# ACTUALITY, QUANTIFIERS, AND ACTUALITY QUANTIFIERS 

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#### Abstract

This paper extends Hazen's [11] work on actuality quantifiers. We present a sound and complete Hilbert-style axiomatization of the logic of actuality quantifiers. This logic is based on a quantified modal logic with actuality that allows for worlds to have completely empty domains.


## 1. Introduction

It is well-known that the language of quantified modal logic is insufficient to express certain natural language sentences pertaining to the actual. To use an often called-upon example, consider the English sentence
(A) It might have been that everyone who is in fact rich was poor.

Straightforward arguments utilizing the standard semantic readings of the quantifiers and modal operators demonstrate that the apparent translations of (A) into the language of quantified modal logic are inadequate to capture its intended meaning. ${ }^{1}$ Furthermore, one can prove that such difficulties are genuine, and (A) cannot be captured, either through apparent or obscure translations, in the formal language. ${ }^{2}$

The problem with such sentences is that they involve a particular intermingling of quantifiers and modal operators, the demands of which outrun the technical apparatus standardly available to us. Specifically, the objects over which we would like to quantify (the actually rich), may no longer be available to us once we make the migration to another possible world (as required by "It might have been ..."). To put it another way, we would like to be able to "remember" at what world we started, and to be able to refer back

[^0]to individuals in that original world once we have made a departure in order to investigate other possible ways things might be. This suggests the solution that is commonly adopted in such circumstances: incorporate into our formal language an actuality operator that allows us to reference our starting world, regardless of where we might currently be located in modal space. By making use of such an operator, we can formally represent (A) as
$$
\text { (B) } \diamond \forall x(@ R x \rightarrow P x) \text {. }
$$

The @ operator allows us to refer back to the original, actual, world, facilitating the requisite comparison between the set of rich people there, and the poor people at the possible world under consideration.

What gets discussed slightly less, however (though still discussed: see, for example, [11], [12], or [7]) is the extent to which the addition of the actuality operator really solves our problems. Briefly (the next section contains a more detailed discussion of these issues), the problem stems from the fact that the quantifier in (B) ranges over the objects that exist at some possible world (since it is inside the scope of the $\diamond$ ) and not the actual world. On the other hand, on most intuitive readings of (A), the quantifier is understood as ranging over the actual individuals (in order to identify the actually rich). If one assumes a constant domain semantics, then these two domains will be the same. But with varying domains, situations exploiting this discrepancy can arise in which (B) diverges from our intuitive understanding of what (A) is supposed to mean.

As such, one possible remedy is to restrict oneself to constant-domain models. Another, as mentioned by Fara and Williamson, is to adopt Hazen's [11] "actuality quantifiers". This paper is devoted to a more thorough exploration of the formal aspects of the second option. In particular, its main contribution will be to extend, and slightly generalize, Hazen's treatment, which concerns a natural deduction system for actuality logics based on S5. We will provide a Hilbert-style axiomatization of actuality quantifiers that can be based on normal modal logics weaker than S5. This will be built upon a quantified modal logic that allows for worlds to have entirely empty domains.

Before presenting the logics and formal results, the next section conducts a slightly more detailed examination of the translational difficulties. We then begin by introducing a freely quantified modal logic with an actuality operator. This logic will be proved sound and complete in sections 6 and 7. The proofs and techniques introduced in those sections can then be easily extended to encompass the actuality quantifiers, which will be introduced in section 8 . Lastly, in section 10, we will briefly consider other approaches that accomplish the same task as the actuality quantifiers, and identify some issues that remain unresolved by the logics put forward in this paper.

## 2. Is the Actuality Operator Enough?

Fara and Williamson diagnose the failure of (B) to adequately translate (A) in the following manner:

It is not often noted, however, that the question of whether [(B)] is the correct formalization of [(A)] is a delicate one. On a standard variable-domained Kripke semantics for quantified modal logic (Kripke 1963), on which the modal operators are interpreted as quantifiers ranging over possible worlds and different objects may exist at different worlds, [(B)] can be true in virtue of the existence of a world none of whose inhabitants exists at the actual world. But the existence of such a world seems insufficient for the truth of [(A)], at least on one of its natural readings.
So much the worse, perhaps, for variable-domained Kripke semantics. [7, p. 5]
The idea here is that the conditional in (B) might be vacuously satised at the kind of world described, one with a domain disjoint from that of the actual world. It also seems correct that the existence of such a world seems to be an odd truthmaker for (A). However, we think it is helpful to look at the problem from a slightly different perspective, one which, perhaps, compounds the problem somewhat.

From a more formal perspective, the problem with (B) seems to be not only about which individuals exist at the two worlds, but also about the predication of properties for those individuals. In the setting of natural language, it may well be that being rich implies existing, but this is not necessarily the case in the formal setting. In the usual formal treatment of quantified modal logic (following Kripke [15]), these two notions come apart by allowing individuals to lie in the extension of a predicate at a particular world even if they don't exist at that world (i.e. are not present in the domain of quantification of that world). ${ }^{3}$ Thus, the conditional in (B) will not be vacuously satisfied solely by virtue of a world with a domain that is disjoint from the actual domain - more is needed.

More formally, letting $W$ represent the set of all worlds, $\delta_{w}$ the domain of individuals existing at world $w$, and $V(P, w)$ the extension of the predicate $P$ at world $w$, treatments following Kripke usually allow $V(P, w) \subseteq$ $\left(\bigcup \delta_{w}\right)^{n}$, for an $n$-place predicate $P$. Note that this does not preclude an $w \in W$
individual $d$, non-existent at $w$, falling under the extension of $P$ at $w$. Thus, strictly speaking, given this kind of formal semantic backdrop, it is not only the non-existence of possible individuals at the actual world that renders the proposed formal translation of (A) true (since even at a world completely inhabited by individuals not existing at the actual world, $\Delta \forall x(@ R x \rightarrow P x)$

[^1]might still be false by virtue of some individual $d$, existing at the possible world in question but not the actual world, lying in the extension of R at the actual world but not in that of P at the possible world), but rather it is the non-actual-richness of any of the possibly-existing individuals that forces the sentence to be true. While this new situation does not bring us closer to a more natural truth condition for (A), it does highlight two, subtly different, aspects to the problem.

It might be reasonable, however, to think that there is an implicit condition present in (A) that makes this analysis somewhat redundant. ${ }^{4}$ As noted above, it could be argued that "Richness" (at least in the context of (A)) is an existence-entailing predicate, and so one cannot be rich at a world unless one is in the domain of existence of that world. This would imply that a more perspicuous translation of (A) would be something like ${ }^{5}$

$$
\text { (C) } \Delta \forall x(@(E x \wedge R x) \rightarrow(E x \wedge P x))
$$

(where $E x$ is the existence predicate) which, given the usual definition for $E x$ and semantic clause for the quantifier, is equivalent to

$$
\text { (D) } \Delta \forall x(@(E x \wedge R x) \rightarrow P x) .
$$

These more transparently reflect the intuition that (A) is unconcerned with the fiscal status of non-existing entities. On this reading, a possible domain disjoint from the actual domain (the problematic situation diagnosed by Fara and Williamson) would indeed result in (D) being true, and, therefore, an undesirable translation of (A). However, importantly, so would a host of other circumstances, including, for example, the situation already mentioned where the people in the domain of the quantifiers at the possible world also exist at the actual world, but are not rich at the actual world.

In fact, (D) brings forth potentially more implausible truth-making scenarios. To consider just one, (D) and (B) will both be true when there exists a possible world at which not all (but perhaps some or most) of the actually rich people exist. All of the actually rich who do exist are poor, but not all of the actually rich people are present in any form. This seems like a more egregious, and explicit violation of what we intend by "all the actually rich are ..." (given that completely vacuous satisfaction of universals is, for better or worse, somewhat familiar, but such partial fulfillments seem stranger), and stems from the inability of the quantifier at play in the actual world to

[^2]tightly bind, during transition, the individuals falling under its domain. Instead, as we move from world to world, the individuals once held firmly by an actual quantifier are allowed to slowly leak out, sometimes ceasing to exist at all.

It is in this setting that Hazen's [11] "actuality quantifiers" arise, providing a type of quantifier able to shepherd its subjects safely from one world to another. Briefly (the full details are explicated below), the proposal is to supplement our quantified modal logic with a special quantifier, $\forall$ @ , which allows us to quantify over the actually extant individuals regardless of what worlds we migrate to via the analyses of modal operators. This addition extends the expressivity of the language sufficiently in order to clearly render a formal version of (A) (and sentences like (A)) that is faithful to the originally intended meaning. ${ }^{6}$

## 3. A Brief Note on the Logics

It is worth very briefly highlighting precisely where the formal treatment presented in this paper differs from Hazen's logic, as well as other treatments of quantifiers and varying-domains found in the literature. The logics of this paper will be based on $\mathbf{K}$, and can be easily extended to incorporate many other normal modal logics. Also, whereas Hazen presents a natural deduction system, this paper offers a Hilbert-style axiomatization which accords more with contemporary approaches to these types of systems.

Lastly, the basic logic of quantification in this paper is one that is free of the existential suppositions implicit in many other treatments of varying domains (such as those, for example, in [14]). Specifically, we forgo a usual restriction preventing worlds from having non-empty domains. One might argue that, philosophically, an empty world is a somewhat dubious entity, but from the technical point of view of trying to create logics that are as general and robust as possible, such freedom seems desirable. Also, it is interesting to see exactly how some of the proofs and axiom systems need to be modified in order to make everything work out (forbidding empty domains makes certain technical constructions much easier). We should emphasize that in this respect we do not part with Hazen, who also allows for worlds to have empty domains.

Before introducing the actuality quantifiers, we first describe a freely quantified modal logic. Once this has been established we show that one can then extend the results to incorporate logics with the actuality quantifiers.

[^3]
## 4. The Language of Quantified Actuality Logic

We begin by looking at quantified actuality logic without actuality quantifiers.
Let $\mathcal{V}$ be a countable set of variables $\left(\left\{x_{0}, x_{1}, \ldots,\right\}\right)$. In addition, we assume, for each $n \in \mathbb{N} \backslash\{0\}$ a countable set of $n$-ary predicates $P, Q, \ldots$. The well-formed formulas of our language $\mathcal{L Q}$ can then be defined recursively:

$$
\varphi::=P\left(x_{1}, \ldots, x_{n}\right)|x=y| \neg \varphi|\varphi \wedge \varphi| \square \varphi|@ \varphi| \forall x \varphi
$$

where $x_{1}, \ldots, x_{n} \in \mathcal{V}$ and $P$ is an $n$-place predicate. The boolean connectives $\vee, \rightarrow$, and $\leftrightarrow$ can be defined as usual, as can the abbreviations $\diamond, \exists$. We can employ our identity symbol to define a monadic existence predicate $E$ : for a variable $x$, we define $E(x)$ as $\exists y(x=y)$.

Let $\varphi[y / x]$ denote the formula $\varphi$ in which all free occurrences of $x$ have been replaced by $y$ such that $\varphi[y / x]$ has free occurrences of $y$ wherever $\varphi$ has free occurrences of $x$ (where bound variables have been renamed, if necessary, to avoid inadvertently binding occurrences of $y$ ).

The axiom system $\mathbf{Q}^{\mathbf{0}}+\mathbf{K}+\mathbf{A}_{\mathbf{1}}$ is laid out below, and comprises axioms governing the propositional and modal components, the actuality operator, and the quantifiers. ${ }^{7}$ We use the same name for the set of formulas that are derivable in the system. We present these components separately for clarity. For the base propositional actuality logic we will follow, albeit with a slight (though inconsequential) change, [10].

