbers satisfies the Required Properties of + (II.10.1).

(V) $\eqcirc^{j}$, j-rep, and rank satisfy the $\backsim^{j}$ Requirements from section II.11. This was established in the $\eqcirc^{j}$ Requirements Lemma (III.16.15), above.

Theorem II.14.5 (Symmetric Difference_2 Theorem)

$\forall a \forall b \exists z \forall w.\ w \in_3 z \equiv (w \in_3 a \not\equiv w \in_3 b)$.

Theorem II.14.7 (Interpretation of the Axiom of Extensionality for Sets)

$\forall a \forall b.\ \text{nonempty}_3(a)\ \&\ \forall z.\ z \in_3 a \equiv z \in_3 b.\ \Rightarrow a = b$.

17.6. *Verification of the New Axioms*

I turn now to the verifications of the interpretation of the new axioms, the first of which was just proven.

17.6.1. Unrestricted Axiom of Symmetric Difference

See above.

17.6.2. Unrestricted Axiom of Pairwise Union

Theorem 17.5 (Interpretation of the Unrestricted Axiom of Pairwise Union)**.** $\forall x \forall y \exists z \forall w.\ w \in z \equiv (w \in x \vee w \in y)$

17.6.3. Purity Lemmata

For $j \in_0 \omega$, define **j-unaltered**(x) iff_{df} $\forall i{\leq}j\ \forall u.\ u \in^i{}_0\ x \equiv u \in^i{}_3\ x$. (Note that subscripts on the implicit "$\in$" in "$\leq$" are unnecessary, since ordinals are unaltered.) Similarly to j-purity, the predicate 0-unaltered is vacuously true and will not be used; being 1-unaltered is equivalent to being unaltered.

Lemma 17.6 (j-Purity Chain Lemma)**.** If x is j-pure_0, then any membership_0 chain from x of length $\leq$ j is also a membership_3 chain, and conversely.

The result will obviously imply the following:

Corollary 17.7 (j-Unaltered j-Purity Corollary)**.** If x is j-pure_0, it is j-unaltered.

Lemma 17.8 (Ill-Founded Level j Lemma)**.** $\forall j{\in}\omega$. ill-founded_3(x) & $x \in^j{}_3$ $y \rightarrow$ ill-founded_3(y)

Lemma 17.9 (Well-Founded Purity Lemma)**.** If b is well-founded_3, it is j-pure_0 for any j.

Theorem 17.10 (j-Pure j-Isomorphism Absoluteness Theorem)**.** $\forall a, b$. j-pure_0(a) $\Rightarrow a \bumpeq^j{}_0\ b \equiv a \bumpeq^j{}_3\ b$.

Corollary, by the Well-Founded Equivalence Relation Theorem (16.3), the j-Pure j-Isomorphism Absoluteness Theorem (17.10), and the Well-Founded Purity Lemma (17.9):

Remark 17.11 (Well-Founded_3 Equivalence Relation_3 Remark)**.** $\bumpeq^j{}_3$ is a restricted equivalence relation on the well-founded_3 sets_3.

The following slightly stronger result follows from the above, but I will not use it; it serves only to justify the use of Frege's name and the use of the word "equivalence": If a is well-founded_3, then (i) $a \bumpeq^j{}_3\ a$, (ii) $a \bumpeq^j{}_3\ b \equiv b \bumpeq^j{}_3\ a$, (iii) $a \bumpeq^j{}_3\ b\ \&\ b \bumpeq^j{}_3\ c \rightarrow a \bumpeq^j{}_3\ c$.

Lemma 17.12 (j-Isomorphism j-Purity Lemma). $\forall j{\in}\omega \forall a\ \forall b$**.** j-$\text{pure}_0$(a) & $a \bumpeq^j{}_0\ b \Rightarrow$ j-pure_0(b).

I.e., something to which something j-pure_0 is j-isomorphic_0, is also j-pure_0.

17.6.4. Frege Cardinals and the Singleton Function

Theorem 17.13 (Interpretation of the Restricted Axiom of Generalized Frege Cardinals)**.** $\forall j \in_3 \omega\ \forall b.\ \text{wf}_3(b) \Rightarrow \exists F\ \forall x.\ x \in_3 F \equiv b \bumpeq^j{}_3\ x$.

Corollary:

Theorem 17.14 (Singleton Function Theorem)**.** The Singleton Function is a Set_3.

This concludes the proof of the relative consistency of CUSι, Q.E.D.

18. Conclusion and Future Work

The construction technique pioneered by Church and followed by Mitchell and myself suffices to rebut naive anti-Platonist arguments against the universal set and Frege-Russell cardinals, but in the long run it seems to be a dead end. Forster's Potemkin Village criticism fairly argues that the technique will not suffice for serious theories, and it is hardly clear that a serious set theory with a universal set must have consistency strength easily comparable to a theory based on the cumulative hierarchy. The approach seems an even worse dead end in terms of manpower; all three consistency proofs involve large amounts of unrewarding complexity without concomitant aesthetic or theoretical benefits.

The paradox involving my partially-specified theory CUSι# seems less profound: merely an instance of the obvious (in retrospect) point that while natural equivalence relations may have equivalence classes which are sets, a relation which can code enough information about the membership relation to emulate the Russell Paradox cannot.

The recent work by neo-Fregeans is to some extent a divergent method of rescuing Frege: Positing representatives for equinumerosity classes, rather than defining them as sets, suffices for much of arithmetic. This presumably would have been considerable consolation to Frege, who seemed willing to abandon set theory with a universal set as a foundation for mathematics, once an inconsistency was found.[24] But I like to think that he would have appreciated the benefit of honest toil in showing that something like his set theory could define Frege cardinals while avoiding the paradoxes.

To those considering doing further research in the field, I would advise against re-traversing Church's, Mitchell's, and my paths. Oberschelp's theory may repay verification and further investigation; perhaps his theory can place the singleton function on a firmer footing than my efforts. Constructions which alter the equality relation, such as Malitz's, and Church's abandoned construction, may allow theories of greater complexity to have their relative consistency proved. My concluding advice echoes and extends Gödel's and

[24] See, e.g., at the end of his career, "A New Attempt at a Foundation for Arithmetic," reprinted in *Posthumous Writings*, pp. 278-281, in which he bases mathematics on the complex numbers and geometry rather than the natural numbers and sets.

Malitz's: What is more important than relative consistency proofs is applying Platonistic intuition to develop new theories with new axioms.

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[26] The second edition contains less detail.

[27] But note that I am a great-grand-student of Church's (via Turing and Gandy), not a student as claimed on the first page.

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[30] I have been unable to verify this independently.