The propositional component ${ }^{8}$ :
$(P C)$ All substitution-instances of theorems of propositional logic
$(K)$ All formulas of the form $\square(\varphi \rightarrow \psi) \rightarrow(\square \varphi \rightarrow \square \psi)$

$$
\begin{equation*}
\frac{\vdash \varphi \quad \vdash \varphi \rightarrow \psi}{\vdash \psi} \tag{MP}
\end{equation*}
$$

(Nec)

$$
\frac{\vdash \varphi}{\vdash \square \varphi}
$$

(@1) @ $(\varphi \rightarrow @ \varphi)$
(@2) $\_\varphi \leftrightarrow @ \neg \varphi$
$(@ 3) @(\varphi \rightarrow \psi) \rightarrow(@ \varphi \rightarrow @ \psi$
(@4) @ $\rightarrow \square @ \varphi$
(@R1)

$$
\frac{\vdash \varphi}{\vdash @ \varphi}
$$

[^4]The quantifier component:
(Reflexivity)

$$
x=x
$$

(Substitutivity) $\quad(x=y \wedge \varphi[x / z]) \rightarrow \varphi[y / z]$
(Free $\forall$-Elimination) $\quad(\forall x \varphi \wedge E(y)) \rightarrow \varphi[y / x]$
(Universal Distribution 1)
(Actual Identity)
$\forall x(\varphi \rightarrow \psi) \rightarrow(\forall x \varphi \rightarrow \forall x \psi)$
$x=y \leftrightarrow @(x=y)$
and the following rules:
(Free $\forall$-Introduction)

$$
\frac{\vdash(\psi \wedge E(y)) \rightarrow \varphi[y / x]}{\vdash \psi \rightarrow \forall x \varphi}
$$

provided $y$ does not occur free in $\psi$ or $\forall x \varphi$. And
(Free @ $\forall$-Introduction)

$$
\frac{\vdash(\psi \wedge @ E(y)) \rightarrow \varphi[y / x]}{\vdash \psi \rightarrow \forall x \varphi}
$$

provided $y$ does not occur free in $\psi$ or $\forall x \varphi$.
The following rules, the collection of which, following the terminology of Corsi [4], we will dub the Extended Barcan Rule (or $E B R$ ) ${ }^{9}$ :
( $\operatorname{BR}(n)$, for $n>0)$

$$
\frac{\vdash \psi_{1} \rightarrow \square\left(\psi_{2} \rightarrow \ldots \rightarrow \square\left(\psi_{n} \rightarrow \square \varphi\right) \ldots\right)}{\vdash \psi_{1} \rightarrow \square\left(\psi_{2} \rightarrow \ldots \rightarrow \square\left(\psi_{n} \rightarrow \square \forall x \varphi\right) \ldots\right)}
$$

for $x$ not free in $\psi_{1}, \ldots, \psi_{n}$.
And, finally, we the set of rules that we will collectively refer to as @ $E B R$ :
(@BR(n), for $n>0$ )

$$
\frac{\vdash @\left(\psi_{1} \rightarrow \square\left(\psi_{2} \rightarrow \ldots \rightarrow \square\left(\psi_{n} \rightarrow \square \varphi\right) \ldots\right)\right)}{\vdash @\left(\psi_{1} \rightarrow \square\left(\psi_{2} \rightarrow \ldots \rightarrow \square\left(\psi_{n} \rightarrow \square \forall x \varphi\right) \ldots\right)\right)}
$$

for $x$ not free in $\psi_{1}, \ldots, \psi_{n}$.
We can also consider a closely related logic by taking the set of $\mathbf{Q}^{\mathbf{0}}+$ $\mathbf{K}+\mathbf{A}_{\mathbf{1}}$ theorems and closing it under the following rule:
(@R2)

$$
\frac{\vdash \mathbf{Q}^{\mathbf{0}}+\mathbf{K}+\mathbf{A}_{\mathbf{1}} @ \varphi}{\vdash \varphi}
$$

[^5]and then closing this second set under modus ponens. We will call this logic $\mathbf{Q}^{\mathbf{0}}+\mathbf{K}+\mathbf{A}_{\mathbf{2}}{ }^{10}$

Unless specifically noted, we will assume $\mathbf{K}$ as our base modal logic. If, for example, we were using $\mathbf{S 5}$ instead, this could be denoted $\mathbf{Q}^{\mathbf{0}}+$ $\mathrm{S5}+\mathrm{A}_{1}$.

Observation 4.1. Let $\alpha, \beta, \gamma$, and $\delta$ be formulas in $\mathcal{L Q}$. If $\alpha \leftrightarrow \beta$ is a theorem of $\mathbf{Q}^{\mathbf{0}}+\mathbf{K}+\mathbf{A}_{\mathbf{1}}$, and if $\gamma$ and $\delta$ differ only in that $\gamma$ contains $\alpha$ at 0 or more places where $\delta$ contains $\beta$, then $\gamma \leftrightarrow \delta$ is also a theorem.
The proof of this more or less follows the standard one presented in [14], but with slight alterations due to the differences in the axiom system.

Observation 4.2. The following are theorems of $\mathbf{Q}^{\mathbf{0}}+\mathbf{K}+\mathbf{A}_{\mathbf{1}}$ :
(a)@ $(\varphi \wedge \psi) \rightarrow(@ \varphi \wedge @ \psi)$,
(b) @ $\rightarrow$ @@
(c) $\forall x E x$ (Universal Existence),
(d) $\forall x(\varphi \rightarrow \psi) \rightarrow(\varphi \rightarrow \forall x \psi)$ for $x$ not free in $\varphi$ (Universal Distribution 2),
(e) $\forall x \neg E(x) \rightarrow \forall x \varphi(x)$ (Empty Universal),
(f) $E(z) \rightarrow \exists y(\varphi[y / x] \rightarrow \forall x \varphi)$ (for y not free in $\varphi$ ) (Universal Witness),
(g) $x \neq y \rightarrow \square(x \neq y)$,
(h) $x=y \rightarrow \square(x=y)$,
(i) $\neg \exists y(\varphi[y / x] \rightarrow \forall x \varphi) \rightarrow \forall z \neg E(z)$ for $y$ not free in $\varphi$.

Establishing these theorems is a straightforward exercise using the axiomatization.
${ }^{10}$ One's decision regarding which is the more appropriate logic will largely depend on the attitude one takes towards sentences of the form @ $\varphi \leftrightarrow \varphi$, all instances of which will be theorems of the " $\mathbf{A}_{2}$-logics" (though not of the logics having $\mathbf{A}_{\mathbf{1}}$ as their actuality component). The motivation for including such formulas amongst our theorems "stems from the observation that the outright assertion of 'Actually p ' (or 'Now p') is tantamount to the assertion of the simple ' p '...", [6, p. 14]. These different logics are accompanied by differing notions of validity, often called "general" and "real-world" (because our semantics will not include a designated actual world, we will instead refer to the second kind as "diagonal" validity). Intuitively, the difference revolves around whether one wishes to evaluate formulas at arbitrary worlds or at a specific, designated actual world: a formula is said to be generally valid when it is holds at every world in every model, whereas it is real-world valid when it holds at every specified actual world in every model. We will not discuss further the merits of either approach, but the interested reader ought to consult Crossley and Humberstone [6] for a more thorough discussion of these issues.

## 5. Semantics

Definition 5.1 (Varying Domain Relational Frame). A varying domain relational frame is a tuple $\mathcal{F}=\left\langle W, R, D,\left\{\delta_{w}\right\}_{w \in W}\right\rangle$ where $W$ is a nonempty set of worlds, $R \subseteq W \times W$ is a binary accessibility relation, $D \neq \emptyset$ is the domain of the frame, and, for each $w \in W, \delta_{w} \subseteq D$.
Note that this definition differs slightly from Kripke's [15] in that nothing prevents individuals from existing in the frame domain without existing in any world domain. ${ }^{11}$ The notion of a frame is extended to that of a model as usual.

Definition 5.2 (Varying Domain Relational Model). A varying domain relational model is a tuple $\mathcal{M}=\left\langle W, R, D,\left\{\delta_{w}\right\}_{w \in W} V\right\rangle$ where $V$ is a function assigning to each predicate an intension. That is, for each world $w$ and n-ary predicate $P, V(P, w) \subseteq D^{n}$. We can allow $V_{w}(P)$ as shorthand for $V(P, w)$.

Definition 5.3 (Assignment). An assignment is a function $\sigma: \mathcal{V} \rightarrow D$.
Definition 5.4 ( $x$-variant). An assignment $\sigma^{\prime}$ is an $x$-variant of $\sigma$ when $\sigma(y)=$ $\sigma^{\prime}(y)$ for all $y \in \mathcal{V} \backslash\{x\}$. This is denoted $\sigma \sim_{x} \sigma^{\prime}$.

Satisfaction with respect to a pair of worlds and an assignment function can now be defined:

```
\(\mathcal{M}, w_{0}, w \vDash_{\sigma} P\left(x_{1}, \ldots, x_{n}\right)\) iff \(\left\langle\sigma\left(x_{1}\right), \ldots, \sigma\left(x_{n}\right)\right\rangle \in V(P, w)\)
\(\mathcal{M}, w_{0}, w \vDash_{\sigma} x=y \quad\) iff \(\quad \sigma(x)=\sigma(y)\)
\(\mathcal{M}, w_{0}, w \vDash_{\sigma} \neg \varphi \quad\) iff \(\mathcal{M}, w_{0}, w \not \models \varphi\)
\(\mathcal{M}, w_{0}, w \vDash_{\sigma} \varphi \wedge \psi \quad\) iff \(\mathcal{M}, w_{0}, w \vDash_{\sigma} \varphi\) and \(\mathcal{M}, w_{0}, w \vDash_{\sigma} \psi\)
\(\mathcal{M}, w_{0}, w \vDash_{\sigma} \square \varphi \quad\) iff for every \(w^{\prime} \in W\), if \(w R w^{\prime}\) then \(\mathcal{M}, w_{0}\),
    \(w^{\prime} \vDash_{\sigma} \varphi\)
\(\mathcal{M}, w_{0}, w \vDash_{\sigma} @ \varphi \quad\) iff \(\mathcal{M}, w_{0}, w_{0} \vDash_{\sigma} \varphi\)
\(\mathcal{M}, w_{0}, w \vDash_{\sigma} \forall x \varphi(x) \quad\) iff \(\quad\) for all \(\sigma^{\prime} \sim_{x} \sigma\) s.t. \(\sigma^{\prime}(x) \in \delta_{w}, \mathcal{M}, w_{0}\),
\(w \vDash_{\sigma^{\prime}} \varphi(x)\)
```

It is useful to note that the semantic clause for an existence predicate $E$ can be understood as follows:

[^6]\[

$$
\begin{aligned}
\mathcal{M}, w_{0}, w \vDash_{\sigma} E(x) \text { iff } & \text { there exists a } y \text {-variant } \sigma^{\prime} \text { of } \sigma \text { s.t. } \sigma^{\prime}(y) \in \delta_{w} \\
& \text { and } \mathcal{M}, w_{0}, w \vDash_{\sigma^{\prime}} x=y
\end{aligned}
$$
\]

Given the semantic clause for equality, this can be simplified, and one can see that $E(x)$ will hold at a pair $w_{0}, w$ just in case $\sigma(x) \in \delta_{w}$, as desired.

Definition 5.5 (General Validity). A formula $\varphi$ is said to be generally valid in a model, denoted $\mathcal{M} \vDash \varphi$, if and only if $\mathcal{M}, w_{0}, w \vDash_{\sigma}$ for all $w_{0}, w \in W$ and all assignments $\sigma$. A formula is said to be generally valid with respect to a frame, denoted $\mathcal{F} \vDash \varphi$, iff it is generally valid on every model based on $\mathcal{F}$.

Definition 5.6 (Diagonal Validity ${ }^{12}$ ). A formula $\varphi$ is said to be diagonally valid in a model, denoted $\mathcal{M} \vDash_{D} \varphi$, if and only if $\mathcal{M}, w_{0}, w \vDash_{\sigma} \varphi$ for all $w \in W$ and all assignments $\sigma$. A formula is said to be diagonally valid with respect to a frame, denoted $\mathcal{F} \vDash_{D} \varphi$, iff it is diagonally valid on every model based on $\mathcal{F}$.
(These definitions give rise to corresponding notions of soundness and completeness.)

## 6. Soundness for Quantified Actuality Logic

We move quite quickly through this section since, as usual, it is far less interesting than the completeness section to which we will devote much more time. However, we do require a few preliminary results.

Observation 6.1. The principles of agreement and replacement hold in the varying domain semantics presented above.

The principle of replacement says that when $\varphi$ is any formula, $\mathcal{M}$ any model, and $\sigma$ any assignment, then if $\sigma^{\prime}$ is exactly like $\sigma$ except that $\sigma^{\prime}(x)=$ $\sigma(y)$, then $\mathcal{M}, w_{0}, w \vDash_{\sigma^{\prime}} \varphi$ iff $\mathcal{M}, w_{0}, w \vDash_{\sigma} \varphi[y / x]$.

The principle of agreement says that if two assignments $\sigma$ and $\sigma^{\prime}$ agree on all free variables in $\varphi$, then $\mathcal{M}, w_{0}, w \vDash_{\sigma} \varphi$ iff $\mathcal{M}, w_{0}, w \vDash_{\sigma^{\prime}} \varphi$.

The soundness proofs are straightforward, and go through as usual with the help of the above observation. Some level of generality can be established immediately in the case of soundness by way of the following theorem, which is established following the usual methods (outlined, for example, in [14]).

Theorem 6.2. Let $\mathbf{S}$ be a propositional modal logic. $F=\langle W, R\rangle$ validates every theorem of $\mathbf{S}$ iff every theorem of $\mathbf{Q}^{0}+\mathbf{S}+\mathbf{A}_{\mathbf{1}}$ is generally valid on

[^7]$\left\langle W, R, D,\left\{\delta_{w}\right\}_{w \in W}\right\rangle$ (for arbitrary $D$ and $\delta_{w}$ ). With $\mathbf{A}_{\mathbf{2}}$, the same result holds, but for diagonal validity.

Corollary 6.3 (Soundness). The following soundness results are then immediate:
(a) $\mathbf{Q}^{\mathbf{0}}+\mathbf{K}+\mathbf{A}_{\mathbf{1}}$ is generally sound with respect to the varying domain semantics outlined above.
(b) $\mathbf{Q}^{\mathbf{0}}+\mathbf{S}+\mathbf{A}_{\mathbf{2}}$ is diagonally sound with respect to the varying domain semantics outlined above.

Obviously, however, due to theorem 6.2, we in fact have a more general soundness result.

Corollary 6.4. If $\mathbf{S}$ is sound with respect to a certain class of frames, then $\mathbf{Q}^{\mathbf{0}}+\mathbf{S}+\mathbf{A}_{\mathbf{1}}$ will be sound with respect to that class as well.

## 7. Completeness for Quantified Actuality Logic

The completeness proof for $\mathbf{Q}^{\mathbf{0}}+\mathbf{K}+\mathbf{A}_{\mathbf{2}}$ (and the other systems) can be conducted using a standard canonical model construction. However, our desire to make the quantified portion of our logic totally free of existential presuppositions complicates this somewhat. Specifically, as one expects, we face the familiar problem of dealing with the quantifiers in our canonical model in such a way that any formulas of the form $\neg \forall x \varphi$ (or $\neg \forall @ x \varphi$, in the next sections) found in a set of maximally consistent sentences must have witnesses within that same set in order for the truth lemma to go through. In a constant domain setting, one simply requires that for every universal formula there exists an appropriate witness $y$. That is, we ensure $\varphi[y / x] \rightarrow \forall x \varphi$ is in every set of our construction for every formula $\varphi$. With varying domains, one can do something similar, however one has the added burden of ensuring that the $y$ acting as a witness to the universal $\forall x \varphi$ exists at the world in question. For example, in their system, Hughes and Cresswell [14] require that every maximal consistent set in their canonical models contain the formulas $E(y) \wedge(\varphi[y / x] \rightarrow \forall x \varphi)$. Unfortunately, an examination of their proofs will reveal that this property will not be sufficient for our needs, specifically because our worlds are permitted to be empty.

Thus, we have to search for new properties for $\forall$ (and $\forall^{@}$ ) that permit the canonical construction. We abstain from a detailed discussion of various options and their eventual shortcomings (which, while technically interesting and illuminating, takes us a bit too far afield at present), though a few remarks seem worth the detour.

Traditionally, there have been two standard approaches to constructing canonical models with varying domains. One involves using a constant language while the other allows the language (in particular the variables) to vary from world to world in the canonical model.

Without the actuality operator, canonical constructions with varying languages can be easier, and more elegant, than with a constant language: we can do without the somewhat inelegant $E B R$ rules. The only constraint needed to be placed upon the construction is that if $w R v$, then $\mathcal{L Q}(w) \subseteq$ $\mathcal{L} \mathcal{Q}(v)$. Intuitively, one allows the set of available variables to grow, denumerably, when we move from one world to an accessible world. This facilitates the proofs of important preliminary constructions used, ultimately, to assemble the canonical model.

However, when we incorporate an @ operator, this approach is complicated. We can understand the necessity operator as forcing our language to grow from world to accessible world in the course of our set constructions. On the other hand, the actuality operator places a complementary constraint on the variable domains. Specifically, one way to think about how the canonical construction works is that one anchors the model around some world which acts as the actual world for the purpose of the truth lemma. ${ }^{13}$ In particular, we require, for every world in the model, if $@ \varphi$ is in that world, then $\varphi$ is in the central, anchor, world. Thus, we cannot simply add more variables whenever they are needed, for this would result in the situation in which we have, at a world in the model, formulas of the form @ $\varphi(x)$ where $x$ is not in the language of the actual world in question.

Thus, whereas we can view $\square$ as placing an increasing restriction on the languages of worlds in the canonical model, we can, similarly, view the @ operator as placing a decreasing restriction on languages. Combined, it seems a constant language approach, its inelegance notwithstanding, is the more fruitful option. Thus, the focus ought to be on obtaining a new quantifier property.

It turns out that this is feasible, though predictably inelegant, as we shall now see. Our basic property will take the following form: for every formula $\varphi$, we will include in our maximal consistent sets the formula $(E(y) \wedge(\varphi[y / x] \rightarrow \forall x \varphi)) \vee \forall x \neg E(x)$. As one can see, this formula explicitly considers the situation where the domain might be empty. Unfortunately, due to the presence of the $E B R$-type rules, as well as complications induced by the actuality operator, we have to require that our sets have several

[^8]varieties of this property. We make use of all the following properties in the course of our canonical construction. ${ }^{14}$

Definition 7.1 ( $E \forall$-property). A set of $w f f \Delta$, in a language $\mathcal{L} \mathcal{Q}$, is said to have the $E \forall$-property iff for every wff of $\mathcal{L Q}$ and variable $x$ there is some variable $y($ in $\mathcal{L Q})$ s.t. $(E(y) \wedge(\alpha[y / x] \rightarrow \forall x \alpha)) \vee \forall x \neg E(x) \in \Delta$.

Definition 7.2 (@E $\forall$-property). A set of wff $\Delta$, in a language $\mathcal{L Q}$, is said to have the @E $\forall$-property iff for every wff and variable x there is some variable $y$ s.t. @ $((E(y) \wedge(\alpha[y / x] \rightarrow \forall x \alpha)) \vee \forall x \neg E(x)) \in \Delta$.

Definition 7.3 ( $\square^{n} \forall$-property). A set of wff $\Delta$, in a language $\mathcal{L Q}$, is said to have the $\square^{n} \forall$-property iff for every wff $\beta_{1}, \ldots, \beta_{n}(n \geq 0)$, and $\alpha$, and every variable $x$ not free in $\beta_{1}, \ldots, \beta_{n}$, there is some variable y s.t. $\square\left(\beta_{1} \rightarrow \ldots \rightarrow \square\right.$ $\left.\left(\beta_{n} \rightarrow \square(E(y) \rightarrow \alpha[y / x])\right) \ldots\right) \rightarrow \square\left(\beta_{1} \rightarrow \ldots \rightarrow \square\left(\beta_{n} \rightarrow \square \forall x \alpha\right) \ldots\right) \in \Delta$.

Definition 7.4 (@ $\square^{n} \forall$-property). A set of wff $\Delta$, in a language $\mathcal{L Q}$, is said to have the @ $\square^{n} \forall$-property iff for every wff $\beta_{1}, \ldots, \beta_{n}(n \geq 0)$, and $\alpha$, and every variable $x$ not free in $\beta_{1}, \ldots, \beta_{n}$, there is some variable y s.t. $@\left(\square\left(\beta_{1} \rightarrow \ldots \rightarrow \square\left(\beta_{n} \rightarrow \square(E(y) \rightarrow \alpha[y / x])\right) \ldots\right) \rightarrow \square\left(\beta_{1} \rightarrow \ldots \rightarrow \square\left(\beta_{n} \rightarrow\right.\right.\right.$ $\square \forall x \alpha) \ldots)) \in \Delta$.

One can show that these properties can all coexist together in a maximal consistent set (the proof is in the appendix).

Theorem 7.5. If $\Delta$ is a consistent set of formulas in our modal language $\mathcal{L Q}$, then there exists a consistent set $\Gamma$ in the language $\mathcal{L} \mathcal{Q}^{+}$(where $\mathcal{L} \mathcal{Q}^{+}$is obtained by adding a countable number of new variables to $\mathcal{L Q}$ ) with the @ $E \forall$-property, $E \forall$-property, $\square^{n} \forall$-property, and @ $\square^{n} \forall$-property such that $\Delta \subseteq \Gamma$.

It is also important to note that the above properties are preserved when moving from consistent sets to maximal consistent sets. That is, if $\Gamma$ has the $E \forall$-property, for example, and $\Gamma^{\prime}$ is a maximal consistent superset of $\Gamma$, then $\Gamma^{\prime}$ also has the $E \forall$-property (such a $\Gamma^{\prime}$ is also guaranteed to exist, as can be demonstrated by the usual methods.)

We also obtain a version of the usual theorem used to manage the modal operators. (If $\Gamma$ is a set of formulas, $\square^{-}(\Gamma)=\{\varphi \mid \square \varphi \in \Gamma\}$.)

Theorem 7.6. Let $\Gamma$ be a maximal consistent set of formulas in $\mathcal{L} \mathcal{Q}^{+}$possessing the $\square^{n} \forall$-property. Furthermore, let $\alpha$ be a formula of $\mathcal{L} \mathcal{Q}^{+}$such

[^9]that $\square \alpha \notin \Gamma$. Then there exists a consistent set of $\mathcal{L} \mathcal{Q}^{+}$formulas, $\Delta$, with the $E \forall$-property and the $\square^{n} \forall$-property such that $\square^{-}(\Gamma) \cup\{\neg \alpha\} \subseteq \Delta$.

However, once all of these set construction theorems have been established, a familiar canonical model approach is readily forthcoming.
$M C E \forall_{\mathbf{Q}^{\mathbf{0}}+\mathbf{K}+\mathbf{A}_{\mathbf{1}}}:=\left\{\right.$ all maximal $\mathbf{Q}^{\mathbf{0}}+\mathbf{K}+\mathbf{A}_{\mathbf{1}}$-consistent sets of wff in $\mathcal{L} \mathcal{Q}^{+}$with the $E \forall$-property and the $\square^{n} \forall$-property $\}$

Given a maximal-consistent set $w_{0}$, with the $E \forall$-property and the $\square^{n} \forall$-property and containing all instances of @ $\varphi \rightarrow \varphi$ (we prove below that such a set exists), define the model $\mathcal{M}=\left\langle W, R, D,\left\{\delta_{w}\right\}_{w \in W}, V\right\rangle$, for the language $\mathcal{L Q}$ with extension $\mathcal{L Q} \mathcal{Q}^{+}$as the tuple

$$
\mathcal{M}_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}=\left\langle W_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}, R_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}, D_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}, \Delta_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}, V_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}\right\rangle
$$

where:

$$
\begin{aligned}
& W_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}=\left\{w \in M C E \forall \mathbf{Q}^{0}+\mathbf{K}+\mathbf{A}_{\mathbf{1}} \mid @\left(@^{-}(w) \subseteq w_{0}\right\}\right. \\
& R_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}=\left\{\left\langle w_{1}, w_{2}\right\rangle \in W_{\mathbf{Q}^{0}{ }^{0} \mathbf{K A}_{1}}^{w_{1}} \times W_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}} \mid \square^{-}\left(w_{1}\right) \subseteq w_{2}\right\} \\
& {[x]:=\left\{y \in \mathcal{V}^{+} \mid x=y \in w_{0}\right\}} \\
& D_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}=\left\{[x] \mid x \in \mathcal{V}^{+}\right\} \\
& V_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}\left\{(P, w)=\left\langle\left[x_{1}\right], \ldots,\left[x_{n}\right]\right\rangle \mid P\left(x_{1}, \ldots, x_{n}\right) \in w\right\} \\
& \Delta_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}=\left\{\delta_{w}\right\}_{w \in W_{\mathbf{Q}^{0}} W_{0}}^{w_{0}} \text { where, for each } \delta_{w}, \\
& \quad \delta_{w}=\left\{[x] \in D_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}} \mid E(x) \in w\right\}
\end{aligned}
$$

Where $@^{-}(w)=\{\varphi \mid @ \varphi \in w\}$. Let the canonical assignment $\sigma$ be the function $\sigma(x)=[x]$.

Usually, when working with identity, one must consider a cohesive submodel of the canonical model in order to ensure that the model is normal with respect to identity. We can forgo this requirement, as the Actual Identity axiom does this work in the current construction.

Theorem 7.7. Given $\mathcal{M}_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}$ and $\sigma$ as defined above, for any $w \in W_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}$, and any formula $\alpha$ :

$$
\mathcal{M}_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}, w_{0}, w \vDash_{\sigma} \alpha \text { iff } \alpha \in w
$$

Proof. We only present the inductive cases for the quantifier, and half of the modal case.

In the case of the quantifier, assume first that $\forall x \varphi \in w$. Let $\sigma^{\prime} \sim_{x} \sigma$ be any $x$-variant s.t. $\sigma^{\prime}(x)=[y]$ for some $y \in \mathcal{V}^{+}$and $[y] \in \delta_{w}$. Assuming, for the moment, that there is some such $y$, since $[y] \in \delta_{w}$ we have $E(y) \in w$. This implies, from Free $\forall$-Elimination, that $\varphi[y / x] \in w$ and so, by the
induction hypothesis, $\mathcal{M}_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}, w_{0}, w \vDash_{\sigma} \varphi[y / x]$. This means $\mathcal{M}_{\mathbf{Q}^{0}{ }^{0} \mathbf{K A}_{1}}^{w_{0}}, w_{0}$, $w \vDash_{\sigma^{\prime}} \varphi[y / x]$, from the principle of agreement. But, since $\sigma^{\prime}$ is any $x$-variant s.t. $\sigma^{\prime}(x) \in \delta_{w}$, we have that $\mathcal{M}_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}} w_{0}, w \vDash_{\sigma} \forall x \varphi$.

If there are no $[y] \in \delta_{w}$, then $\mathcal{M}_{\mathbf{Q}^{{ }^{0}} \mathbf{K} \mathbf{K A}_{1}}^{w_{1}}, w_{0}, w \vDash_{\sigma} \forall x \varphi$ holds true by vacuous satisfaction of the quantifier.

In the other direction, if $\forall x \varphi(x) \notin w$, then, since $w$ is maximal, $\neg \forall x \varphi(x) \in w$, and so, since $\neg \forall x \varphi(x) \rightarrow \neg \forall x \neg E(x)$ is a theorem, we have that $\neg \forall x \neg E(x) \in w$. Then, since $w$ possesses the $E \forall$-property, it must be that there is some variable $y \in \mathcal{V}^{+}$s.t. $E(y) \wedge([y / x] \rightarrow \forall x) \in w$, which means $E(y) \in w$ but $\varphi[y / x] \notin w$. Then, by the induction hypothesis, $\mathcal{M}_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}, w_{0}$, $w \not \not_{\sigma} \varphi[y / x]$. Taking $\sigma^{\prime} \sim_{x} \sigma$ s.t. $\sigma^{\prime}(x)=\sigma(y) \in \delta_{w}, \mathcal{M}_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}, w_{0}, w \vDash_{\sigma^{\prime}} \varphi$, by the principle of replacement. Therefore, since $\sigma^{\prime}(x)=[y] \in \delta_{w}$, we have, by the definition of the semantics, $\mathcal{M}_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}, w_{0}, w \not \not_{\sigma} \forall x \varphi$.

When $\alpha$ is of the form $\square \varphi$, the right to left direction is straightforward, and demonstrated as usual.

From left to right, we assume that $\square \varphi \notin w$. From 7.6 we have the existence of a maximal consistent $\mathcal{L} \mathcal{Q}^{+}$set of wff $w^{\prime}$, with the $E \underset{w_{0}}{\forall}$-property and the $\square^{n} \forall$-property, s.t. $\square^{-}(w) \cup\{\neg \varphi\} \subseteq w^{\prime}$. Thus $\varphi \not w_{0} w^{\prime}, \mathcal{M}_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}, w_{0}, w^{\prime} \nvdash_{\sigma}$ $\varphi(\mathrm{IH})$, and $w R w^{\prime}$ (by construction). Thus $\mathcal{M}_{\mathbf{Q}^{0} \mathrm{KA}_{1}}^{w_{0}}, w_{0}, w \nvdash_{\sigma} \square \varphi$. As usual, since we are using a restricted set of worlds in our model, we must ensure that this $w^{\prime}$ actually exists in $W$. However, this demonstration proceeds, unproblematically. If $w^{\prime}$ were not in $W$, i.e. @ ${ }^{-}(w) \nsubseteq w_{0}$, there would have to be some $\psi$ s.t. @ $\psi \in w^{\prime}$ but $\psi \notin w_{0}$. Then $\neg \psi \in w_{0}$ and @ $\neg \psi \in w_{0}$ and @ $\neg \psi \in w$ (it is easy to show that $w$ and $w_{0}$ will agree on @--formulas), from which it follows that $\square @ \neg \psi \in w$ and so $\square @ \neg \psi \in w^{\prime}$, contradicting the consistency of $w^{\prime}$.

Finally, the case for @ $\varphi$ is easy.
Observation 7.8 (Gregory [10]). If $w$ is a maximal consistent set, so is @ ${ }^{-}(w)$.

It is straightforward to adapt the proof given in [10] to the present setting.

Theorem 7.9. Let $\varphi$ be an $\mathbf{Q}^{\mathbf{0}}+\mathbf{K}+\mathbf{A}_{\mathbf{1}}$-consistent formula. Then there is a model $\mathcal{M}$, worlds $w_{0}$ and $w$, and assignment $\sigma$, such that $\mathcal{M}, w_{0}, w \vDash \varphi$.

Proof. Since $\varphi$ is consistent, we have, from 7.5, that we can construct a maximal consistent set $w$, containing $\varphi$, s.t. $w$ has the @ $E \forall$-property, $E \forall$-property, $\square^{n} \forall$-property, and @ $\square^{n} \forall$-property. Let $w_{0}=@{ }^{-}(w)$. Then $w_{0}$ is also maximal consistent and possesses the $E \forall$-property and the $\square^{n} \forall$-property. Thus we can construct the canonical model, as described above, centered around $w_{0}$. Note that both $w_{0}$ and $w$ possess all the correct properties to be included in the model. So, from 7.7 we have that $\mathcal{M}, w_{0}$, $w \vDash_{\sigma} \varphi$ iff $\varphi \in w$. Therefore $\mathcal{M}, w_{0}, w \vDash_{\sigma} \varphi$.

Corollary 7.10. $\mathbf{Q}^{\mathbf{0}}+\mathbf{K}+\mathbf{A}_{\mathbf{1}}$ is generally complete.
Corollary 7.11. $\mathbf{Q}^{0}+\mathbf{K}+\mathbf{A}_{\mathbf{2}}$ is diagonally complete.
For soundness, we could easily generalize our result by using theorem 6.2. As is usual (see [14] for a detailed discussion of this), such generalization is not available to us for completeness, though a more relative version is. Specifically, from the completeness of a propositional modal logic $\mathbf{S}$ with respect to a certain class of frames, we do not immediately obtain completeness for $\mathbf{Q}^{\mathbf{0}}+\mathbf{S}+\mathbf{A}_{\mathbf{1}}$ with respect to that same class. Rather, we get the more limited result that if the frame of our canonical model for $\mathbf{Q}^{\mathbf{0}}+\mathbf{S}+\mathbf{A}_{\mathbf{1}}$ is also an $\mathbf{S}$-frame, then $\mathbf{Q}^{\mathbf{0}}+\mathbf{S}+\mathbf{A}_{\mathbf{1}}$ will be characterized be the class of $\mathbf{S}$-frames and, more specifically, by any class of frames for $\mathbf{S}$ containing the canonical model of $\mathbf{Q}^{0}+\mathbf{S}+\mathbf{A}_{\mathbf{1}}$.

While this is not as general as one might want, completeness results for quantified actuality logics based on many normal modal logics (e.g. T, S4, S5, etc.) are almost immediately attainable using this result.

## 8. Syntax and Semantics for Actuality Quantifiers

We will now introduce actuality quantifiers to our language. The well-formed formulas of our language $\mathcal{L} \mathcal{Q}_{\forall}$ @ are:

$$
\varphi::=P\left(x_{1}, \ldots, x_{n}\right)|x=y| \neg \varphi|\varphi \wedge \varphi| \square \varphi|@ \varphi| \forall x \varphi \mid \forall @ x \varphi
$$

We use $\exists^{@}$ as an abbreviation of $\neg \forall^{@} \neg$.
The new axiom system, $\mathbf{Q}_{\forall^{@}}^{0}+\mathbf{K}+\mathbf{A}_{1}$, is obtained by including the following new axioms and rule:
(Free $\forall^{@}$-Elimination) $\quad\left(\forall^{@} x \varphi \wedge @ E(y)\right) \rightarrow \varphi[y / x]$
$\left(\forall^{@}\right.$ Distribution 1) $\quad \forall{ }^{@} x(\varphi \rightarrow \psi) \rightarrow\left(\forall^{@} x \varphi \rightarrow \forall^{@} x \psi\right)$
$(\forall @ \mathrm{BF}) \quad \forall{ }^{@} x \square \varphi \leftrightarrow \square \forall @ x \varphi$
( $\left.\forall^{@} \mathrm{BF} @\right) \quad \forall{ }^{@} x @ \varphi \leftrightarrow @ \forall^{@} x \varphi$
and the following rule ${ }^{15}$ :
(Free $\forall^{@}$-Introduction)

$$
\frac{\vdash(\psi \wedge @ E(y)) \rightarrow \varphi[y / x]}{\vdash \psi \rightarrow \forall @ x \varphi}
$$

provided $y$ does not occur free in $\psi$ or $\forall{ }^{@} x \varphi$.

[^10]Observation 8.1. The following are derivable in $\mathbf{Q}_{\forall ®}^{0}+\mathbf{K}+\mathbf{A}_{\mathbf{1}}$ :
(a) $\forall^{@} x @ E(x)$ (Universal $\forall^{@}$-Existence);
(b) $\forall^{@} x(\varphi \rightarrow \psi) \rightarrow\left(\varphi \rightarrow \forall^{@} x \psi\right)$ for $x$ not free in $\varphi\left(\forall^{@}\right.$ Distribution 2$)$;
(c) $\forall{ }^{@} x @ \neg E(x) \rightarrow \forall @ x \varphi(x)$ (Empty $\left.\forall @\right)$;
(d) @ $E(z) \rightarrow \exists{ }^{@} y\left(\varphi[y / x] \rightarrow \forall{ }^{@} x \varphi\right)$ (for $y$ not free in $\varphi$ ) (Universal $\forall^{@}$ _ Witness);
(e) $\neg \exists{ }^{@} y(\varphi[y / x] \rightarrow \forall @ x \varphi) \rightarrow \forall @ z \neg @ E(z)$ for $y$ not free in $\varphi$.

Semantically, we just have to add a clause for the actuality quantifier, which is:
$\mathcal{M}, w_{0}, w \vDash \forall @ x \varphi(x) \quad$ iff $\quad$ for each $\sigma^{\prime} \sim_{x} \sigma$ s.t. $\sigma^{\prime}(x) \in \delta_{w_{0}}, \mathcal{M}, w_{0}, w \vDash{ }_{\sigma^{\prime}}$ $\varphi(x)$

## 9. Soundness and Completeness for Actuality Quantifiers

### 9.1. Soundness

This is uncomplicated, and things work as usual.
Theorem 9.1. Let $\mathbf{S}$ be a propositional modal logic. $F=\langle W, R\rangle$ validates every theorem of $\mathbf{S}$ iff every theorem of $\mathbf{Q}_{\forall}^{0}+\mathbf{S}+\mathbf{A}_{\mathbf{1}}$ is generally valid on $\left\langle W, R, D,\left\{\delta_{w}\right\}_{w \in W}\right\rangle$ (for arbitrary $D$ and $\delta_{w}$ ). For $\mathbf{A}_{2}$, the same result holds, just with diagonal validity.

Corollary 9.2 (Soundness). The following soundness results are then immediate:
(a) $\mathbf{Q}_{\forall^{\circledR}}^{0}+\mathbf{K}+\mathbf{A}_{\mathbf{1}}$ is generally sound with respect to the varying domain semantics outlined above.
(b) $\mathbf{Q}_{\forall^{@}}^{0}+\mathbf{K}+\mathbf{A}_{\mathbf{2}}$ is diagonally sound with respect to the varying domain semantics outlined above.

As before, this can be generalized, and we obtain a soundness theorem for any $\mathbf{Q}^{\mathbf{0}}+\mathbf{S}+\mathbf{A}_{\mathbf{1}}$ where $\mathbf{S}$ is sound with respect to a particular class of frames.

### 9.2. Completeness for Actuality Quantifiers

A benefit of the proof methods used in the previous sections, particularly the canonical model construction, is their flexibility. As such, extending our results to include actuality quantifiers turns out to be reasonably straightforward. However, we do require additional set-properties for the new quantifiers.

Definition 9.3 ( $E \forall^{@}$-property). A set of wff $\Delta$, in a language $\mathcal{L} \mathcal{Q}_{\forall ®}^{+}$, is said to have the $E \forall @$-property iff for every wff $\alpha$ of and variable $x$ there is some variable y s.t. $\left((@ E(y) \wedge(\alpha[y / x] \rightarrow \forall @ x \alpha)) \vee \forall^{@} x \neg @ E(x)\right) \in \Delta$.

Definition 9.4 ( $\square \forall^{@}$-property). A set of wff $\Delta$, in a language $\mathcal{L} \mathcal{Q}_{\forall^{@}}^{+}$, is said to have the $\square \forall^{@}$-property iff for every wff $\alpha$ and variable $x$ there is some variable y s.t. $\square(@ E(y) \rightarrow \alpha[y / x]) \rightarrow \square \forall^{@} x \alpha \in \Delta$.

Definition 9.5 (@ $\square \forall$ @-property). A set of wff $\Delta$, in a language $\mathcal{L} \mathcal{Q}_{\forall^{+} \text {@ }}^{+}$, is said to have the @ $\square \forall @$-property iff for every wff $\alpha$ and variable $x$ there is some variable y s.t.@( $\square(@ E(y) \rightarrow \alpha[y / x]) \rightarrow \square \forall @ x \alpha) \in \Delta$.

There are a few small positive aspects to note here. First, because of the presence of Barcan-like schemas in the case of the actuality quantifiers, the $\square \forall^{@}$ property is simpler, and more intuitive, than its $\square^{n} \forall$ counterpart. In addition, because of $\forall^{@} B F @$, we needn't include an $@$-version of the $E \forall{ }^{@}$-property. ${ }^{16}$

We now need an updated version of theorem 7.5.
Theorem 9.6. If $\Delta$ is a consistent set of formulas in our modal language $\mathcal{L} \mathcal{Q}_{\forall @}$, then there exists a consistent set $\Gamma$ in the language $\mathcal{L} \mathcal{Q}_{\forall^{@}}^{+}$(where $\mathcal{L} \mathcal{Q}_{\forall^{@}}^{+}$is obtained by adding a countable number of new variables to $\mathcal{L} \mathcal{Q}_{\forall @}$ ) with the @E $\forall$-property, $E \forall$-property, $\square^{n} \forall$-property, @ $\square^{n} \forall$-property, $E \forall^{@}$ property, $\square \forall @$-property and @ $\square \forall @$-property such that $\Delta \subseteq \Gamma$.

We also need a new version theorem 7.6.
Theorem 9.7. Let $\Gamma$ be a maximal consistent set of formulas in $\mathcal{L} \mathcal{Q}_{\forall}^{+}$possessing the $\square^{n} \forall$-property and the $\square \forall @$-property. Furthermore, let $\alpha$ be a formula of $\mathcal{L} \mathcal{Q}_{\Downarrow^{@}}^{+}$such that $\square \alpha \notin \Gamma$. Then there exists a consistent set of $\mathcal{L} \mathcal{Q}_{\forall ®}^{+}$formulas, $\Delta$, with the $E \forall$-property, the $\square^{n} \forall$-property, the $E \forall{ }^{@}$-property, and the $\square \forall^{@}$-property such that $\square^{-}(\Gamma) \cup\{\neg \alpha\} \subseteq \Delta$.

The canonical model construction can now proceed as before.
$M C E \forall_{\mathbf{Q}^{\circ}{ }^{\circ}+\mathbf{K}+\mathbf{A}_{1}}^{@}:=\left\{\right.$ all maximal $\mathbf{Q}_{\forall ®}^{0}+\mathbf{K}+\mathbf{A}_{\mathbf{1}}$-consistent sets in $\mathcal{L} \mathcal{Q}_{\forall ®}^{+}$of wff with the $E \forall$-property, the $\square^{n} \forall$-property, the $E \forall @$-property, and the $\square \forall{ }^{@}$-property $\}$

Given a maximal-consistent set $w_{0}$, with the $E \forall$-property, the $\square^{n} \forall$-property, the $E \forall{ }^{@}$-property, and the $\square \forall @$-property (we prove below such a set exists), containing all instances of @ $\varphi \rightarrow \varphi$, define the model $\mathcal{M}=\langle W, R$, $\left.D,\left\{\delta_{w}\right\}_{w \in W}, V\right\rangle$, for the language $\mathcal{L} \mathcal{Q}_{\forall} @$ with extension $\mathcal{L} \mathcal{Q}_{\forall^{\circledR}}^{+}$as the tuple

[^11]where:
\[

$$
\begin{aligned}
& W_{\mathbf{Q}_{{ }_{e}(\mathbf{C}} \mathbf{K A}_{1}}^{w_{0}}=\left\{w \in M C E \forall_{\mathbf{Q}_{v^{\oplus}}+\mathbf{K}+\mathbf{A}_{1}}^{@} \mid @^{-}(w) \subseteq w_{0}\right\} \\
& R_{\mathbf{Q}^{6} \mathbf{K A}_{1}}^{w_{0}}=\left\{\left\langle w_{1}, w_{2}\right\rangle \in W_{\mathbf{Q}^{0}{ }^{0} \mathbf{K A}_{1}}^{w_{0}} \times W_{\mathbf{Q}^{6}{ }^{0}}^{w_{0}} \mathbf{K A}_{1} \mid \square^{-}\left(w_{1}\right) \subseteq w_{2}\right\} \\
& {[x]:=\left\{y \in \mathcal{V}^{+} \mid x=y \in w_{0}\right\}} \\
& D_{\mathbf{Q}_{8}{ }^{6} \mathbf{K A}_{1}}^{w_{0}}=\left\{[x] \mid x \in \mathcal{V}^{+}\right\} \\
& V_{\mathbf{Q}^{*} \mathbf{K A}_{1}}^{w_{0}}(P, w)=\left\{\left\langle\left[x_{1}\right], \ldots,\left[x_{n}\right]\right\rangle \mid P\left(x_{1}, \ldots, x_{n}\right) \in w\right\} \\
& \Delta_{\mathbf{Q}_{\gamma^{0}} \mathbf{K A}_{1}}^{w_{0}}=\left\{\delta_{w}\right\}_{w \in W_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}}^{w_{0}} \text { where, for each } \delta_{w} \text {, } \\
& \delta_{w}=\left\{[x] \in D_{\mathbf{Q}_{0^{0}} \mathbf{K A}_{1}}^{w_{0}} \mid E(x) \in w\right\}
\end{aligned}
$$
\]

Let the canonical assignment $\sigma$ be the function $\sigma(x)=[x]$.
Now the standard truth lemma follows nicely.
Theorem 9.8. Given $\mathcal{M}_{\mathbf{Q}^{6}{ }^{*} \mathbf{K A}_{1}}^{w_{0}}$ and $\sigma$ as defined, for any $w \in W_{\mathbf{Q}^{6}{ }^{0} \mathbf{K A}_{1}}^{w_{0}}$, and any formula $\alpha$ :

$$
\mathcal{M}_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}, w_{0}, w \vDash_{\sigma} \alpha \text { iff } \alpha \in w
$$

Proof. We only present the inductive cases for the actuality quantifiers. The rest of the proof is as before.

Assume that $\forall @ x \varphi \in w$. If there is no $y \in \mathcal{V}^{+}$s.t. $[y] \in \delta_{w_{0}}, \mathcal{M}_{\mathbf{Q}_{\bullet} \mathbf{K A}_{1}}^{w_{0}}$, $w_{0}, w \vDash_{\sigma} \forall^{@} x \varphi$. So, assuming there is some appropriate $y$, take $\sigma^{\prime} \sim_{x} \sigma$ s.t. $\sigma^{\prime}(x)=[y]$ for $[y] \in \delta_{w_{0}}$. Therefore $E(y) \in w_{0}$, and so @ $E(y) \in w_{0}$ and $@ E(y) \in w$. Therefore, from Free $\forall @$-elimination, we have $\varphi[y / x] \in w$. By the induction hypothesis this implies $\mathcal{M}_{\mathbf{Q}_{\theta^{*}} \mathbf{K A}_{1}}^{w_{0}}, w_{0}, w \vDash_{\sigma} \varphi[y / x]$. From the principle of agreement, $\mathcal{M}_{\mathbf{Q}^{0} \mathbf{K A}_{1}}^{w_{0}}, w_{0}, w \vDash_{\sigma^{\prime}} \varphi[y / x]$. Therefore, since $\sigma^{\prime}$ is an arbitrary $x$-variant of $\sigma$ with $\sigma^{\prime}(x) \in \delta w_{0}$, we have $\mathcal{M}_{\mathbf{Q}_{\gamma^{6}} \mathbf{K A}_{1}}^{w_{0}}, w_{0}$, $w \vDash_{\sigma} \forall{ }^{@} x \varphi$.

In the other direction, take $\forall @ x \varphi \notin w$. Thus, from the Empty $\forall @$ axiom, we have $\neg \forall^{@} x \neg @ E x \in w$, and so, from the $E \forall^{@}$-property, (@Ey $\wedge$ $(\varphi[y / x] \rightarrow \forall @ x \varphi)) \in w$. From $@ E y \in w$ we know that $E y \in w_{0}$, and so $[y] \in w_{0}$. From the conditional half of the conjunction, we can conclude
 $\varphi[y / x]$. Taking $\sigma^{\prime} \sim_{x} \sigma$ with $\sigma^{\prime}(x)=\sigma(y)=[y]$, the principle of replacement gives us $\mathcal{M}_{\mathbf{Q}_{\nabla^{6}} \mathbf{K A}_{1}}^{w_{0}}, w_{0}, w \nvdash_{\sigma^{\prime}} \varphi$. But, since $\sigma^{\prime}(x) \in \delta_{w_{0}}, \mathcal{M}_{\mathbf{Q}_{\nabla^{*}} \mathbf{K A}_{1}}^{w_{0}}, w_{0}$, $w \nvdash_{\sigma} \forall @ x \varphi$, as desired.

Observation 9.9. If a maximal consistent set, $w$, has the $E \forall @$-property, then so does @ ${ }^{-}(w)$.
(The proof is in the appendix.)

Theorem 9.10. Let $\varphi$ be a $\mathbf{Q}_{\forall ®}^{0}+\mathbf{K}+\mathbf{A}_{\mathbf{1}}$-consistent formula. Then there is a model $\mathcal{M}$, worlds $w_{0}$ and $w$, and an assignment $\sigma$ s.t. $\mathcal{M}, w_{0}, w \nvdash_{\sigma} \varphi$.

Proof. Since $\varphi$ is consistent, we have, from 9.6, that we can construct a maximal consistent set $w$, containing $\varphi$, s.t. $w$ has the @ $E \forall$-property, $E \forall$-property, $\square^{n} \forall$-property, @ $\square^{n} \forall$-property, $E \forall^{@}$-property, $\square \forall^{@}$-property, and @ $\square \forall^{@}$-property. Let $w_{0}=@^{-}(w)$. Then $w_{0}$ is also maximal consistent and possesses the $E \forall$-property, the $\square^{n} \forall$-property, the $E \forall @$-property (from observation 9.9), and the $\square \forall^{@}$-property. We can construct the canonical model, as described above, centered around $w_{0}$. Note that both $w_{0}$ and $w$ possess all the correct properties to be included in the model. Thus, from 9.8 we have that $\mathcal{M}, w_{0}, w \vDash_{\sigma} \varphi$ iff $\varphi \in w$. Therefore $\mathcal{M}, w_{0}, w \vDash_{\sigma} \varphi$.

Corollary 9.11. $\mathbf{Q}_{\forall^{@}}^{0}+\mathbf{K}+\mathbf{A}_{\mathbf{1}}$ is generally complete.
Corollary 9.12. $\mathbf{Q}_{\forall ®}^{0}+\mathbf{K}+\mathbf{A}_{\mathbf{2}}$ is diagonally complete.
Once again, these results are not generalizable to the same extent as our soundness results, and for the same reasons. However, the same limited form of generalization still applies by virtue of theorem 9.1.

## 10. Conclusion

Lastly, let us return to the motivating examples of the first section. Recall our sentence
(A) It might have been that everyone who is in fact rich was poor.

Now, with the actuality operators, we can formalize this as

$$
\text { (E) } \diamond \forall @ x(@ R x \rightarrow(E x \wedge P x))
$$

(E) avoids our previous problems. First, it isn't satisfied at a world where the domain excludes all actual individuals. Also, (E) will be false at a world at which some of the actually rich people don't exist (due to the consequent of the conditional failing to hold true). In addition, we can now articulate other sentences that were previously unavailable to us.

Theorem 10.1 (Hodes [12]). There is no wff $\alpha$ in the language of quantified modal logic with an actuality operator $(\mathcal{L Q})$ such that:
(i) for all models, and all worlds $w_{0}, w_{1}, \mathcal{M}, w_{0}, w_{1} \vDash \alpha$ iff $\delta\left(w_{0}\right) \subsetneq \delta\left(w_{1}\right)$;
(ii) for all models, and every world $w_{0}, \mathcal{M}, w_{0}, w_{0} \vDash \alpha$ iff $\exists w \in W$ s.t. $\delta\left(w_{0}\right) \subsetneq \delta(w)$.

With the inclusion of $\forall^{@}$, this is easily overcome. Consider the following formulas:
(I) $\forall{ }^{@} x E x \wedge \exists y \neg @ E y$;
(II) $\diamond(\forall @ x E x \wedge \exists y \neg @ E y)$.

If we just assume $\mathbf{K}$ then these will not suffice. However, if we upgrade our modal logic to $\mathbf{S 5}$, and take $\square$ to range over all worlds in the frame (as Hodes does), then these formulas express precisely the properties we were formerly unable to articulate.

It must also be pointed out that actuality quantiers are not the only formal mechanism that can be introduced to deal with the kind of expressivity issues with which we have been concerned. Consider the formula $\triangle \forall @ x(@ R x \rightarrow$ $P x$ ). Using the Vlach-operators $\uparrow$ and $\downarrow$, this can be rendered as $\diamond \uparrow$ @ $\forall x(R x \rightarrow \downarrow P x)$ (see [8], for example, for more on these operators). Alternatively, one could also adopt indexed actuality operators (introduced by Peacocke [16], and given a detailed study by Stephanou in [17]). In this case, the same sentence could be expressed by $\diamond_{1} @ \forall x\left(R x \rightarrow @_{1} P x\right)$.

However, as has been noted by several authors (including, for example, Bricker [2], Cresswell [5], and Williamson [18]), while the actuality quantifiers do increase the expressivity of quantified modal logic, they by no means solve all the problems. Specifically, when dealing with natural language sentences involving iterated modality, actuality quantifiers are no longer sufficient to provide a formal translation. ${ }^{17}$ To use an example from [2]:
(F) It might have been the case that some person in the room had to win
involves just this kind of iterated modality that outruns the capabilities of the actuality quantifiers. The problem is that evaluating these types of sentences involves making comparisons across multiple worlds, specifically, more than two. While the actuality quantifiers, in conjunction with the actuality operator, allow one to make more precise comparisons between two worlds (the actual and some other), they do not help when more involved bookkeeping is called for. (It should also be noted that a similar malady faces $\uparrow$ and $\downarrow$.)

There are solutions to this. One is to adopt many Vlach-operators: $\uparrow_{1}, \uparrow_{2}, \uparrow_{3}, \ldots, \downarrow_{1}, \downarrow_{2}, \downarrow_{3}, \ldots$ (see Correia [3]). Another calls for the indexed operators studied by Stephanou [17]. Both techniques allow us to overcome the problems presented by the aforementioned cases of iterated modality. ${ }^{18}$

[^12]And, indeed, in both settings the actuality quantifiers become a mere fragment of the richer language. ${ }^{19}$

This leaves the actuality quantifiers as a stepping stone on a path of increasing expressivity: they add, in a reasonably intuitive way, to the expressive power of quantified modal logic while failing to remedy all the ailments. Our goal in this paper has been to demonstrate more thoroughly, and formally, how this intermediate step might work.

## A. Proofs

Theorems 7.5 and 9.6. Consider first theorem 7.5. Assume an enumeration of wff of $\mathcal{L Q} \mathcal{Q}^{+}$that begin with a universal quantifier as well as an enumeration of all wff of the form $\square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{n} \rightarrow \square \forall x \varphi\right) \ldots\right)$. In addition, assume a double enumeration ( $Z$ and $U$, say) of all variables in $\mathcal{V}^{+} \backslash \mathcal{V}$ (which may be assumed to be countably infinite) such that both $Z$ and $U$ are disjoint and countably infinite. Let $\Gamma_{0}=\Delta$. As usual, we construct $\Gamma$ one formula at a time. Specifically, for $\Gamma_{n}$, we let $\Gamma_{n+1}=\Gamma_{n} \cup\{[(E(y) \wedge(\varphi[y / x]$ $\rightarrow \forall x \varphi)) \vee \forall x \neg E(x)]$, @ [(E(z) $\wedge(\varphi[z / x] \rightarrow \forall x \varphi)) \vee \forall x \neg E(x)]$, $\square\left(\psi_{1}\right.$ $\left.\rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square(E(u) \rightarrow \chi[u / x])\right) \ldots\right) \rightarrow \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall x\right.\right.$ $\chi) \ldots), @\left[\square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square(E(t) \rightarrow \chi[t / x])\right) \ldots\right) \rightarrow \square\left(\psi_{1} \rightarrow \ldots \rightarrow\right.\right.$ $\left.\left.\left.\square\left(\psi_{h} \rightarrow \square \forall x \chi\right) \ldots\right)\right]\right\}$, for $\forall x \varphi$ the $n+1^{t h}$ member of the universal enumeration and $\square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall x \chi\right) \ldots\right)$ the $n+1^{\text {th }}$ of its enumeration. Finally, $y$ and $z$ are the first variables in $Z$, and $u$ and $t$ the first in $U$ not occurring in $\Gamma_{n}$ or $\forall x \varphi$, or $\forall x \chi$, or $\psi_{1}, \ldots, \psi_{h}$. We show that $\Gamma_{n+1}$ is consistent when $\Gamma_{n}$ is. Finally, we let $\Gamma=\bigcup_{n \in \omega} \Gamma_{n}$.
So, assuming $\Gamma_{n}$ is consistent, we will demonstrate $\Gamma_{n+1}$ is as well.
We make use of the fact that, for a formula $\varphi$ and a set of formulas $\Sigma$, $\Sigma \cup\{\varphi\}$ is consistent iff it is not the case that $\Sigma \vdash \neg \varphi$. So assume $\Gamma_{n} \vdash \neg[(E(y) \wedge(\varphi[y / x] \rightarrow \forall x \varphi)) \vee \forall x \neg E(x)]$.
$\Longrightarrow \Gamma_{n} \vdash \neg(E(y) \wedge(\varphi[y / x] \rightarrow \forall x \varphi))$
$\Longrightarrow \Gamma_{n} \vdash \neg \forall x \neg E(x)$
$\Longrightarrow \Gamma_{n} \vdash \exists x E(x)$
$\Longrightarrow \Gamma_{n} \vdash E(y) \rightarrow \neg(\varphi[y / x] \rightarrow \forall x \varphi)$
$\Longrightarrow \Gamma_{n} \vdash \forall y \neg(\varphi[y / x] \rightarrow \forall x \varphi)$ (Free $\forall$-intro, $y$ not in $\Gamma_{n}$ )
$\Longrightarrow \Gamma_{n} \vdash \neg \exists y(\varphi[y / x] \rightarrow \forall x \varphi)$
$\Longrightarrow \Gamma_{n} \vdash \neg E(y)$ (Universal Witness Axiom)
$\Longrightarrow \Gamma_{n} \vdash \forall x \neg E(x)$ (Free $\forall$-intro, $y$ not in $\Gamma_{n}$ )
cannot adequately be overcome without reverting to the use of constant domains and outer quantifiers (see Williamson [19] and Fritz [9]).
${ }^{19}$ See Hazen [11] for more discussion on this.

Contradicting $\Gamma_{n}$ 's consistency. So then $\Gamma_{n} \cup\{[(E(y) \wedge(\varphi[y / x] \rightarrow$ $\forall x \varphi)) \vee \forall x \neg E(x)]\}$ must be consistent.

We must now show that we can consistently add $@[(E(z) \wedge(\varphi[z / x] \rightarrow$ $\forall x \varphi)) \vee \forall x \neg E(x)]$ to this. (For simplicity, let $\beta$ be a meta-variable standing for the formula $[(E(y) \wedge(\varphi[y / x] \rightarrow \forall x \varphi)) \vee \forall x \neg E(x)]$.) Assume not. This would mean that

```
\(\Longrightarrow \Gamma_{n} \cup\{\beta\} \vdash \neg @[(E(z) \wedge(\varphi[z / x] \rightarrow \forall x \varphi)) \vee \forall x \neg E(x)]\)
\(\Longrightarrow \Gamma_{n} \cup\{\beta\} \vdash @[\neg(E(z) \wedge(\varphi[z / x] \rightarrow \forall x \varphi)) \wedge \neg \forall x \neg E(x)]\)
\(\Longrightarrow \Gamma_{n} \cup\{\beta\} \vdash @ \neg(E(z) \wedge(\varphi[z / x] \rightarrow \forall x \varphi))\)
\(\Longrightarrow \Gamma_{n} \cup\{\beta\} \vdash @ \exists x E(x)\)
\(\Longrightarrow \Gamma_{n} \cup\{\beta\} \vdash @(E(z) \rightarrow \neg(\varphi[z / x] \rightarrow \forall x \varphi))\)
\(\Longrightarrow \Gamma_{n} \cup\{\beta\} \vdash @ E(z) \rightarrow @ \neg(\varphi[z / x] \rightarrow \forall x \varphi)\)
\(\Longrightarrow \Gamma_{n} \cup\{\beta\} \vdash @ \forall z \neg(\varphi[z / x] \rightarrow \forall x \varphi)\) (free @ \(\forall\)-intro, \(z\) not in \(\Gamma_{n} \cup\{\beta\}\) )
\(\Longrightarrow \Gamma_{n} \cup\{\beta\} \vdash @ \neg \exists z(\varphi[z / x] \rightarrow \forall x \varphi)\)
\(\Longrightarrow \Gamma_{n} \cup\{\beta\} \vdash @ \neg E(z)\) (Universal witness, @R1, @3)
\(\Longrightarrow \Gamma_{n} \cup\{\beta\} \vdash @ \forall x \neg E(x)\) (free @ \(\forall\)-intro, \(z\) not in \(\Gamma_{n} \cup\{\beta\}\) )
\(\Longrightarrow \Gamma_{n} \cup\{\beta\} \vdash @ \neg \exists x E(x)\)
\(\Longrightarrow \Gamma_{n} \cup\{\beta\} \vdash \neg @ \exists x E(x)\)
```

A contradiction.
As before, for simplicity, let $\alpha$ be a meta-variable for @ $[(E(z) \wedge(\varphi[z / x]$ $\rightarrow \forall x \varphi)) \vee \forall x \neg E(x)]$, we show that we can consistently add $\square\left(\psi_{1} \rightarrow \ldots\right.$
$\left.\rightarrow \square\left(\psi_{h} \rightarrow \square(E(u) \rightarrow \chi[u / x])\right) \ldots\right) \rightarrow \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall x \chi\right) \ldots\right)$.
Assuming not, we have

$$
\Gamma_{n} \cup\{\alpha, \beta\} \vdash \neg \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall x \chi\right) \ldots\right)
$$

and

$$
\Gamma_{n} \cup\{\alpha, \beta\} \vdash \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square(E(u) \rightarrow \chi[u / x])\right) \ldots\right)
$$

Then, since $u$ is not free in $\Gamma_{n}, \alpha, \beta$, or $\psi_{i}$, we obtain from the Extended Barcan Rule that

$$
\Gamma_{n} \cup\{\alpha, \beta\} \vdash \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall u(E(u) \rightarrow \chi[u / x])\right) \ldots\right) .
$$

Using propositional logic and instances of $\forall x(\varphi \rightarrow \psi) \rightarrow(\forall x \varphi \rightarrow \forall x \psi)$, we may conclude that

$$
\begin{aligned}
& \Gamma_{n} \cup\{\alpha, \beta\} \vdash \square^{h} \forall u E(u) \rightarrow \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall u \chi[u / x]\right) \ldots\right) \\
& \Gamma_{n} \cup\{\alpha, \beta\} \vdash \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall x \chi\right) \ldots\right)
\end{aligned}
$$

(The last step follows from the Universal Existence theorem and Necessitation.) This, however, is a contradiction.

Finally, let $\gamma$ be a meta-variable for $\square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square(E(u) \rightarrow\right.\right.$ $\chi[u / x])) \ldots) \rightarrow \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall x \chi\right) \ldots\right)$. We show that we can consistently add @( $\square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square(E(t) \rightarrow \chi[t / x])\right) \ldots\right) \rightarrow$ $\left.\square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall x \chi\right) \ldots\right)\right)$

So, assuming not:

$$
\Gamma_{n} \cup\{\alpha, \beta, \gamma\} \vdash @ \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square(E(t) \rightarrow \chi[t / x])\right) \ldots\right)
$$

and

$$
\Gamma_{n} \cup\{\alpha, \beta, \gamma\} \vdash \neg @ \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall x \chi\right) \ldots\right)
$$

From@ $E B R$ we have

$$
\Gamma_{n} \cup\{\alpha, \beta, \gamma\} \vdash @ \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall t(E(t) \rightarrow \chi[t / x])\right) \ldots\right)
$$

Using the result from propositional modal logic, as we did before, that $\square\left(\psi_{1}\right.$ $\left.\rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall t(E(t) \rightarrow \chi[t / x])\right) \ldots\right) \rightarrow \square^{h} \forall t E t \rightarrow \square\left(\psi_{1} \rightarrow \ldots \rightarrow\right.$ $\left.\square\left(\psi_{h} \rightarrow \square \forall t \chi[t / x]\right) \ldots\right)$, we obtain the result that @ $\left[\square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h}\right.\right.\right.$ $\rightarrow \square \forall t(E(t) \rightarrow \chi[t / x])) \ldots)] \rightarrow\left[\square^{h} \forall t E t \rightarrow \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall\right.\right.\right.$ $t \chi[t / x]) \ldots)]$. Thus

$$
\Gamma_{n} \cup\{\alpha, \beta, \gamma\} \vdash @ \square^{h} \forall t E t \rightarrow @ \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall t \chi[t / x]\right) \ldots\right)
$$

From which, using Universal Existence, Necessitation, and @ $R 1$, we may conclude:

$$
\Gamma_{n} \cup\{\alpha, \beta, \gamma\} \vdash @ \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall t \chi[t / x]\right) \ldots\right)
$$

Contradicting the consistency of $\Gamma_{n} \cup\{\alpha, \beta, \gamma\}$.
Turning our attention to theorem 9.6, we simply supplement this construction to account for the other formulas. First off, we need to assume yet another enumeration of formulas in $\mathcal{L} \mathcal{Q}_{\forall^{@}}^{+}$, this time all those beginning with $\forall^{@}$. Also, we can assume, for ease, yet another countable set of variables $X \in \mathcal{V}^{+} \backslash \mathcal{V}$ which are not in our other variable enumerations.

As usual, we take $\Delta=\Gamma_{0}$, and build upwards, adding, at each stage, an instance of each desired property. Since we have already shown that at each step of the construction of $\Gamma$ we can add the relevant instances of the @ $E \forall$-property, $E \forall$-property, $\square^{n} \forall$-property, and the $@ \square^{n} \forall$-property, we just have to show that we can also add the properties relevant for $\forall @$. So, letting $\alpha, \beta, \gamma$, and $\zeta$ denote the formulas already added at the $(n+1)^{t h}$ step, we will show we can also add $[(@ E(r) \wedge(\varphi[r / x] \rightarrow \forall @ x \varphi)) \vee \forall @ x \neg @ E(x)]$ and $\square(@ E(s) \rightarrow \varphi[s / x]) \rightarrow \square \forall^{@} x \varphi$ and @ $\left(\square(@ E(t) \rightarrow \varphi[t / x]) \rightarrow \square \forall^{@}\right.$ $x \varphi$ ), where $r, s$, and $t$ are new variables from $X$, currently unused.

So assume that $\Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta\} \vdash \neg[(@ \operatorname{Er} \wedge(\varphi[r / x] \rightarrow \forall @ x \varphi)) \vee \forall @$ $x \neg @ E x]$.
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta\} \vdash \neg\left(@ \operatorname{Er} \wedge\left(\varphi[r / x] \rightarrow \forall^{@} x \varphi\right)\right)$
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta\} \vdash \neg{ }^{@} x \neg @ E x$
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta\} \vdash @ E r \rightarrow \neg(\varphi[r / x] \rightarrow \forall @ x \varphi)$
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta\} \vdash \forall{ }^{@} r \neg\left(\varphi[r / \mathrm{x}] \rightarrow \forall{ }^{@} x \varphi\right)$ (Free $\forall @$ Intro)
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta\} \vdash \neg @ E r$ (Universal $\forall^{@}$ Witness)
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta\} \vdash \forall{ }^{@} x \neg @ E x$ (Free $\forall^{@}$ Intro)
A contradiction. Note that the instances of Free $\forall @$ Introduction may be invoked as $r$ is new.

Let $\xi$ denote $(@ \operatorname{Er} \wedge(\varphi[r / x] \rightarrow \forall @ x \varphi)) \vee \forall @ x \neg @ E x$. We must now show that we can consistently add $\square(@ E(s) \rightarrow \varphi[s / x]) \rightarrow \square \forall^{@} x \varphi$. Assuming not, we would have that $\Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi\} \vdash \neg \square(@ E(s) \rightarrow$ $\left.\varphi[s / x]) \rightarrow \square \forall^{@} x \varphi\right)$.
$\left.\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi\} \vdash \neg(\square(@ E(s) \rightarrow \varphi[s / x]) \rightarrow \square \forall @ x \varphi)\right)$
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi\} \vdash \neg \square \forall @ x \varphi$
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi\} \vdash \square(@ E(s) \rightarrow \varphi[s / x])$
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi\} \vdash \forall^{@} s \square(@ E(s) \rightarrow \varphi[s / x])$ (since $s$ is new)
$\left.\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi\} \vdash \square \forall @ s(@ E(s) \rightarrow \varphi[s / x])\left(\forall^{@} \mathrm{BF}\right)\right)$
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi\} \vdash \square\left(\forall^{@} s @ E(s) \rightarrow \forall @ s \varphi[s / x]\right)\left(\forall^{@}\right.$ Dist. 1)
$\left.\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi\} \vdash \square \forall^{@} s @ E(s) \rightarrow \square \forall^{@} s \varphi[s / x]\right)$
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi\} \vdash \square \forall{ }^{@} s \varphi[s / x]$ (Universal $\forall^{@}$-Exist.)
Contradicting the consistency of $\Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi\}$.
Finally, letting $\eta$ represent $\square(@ E(s) \rightarrow \varphi[s / x]) \rightarrow \square \forall @ x \varphi$, we show that we can consistently add @( $\square(@ E(t) \rightarrow \varphi[t / x]) \rightarrow \square \forall @ x \varphi)$ ). Assuming not:
$\Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi, \eta\} \vdash \neg @(\square(@ E(t) \rightarrow \varphi[t / x]) \rightarrow \square \forall @ x \varphi)$
$\left.\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi, \eta\} \vdash \neg\left(@ \square(@ E(t) \rightarrow \varphi[t / x]) \rightarrow @ \square \forall^{@} x \varphi\right)\right)$
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi, \eta\} \vdash @ \square(@ E(t) \rightarrow \varphi[t / x])$
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi, \eta\} \vdash \neg @ \square \forall @ x \varphi$
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi, \eta\} \vdash \forall @ t @ \square(@ E(t) \rightarrow \varphi[t / x])$ (since $t$ is new)
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi, \eta\} \vdash @ \square \forall^{@} t(@ E(t) \rightarrow \varphi[t / x])\left(\forall^{@} \mathrm{BF}\right.$ and $\left.\forall @ \mathrm{BF} @\right)$
$\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi, \eta\} \vdash @ \square\left(\forall{ }^{@} t @ E(t) \rightarrow \forall{ }^{@} t \varphi[t / x]\right)$
$\left.\Longrightarrow \Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi, \eta\} \vdash @ \square \forall @ t @ E(t) \rightarrow @ \square \forall @ t \varphi[t / x]\right)$
Thus, since $\vdash @ \square \forall @ t @ E(t)$, we have $\Gamma_{n} \cup\{\alpha, \beta, \gamma, \zeta, \xi, \eta\} \vdash @ \square \forall @ t \varphi[t / x]$, a contradiction.

Theorems 7.6 and 9.7. Again, we consider the segment of the language without actuality quantifiers first. Enumerate all formulas of the form $\forall x \varphi$ and then all formulas of the form $\square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{n} \rightarrow \square \forall x \varphi\right) \ldots\right)$. In
addition, assume an enumeration of all variables. Let $\gamma_{0}=\neg \varphi$. Given $\gamma_{n}$, define $\gamma_{n+1}$ as follows. Letting $\forall x \varphi$ be the $n+1^{\text {th }}$ wff of the enumeration of such wff, and $y$ the first variable such that
$\square^{-}(\Gamma) \cup\left\{\gamma_{n} \wedge[(E(y) \wedge(\varphi[y / x] \rightarrow \forall x \varphi)) \vee \forall x \neg E(x)]\right\}$ is consistent.
Let $\gamma_{n}^{+}=\gamma_{n} \wedge[(E(y) \wedge(\varphi[y / x] \rightarrow \forall x \varphi)) \vee \forall x \neg E(x)]$.
We must demonstrate that such a $y$ always exists. By a standard result in modal logic, we know that $\square^{-}(\Gamma) \cup\left\{\gamma_{0}\right\}$ is consistent. We will demonstrate that if $\left.\square^{-}(\Gamma) \cup\right)\left\{\gamma_{n}\right\}$ is consistent, there will exist the appropriate $y$.

Suppose not. That is, $\left.\square^{-}(\Gamma) \cup\right)\left\{\gamma_{n}\right\}$ is consistent, but there does not exist a $y$ such that $\square^{-}(\Gamma) \cup\left\{\gamma_{n} \wedge[(E(y) \wedge(\varphi[y / x] \rightarrow \forall x \varphi)) \vee \forall x \neg E(x)]\right\}$ is consistent. Then, for all $y$, there must be a finite $\Lambda \subset \square^{-}(\Gamma)$ s.t.:

$$
\begin{aligned}
& \Lambda \vdash \neg\left(\gamma_{n} \wedge[(E(y) \wedge(\varphi[y / x] \rightarrow \forall x \varphi)) \vee \forall x \neg E(x)]\right. \\
& \Lambda \vdash \gamma_{n} \rightarrow \neg[(E(y) \wedge(\varphi[y / x] \rightarrow \forall x \varphi)) \vee \forall x \neg E(x)] \\
& \left.\Lambda \vdash \gamma_{n} \rightarrow \neg \forall x \neg E(x)\right] \\
& \Lambda \vdash \gamma_{n} \rightarrow \neg[(E(y) \wedge(\varphi[y / x] \rightarrow \forall x \varphi)) \\
& \Lambda \vdash E(y) \rightarrow\left(\gamma_{n} \rightarrow[(E(y) \rightarrow \neg(\varphi[y / x] \rightarrow \forall x \varphi))]\right)
\end{aligned}
$$

Therefore, since $\Lambda \subseteq \square^{-}(\Gamma)$, we will have, for each $y$,

$$
\square\left(E(y) \rightarrow\left(\gamma_{n} \rightarrow[(E(y) \rightarrow \neg(\varphi[y / x] \rightarrow \forall x \varphi))]\right)\right) \in \Gamma
$$

Thus, since $\Gamma$ has the $\square^{n} \forall$-property we have, letting $z$ be s.t. it doesn't occur in $\varphi$ or $\gamma_{n}$,
$\square \forall z\left(\gamma_{n} \rightarrow[(E(z) \rightarrow \neg(\varphi[z / x] \rightarrow \forall x \varphi))]\right) \in \Gamma$
$\square \forall z\left(E(z) \rightarrow\left[\left(\gamma_{n} \rightarrow \neg(\varphi[z / x] \rightarrow \forall x \varphi)\right)\right]\right) \in \Gamma$
$\square \forall z E(z) \rightarrow \square \forall z\left(\left(\gamma_{n} \rightarrow \neg(\varphi[z / x] \rightarrow \forall x \varphi)\right)\right) \in \Gamma$
$\square \forall z\left(\left(\gamma_{n} \rightarrow \neg(\varphi[z / x] \rightarrow \forall x \varphi)\right)\right) \in \Gamma$ (from Universal Existence)
Then, from the axiom $\forall x(\varphi \rightarrow \psi) \rightarrow(\varphi \rightarrow \forall x \psi)$ where $x$ is not free in $\varphi$,

$$
\begin{aligned}
& \square\left(\left(\gamma_{n} \rightarrow \forall z \neg(\varphi[z / x] \rightarrow \forall x \varphi)\right) \in \Gamma\right. \\
& \square\left(\left(\gamma_{n} \rightarrow \neg \exists z(\varphi[z / x] \rightarrow \forall x \varphi)\right) \in \Gamma\right.
\end{aligned}
$$

From which, using propositional modal logic and the fact that $\vdash \neg \exists y(\varphi[y / x]$ $\rightarrow \forall x \varphi \rightarrow \forall z \neg E(z)$ for $y$ not free in $\varphi$ (from observation 4.2), we may conclude that:

$$
\square\left(\gamma_{n} \rightarrow \forall x \neg E(x)\right) \in \Gamma
$$

Since if $\vdash B \rightarrow C$ then $\vdash(A \rightarrow B) \rightarrow(A \rightarrow C)$, and so, using necessitation and distribution, $\vdash \square(A \rightarrow B) \rightarrow \square(A \rightarrow C)$.

Thus, we have the following two facts:
$\square^{-}(\Gamma) \vdash \gamma_{n} \rightarrow \neg \forall x \neg E(x)$ (from above), and
$\square^{-}(\Gamma) \vdash \gamma_{n} \rightarrow \forall x \neg E(x)$.
Together, these mean that $\square^{-}(\Gamma) \vdash \gamma_{n} \rightarrow \neg \gamma_{n}$, and so $\square^{-}(\Gamma) \vdash \neg \gamma_{n}$, contradicting the consistency of $\square^{-}(\Gamma) \cup\left\{\gamma_{n}\right\}$. Thus, if $\square^{-}(\Gamma) \cup\left\{\gamma_{n}\right\}$ is consistent, so is $\square^{-}(\Gamma) \cup\left\{\gamma_{n}^{+}\right\}$.

We can now extend $\gamma_{n}^{+}$to $\gamma_{n+1}$ as follows. Let $\square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow\right.\right.$ $\square \forall x \chi) \ldots$ ) be the $n+1^{\text {th }}$ such formula and $z$ the first variable such that $\square^{-}(\Gamma) \cup\left\{\gamma_{n}^{+} \wedge \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square E(z) \rightarrow \chi[z / x]\right)\right) \ldots\right) \rightarrow \square\left(\psi_{1} \rightarrow \ldots\right.$ $\left.\left.\rightarrow \square\left(\psi_{h} \rightarrow \square \forall x \chi\right) \ldots\right)\right\}$ is consistent. Then we let $\gamma_{n+1}=\gamma_{n}^{+} \wedge\left(\square\left(\psi_{1} \rightarrow \ldots\right.\right.$ $\left.\rightarrow \square E(z) \rightarrow \chi[z / x])) \ldots) \rightarrow \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall x \chi\right) \ldots\right)\right)$.

Without loss of generality, we may assume $x$ is not free in any $\psi$. So, suppose there is no such $z$. In that case, we would have that for some finite $\Lambda \subseteq \square^{-}(\Gamma):$

$$
\begin{aligned}
& \Lambda \vdash \gamma_{n}^{+} \rightarrow\left(\square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square(E(z) \rightarrow \chi[z / x])\right) \ldots\right)\right) \\
& \Lambda \vdash \gamma_{n}^{+} \rightarrow \neg \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall x \chi\right) \ldots\right)
\end{aligned}
$$

Then, from the first line, we have that, for every $z$,

$$
\left.\square\left(\gamma_{n}^{+} \rightarrow \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square(E(z) \rightarrow \chi[z / x])\right) \ldots\right)\right)\right) \in \Gamma
$$

Which gives us

$$
\square\left(\gamma_{n}^{+} \rightarrow \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall x \chi\right) \ldots\right)\right) \in \Gamma\left(\square^{n} \forall \text {-property }\right)
$$

And so

$$
\gamma_{n}^{+} \rightarrow \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall x \chi\right) \ldots\right) \in \square^{-}(\Gamma)
$$

This, however, implies that $\square^{-}(\Gamma) \cup\left\{\gamma_{n}^{+}\right\}$is inconsistent. Thus $\square^{-}(\Gamma) \cup\left\{\gamma_{n+1}\right\}$ is consistent if $\square^{-}(\Gamma) \cup\left\{\gamma_{n}^{+}\right\}$is.

Thus, we have that if $\square^{-}(\Gamma) \cup\left\{\gamma_{n}\right\}$ is consistent, so is $\square^{-}(\Gamma) \cup\left\{\gamma_{n+1}\right\}$.
Finally, let $\Delta=\square^{-}(\Gamma) \cup \bigcup_{n \in \omega} \Gamma_{n}$. Since $\square^{-}(\Gamma) \cup\left\{\gamma_{n}\right\}$ is consistent for each $n$, and $\gamma_{n} \rightarrow \gamma_{m}$ for $m \leq n$, so is the union. Finally, by construction, any maximal consistent extension of $\Delta$ has the $E \forall$-property and the $\square^{n} \forall$ property.

For theorem 9.7 we again just supplement the proof of 7.6 . Specifically, starting where we just left of, we can begin again by letting $\gamma_{n}^{*}=\gamma_{n} \wedge$ $[(E(y) \wedge(\varphi[y / x] \rightarrow \forall x \varphi)) \vee \forall x \neg E(x)] \wedge\left(\square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square(E(z)\right.\right.\right.$ $\rightarrow \chi[z / x])) \ldots) \rightarrow \square\left(\psi_{1} \rightarrow \ldots \rightarrow \square\left(\psi_{h} \rightarrow \square \forall x \chi\right) \ldots\right)$ ), and then show, in exactly the same manner, that we can extend $\gamma_{n}^{*}$ to $\gamma_{n+1}$.

Assuming an enumeration of $\forall^{@}$ formulas, where $\forall @ x \varphi$ is the $\mathrm{n}+1^{\text {th }}$ such formula, consider $\left(@ E s \wedge\left(\varphi[s / x] \rightarrow \forall^{@} x \varphi\right)\right) \vee \forall @ x \neg @ E x$ where $s$ is the first variable s.t. $\square^{-}(\Gamma) \cup\left\{\gamma_{n}^{*} \wedge\left[(@ E s \wedge(\varphi[s / x] \rightarrow \forall @ x \varphi)] \vee \forall^{@} x \neg @ E x\right]\right\}$
is consistent. We show that there must be such an $s$. For, if not, we would have for every $s$, some nite $\Lambda \subseteq \square^{-}(\Gamma)$ :

$$
\begin{aligned}
& \Longrightarrow \Lambda \vdash \neg\left(\gamma_{n}^{*} \wedge[(@ E s \wedge(\varphi[s / x] \rightarrow \forall @ x \varphi)) \vee \forall @ x \neg @ E x]\right) \\
& \Longrightarrow \Lambda \vdash \gamma_{n}^{*} \rightarrow \neg\left[(@ E s \wedge(\varphi[s / x] \rightarrow \forall @ x \varphi)) \vee \forall @^{@} x \neg @ E x\right] \\
& \Longrightarrow \Lambda \vdash \gamma_{n}^{*} \rightarrow \neg \forall @ \neg \neg @ E x \\
& \Longrightarrow \Lambda \vdash \gamma_{n}^{*} \rightarrow \neg(@ E s \wedge(\varphi[s / x] \rightarrow \forall @ x \varphi)) \\
& \Longrightarrow \Lambda \vdash \gamma_{n}^{*} \rightarrow(@ E s \rightarrow \neg(\varphi[s / x] \rightarrow \forall @ x \varphi)) \\
& \Longrightarrow \Lambda \vdash @ E s \rightarrow\left(\gamma_{n}^{*} \rightarrow \neg(\varphi[s / x] \rightarrow \forall @ x \varphi)\right)
\end{aligned}
$$

Where the last stop follows from the propositional equivalence of $\mathrm{A} \rightarrow(B$ $\rightarrow C)$ and $B \rightarrow(A \rightarrow C)$.
Continuing, then,

$$
\Longrightarrow \square\left(@ E s \rightarrow\left(\gamma_{n}^{*} \rightarrow \neg(\varphi[s / x] \rightarrow \forall @ x \varphi)\right)\right) \in \Gamma
$$

But, since this is for all variables $s$, and $\Gamma$ is assumed to have the $\square \forall @_{-}$ property, we can take $z$ to be a variable not appearing in $\varphi$ or $\gamma_{n}^{*}$ :

$$
\begin{aligned}
& \Longrightarrow \square \forall @ z\left(\gamma_{n}^{*} \rightarrow \neg(\varphi[z / x] \rightarrow \forall @ x \varphi)\right) \in \Gamma \\
& \Longrightarrow \square\left(\gamma_{n}^{*} \rightarrow \forall \forall^{@} z \neg\left(\varphi[z / x] \rightarrow \forall{ }^{@} x \varphi\right)\right) \in \Gamma\left(\forall^{@}\right. \text {-Distribution 2) }
\end{aligned}
$$

Then, since $\vdash \neg \exists{ }^{@} z(\varphi[z / x] \rightarrow \forall @ x \varphi) \rightarrow \forall @ x \neg @ E x$ is a theorem, for $z$ not free in $\varphi$ (observation 8.1),

$$
\begin{aligned}
& \Longrightarrow \square\left(\gamma_{n}^{*} \rightarrow \forall^{@} x \neg @ E x\right) \in \Gamma \\
& \Longrightarrow \square^{-}(\Gamma) \vdash \gamma_{n}^{*} \rightarrow \forall @ x \neg @ E x
\end{aligned}
$$

But, from above, we also have that $\Lambda \vdash \gamma_{n}^{*} \rightarrow \neg \forall @ x \neg @ E x$. Thus, $\square^{-}(\Gamma) \vdash$ $\neg \gamma_{n}^{*}$, a contradiction. Thus, take $\gamma_{n}^{\dagger}=\gamma_{n}^{*} \wedge[(@ E s \wedge(\varphi[s / x] \rightarrow \forall @ x \varphi)) \vee$ $\left.\left.\forall{ }^{-} x\right\urcorner @ E x\right]$.

We now wish to add the appropriate $\square(@ E r \rightarrow \varphi[r / x]) \rightarrow \square \forall @ x \varphi$ formula. Assuming there is no variable $r$ for which this can be added consistently, we would have, for all $r$, a finite $\Lambda \subseteq \square^{-}(\Gamma)$ s.t.

$$
\begin{aligned}
& \Lambda \vdash \gamma_{n}^{\dagger} \rightarrow \neg \square(@ E r \rightarrow \varphi[r / x]) \rightarrow \square \forall @ x \varphi \\
\Longrightarrow & \Lambda \vdash \gamma_{n}^{\dagger} \rightarrow \square(@ E r \rightarrow \varphi[r / x]) \\
\Longrightarrow & \Lambda \vdash \gamma_{n}^{\dagger} \rightarrow \neg \square \forall @ x \varphi \\
\Longrightarrow & \Lambda \vdash @ E r \rightarrow\left(\gamma_{n}^{\dagger} \rightarrow \square(@ E r \rightarrow \varphi[r / x])\right)
\end{aligned}
$$

Since $\Gamma$ is assumed to have the $\square \forall^{@}$-property, we have, for a $z$ not existing in $\gamma_{n}^{\dagger}$ or $\varphi$ :

$$
\begin{aligned}
& \Longrightarrow \square \forall @_{z}\left(\gamma_{n}^{\dagger} \rightarrow \square(@ E z \rightarrow \varphi[z / x])\right) \in \Gamma \\
& \Longrightarrow \square\left(\gamma_{n}^{\dagger} \rightarrow \forall^{@} \square(@ E z \rightarrow \varphi[z / x])\right) \in \Gamma \quad(\forall @ \text {-Distribution 2) }
\end{aligned}
$$

```
\(\Longrightarrow \square\left(\gamma_{n}^{\dagger} \rightarrow \square \forall @ z(@ E z \rightarrow \varphi[z / x])\right) \in \Gamma \quad\left(\mathrm{BF} \forall{ }^{@}\right)\)
\(\Longrightarrow \square\left(\gamma_{n}^{\dagger} \rightarrow \square\left(\forall^{@} z(@ E z \rightarrow \forall @ z \varphi[z / x])\right) \in \Gamma \quad\left(\forall^{@}\right.\right.\)-Distribution 1)
\(\Longrightarrow \square\left(\gamma_{n}^{\dagger} \rightarrow\left(\square \forall @ z @ E z \rightarrow \square \forall^{@} z \varphi[z / x]\right)\right) \in \Gamma\)
\(\Longrightarrow \square\left(\gamma_{n}^{\dagger} \rightarrow \square \forall^{@} z \varphi[z / x]\right) \in \Gamma \quad(\vdash \square \forall @ z @ E z)\)
\(\Longrightarrow \square^{-}(\Gamma) \vdash \gamma_{n}^{\dagger} \rightarrow \square \forall^{@} z \varphi[z / x]\)
```

But this implies the inconsistency of $\gamma_{n}^{\dagger}$ with $\square^{-}(\Gamma)$ since we have from above that $\Lambda \vdash \gamma_{n}^{\dagger} \rightarrow \neg \square \vdash^{@} x \varphi$, a contradiction.

Thus, let $\gamma_{n+1}=\gamma_{n}^{\dagger} \wedge \square(@ E z \rightarrow \varphi[r / x]) \rightarrow \square \forall @ x \varphi$. The rest of the proof proceeds as before.

Observation 9.9. We use $\forall^{@} B F @$ and $\forall^{@} C B F @$. Assume @ ${ }^{-}(w)$ does not have the $E \forall^{@}$-property. This would mean that for some formula $\forall^{@} x \varphi$, and every variable $y,(@ E(y) \wedge(\varphi[y / x] \rightarrow \forall @ x \varphi)) \vee \forall @ x \neg @ E(x) \notin @)^{-}(w)$.

$$
\left.\begin{array}{c}
\Longrightarrow \neg\left(\left(@ E(y) \wedge\left(\varphi[y / x] \rightarrow \forall^{@} x \varphi\right)\right) \vee \forall^{@} x \neg @ E(x)\right) \in @^{-}(w) \\
(\text { max. of @ } \\
\Longrightarrow @ \neg\left(\left({ }^{-}(w)\right)\right.
\end{array}\right)
$$

Where the last step follows via the @ axioms along with $\forall$ @ $\mathrm{BF} @$ and $\forall{ }^{@} C B F @$. This is contradictory to $w$ having the $E \forall @$-property.

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[^0]:    * I am grateful to Ed Mares, Rob Goldblatt, Max Cresswell, and two anonymous referees for incredibly helpful comments and conversations.
    ${ }^{1}$ See [7] for a clear discussion. For a wide-ranging, and incredibly clear, discussion of the types of expressivity issues we will be concerned with, one should consult Bricker [2].
    ${ }^{2}$ For the details, consult [13].

[^1]:    ${ }^{3}$ This allows one to maintain the rule of uniform substitution, which would otherwise have to be jettisoned.

[^2]:    ${ }^{4}$ I would like to thank Max Cresswell for interesting comments on this issue, and for pointing out aspects of this analysis I was overlooking or neglecting.
    ${ }^{5}$ If it is assumed that being rich logically entails existing, then (B) and (C) are logically equivalent.

[^3]:    ${ }^{6}$ Though, as will be noted later, it by no means solves all problems concerning expressivity.

[^4]:    ${ }^{7}$ The upper index " 0 " indicates that the quantified component of the logic is a free logic.
    ${ }^{8}$ The axiom system for just the propositional component will be called $\mathbf{K}+\mathbf{A}_{1}$.

[^5]:    ${ }^{9}$ Hughes and Cresswell call this $U G L \forall^{n}$.

[^6]:    ${ }^{11}$ This choice is partly motivated by the desire to keep things as general as possible and avoid any unnecessary restrictions. In a similar manner, like Kripke [15], we will allow predication over non-existent individuals. This has the added benefit of simplifying certain formal aspects (for example, substitution). In addition, though it is not our concern here, both of these assumptions allow for a more straightforward translation into two-sorted firstorder logic [1].

[^7]:    12 As discussed above, diagonal validity corresponds to what is often called real-world validity in the literature.

[^8]:    ${ }^{13}$ This is not the only way to proceed. One can also prove the truth lemma for specific pairs of worlds with the property that one of the pair serves as the actual world, though the same problem, with respect to allowing the language to vary, emerges with these approaches as well.

[^9]:    ${ }^{14}$ The definitions are given here in terms of $\mathcal{L Q}$. In subsequent sections, where we are working with languages also containing the actuality quantifiers, on should understand the definitions as being modified appropriately.

[^10]:    ${ }^{15}$ Note that we can do without analogues of the $E B R$-rules of the quantifier segment due to the validity of the $\forall @$ versions of the Barcan Schemata.

[^11]:    ${ }^{16}$ The required proof of that fact that if $w$ has the $E \forall^{@}$-property then so will @ ${ }^{-}(w)$ can be completed with just $\forall^{@} B F @$.

[^12]:    ${ }^{17}$ A more abstract, and formal, example illustrating a situation we are unable to articulate can be found in Hodes [12]:

    For all $w \in W$, there exists $w^{\prime} \in W$ s.t. $\delta\left(w^{\prime}\right) \nsubseteq \delta(w)$.
    ${ }^{18}$ However, there is also some reason to think that, when one moves beyond first-order quantification, these types of issues relating to the expressivity of our modal languages

