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TWO KINDS OF (BINARY) KRIPKE-STYLE SEMANTICS FOR THREE-VALUED LOGIC

EUNSUK YANG*

ABSTRACT

This paper deals with two sorts of binary Kripke-style semantics, i.e., algebraic and non-algebraic semantics, for three-valued logic. We first introduce three systems, their corresponding algebraic structures, and associated algebraic completeness results. We next introduce various types of algebraic and non-algebraic binary relational Kripke-style semantics.

Keywords: (binary) Kripke-style semantics, algebraic semantics, three-valued logic, fuzzy logic.

1. Introduction

The aim of this paper is to introduce two types of (binary) Kripke-style semantics, i.e., algebraic and non-algebraic semantics, for three-valued logic. We have two reasons why we consider three-valued logics and binary Kripke-style semantics. First, the logic and semantics are very simple. Namely, three-valued logic is the most simple among fuzzy logics, and binary Kripke-style semantics are also simple Kripke-style semantics. Thus, for ease and clarity we consider three-valued logic and binary semantics. Secondly, although algebraic and non algebraic Kripke-style semantics are both binary, they are quite different. That is, algebraic Kripke-style semantics is a semantics whose frames are (reducts of) corresponding algebraic structures, whereas non-algebraic Kripke-style semantics is a semantics whose frames are not (see Remark 4 below). Thus, the investigation of these two sorts of semantics can illustrate the differences between them. Therefore, we investigate the two sorts of binary Kripke-style semantics for three-valued logic.

In this paper, we introduce the well-known systems $\mathbf{Ł}_3$ (Łukasiewicz three-valued logic), $\mathbf{G}_3$ (Dummett-Gödel three-valued logic), and the

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system $\mathbf{IUML}_3$ as the three-valued extension of the fuzzy logic **IUML** (Involutive uninorm mingle logic) introduced in [13]. The system **IUML** $\mathbf{IUML}_3$ also can be regarded as a version of $\mathbf{RM}_3$ (Three-valued $\mathbf{R}$ of relevant implication with mingle), $\mathbf{RM}_3^{\mathbf{T}}$.[1]

The paper is organized as follows. First, in Section 2, we introduce these systems, their corresponding algebraic structures, and their algebraic completeness results. Next, in Section 3, we introduce one kind of binary relational Kripke-style semantics, algebraic Kripke-style semantics, for the above mentioned three-valued systems. We then connect them with algebraic semantics. Finally, in Section 4, we introduce the other kind, non-algebraic Kripke-style semantics, for the systems. To the best of our knowledge, this is the first introduction of a non-algebraic binary relational Kripke-style semantic for $\mathbf{Ł}_3$.

For convenience, we adopt the notation and terminology similar to those in [8, 13, 15, 16, 18] and assume familiarity with them (together with the results found therein).

2. Three-valued logics and algebraic semantics

2.1. Axiomatizations

We base three-valued logics on a countable propositional language with formulas *Fm* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\rightarrow, \&, \wedge, \vee$, and constants $\mathbf{f}$ and $\mathbf{F}$, with defined connectives:

df1. $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$
df2. $\neg A := A \rightarrow \mathbf{f}$.

We further define $\mathbf{t}$, $\mathbf{T}$ and $A_{\mathbf{t}}$ as $\mathbf{f} \rightarrow \mathbf{f}$, $\mathbf{F} \rightarrow \mathbf{F}$ and $A \wedge \mathbf{t}$, respectively. We use the axiom systems to provide a consequence relation.

Definition 1. *(i)* (*Cf.* [*1*, *5*, *13*]) $\boldsymbol{IUML}_3$ *consists of the following axiom schemes and rules:*

$A \rightarrow A$ *(self-implication, SI)*
$(A \wedge B) \rightarrow A, (A \wedge B) \rightarrow B$ *($\wedge$-elimination, $\wedge$-E)*

[1] As mentioned in [20], **R** has the three versions $\mathbf{R^0}$ (**R** without constants), $\mathbf{R^t}$ (**R** with constants **t** and **f**), and $\mathbf{R^T}$ (**R** with constants **t**, **f**, **T**, and **F**); therefore, **RM** has the corresponding three versions $\mathbf{RM^0}$, $\mathbf{RM^t}$, and $\mathbf{RM^T}$. Note that, while $\mathbf{RM}_3^0$, which is generally expressed as $\mathbf{RM}_3$, was already introduced, the other versions have not yet been (see [1, 5]). Thus, since the system $\mathbf{RM}_3^0$ is the famous one in the tradition of relevance logic and semantics for the three versions are very similar, here we introduce $\mathbf{IUML}_3$ as another representation of the three versions.

$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$ *($\wedge$-introduction, $\wedge$-I)*

$A \to (A \vee B)$, $B \to (A \vee B)$ *($\vee$-introduction, $\vee$-I)*

$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$ *($\vee$-elimination, $\vee$-E)*

$(A \& B) \to (B \& A)$ *(&-commutativity, &-C)*

$(A \& \boldsymbol{t}) \leftrightarrow A$ *(push and pop, PP)*

$\boldsymbol{F} \to A$ *(ex falsum quodlibet, EF)*

$A \to \boldsymbol{T}$ *(verum ex quolibet, VE)*

$(A \to (B \to C)) \leftrightarrow ((A \& B) \to C)$ *(residuation, RE)*

$(A \to B) \to ((B \to C) \to (A \to C))$ *(suffixing, SF)*

$(A \to B)_{\boldsymbol{t}} \vee (B \to A)_{\boldsymbol{t}}$ *($\boldsymbol{t}$-prelinearity, PL$_{\boldsymbol{t}}$)*

$\neg\neg A \to A$ *(double negation elimination, DNE)*

$(A \& A) \leftrightarrow A$ *(idempotence, ID)*

$t \leftrightarrow f$ *(fixed-point, FP)*

$A \to (\neg A \to A)$ *(RM3(1))*

$A \vee (A \to B)$ *(RM3(2))*

$A \to B, A \vdash B$ *(modus ponens, mp)*

$A, B \vdash A \wedge B$ *(adjunction, adj)*

(ii) (*See e.g.* [*11*, *12*]) $\boldsymbol{Ł}_3$ *consists of SI, $\wedge$-E, $\wedge$-I, $\vee$-I, $\vee$-E, &-C, PP, EF, VE, RE, SF, PL$_{\boldsymbol{t}}$, DNE, (mp), (adj), and*

$A \to (B \to A)$ *(weakening, W)*

$(A \wedge B) \to (A \& (A \to B))$ *(divisibility, DIV)*

$((A \to \neg A) \to A) \to A$ *(Ł3)*

(iii) (*See e.g.* [*11*, *12*]) $\boldsymbol{G}_3$ *consists of SI, $\wedge$-E, $\wedge$-I, $\vee$-I, $\vee$-E, &-C, PP, EF, VE, RE, SF, PL$_{\boldsymbol{t}}$, ID, W, (mp), (adj), and*

$(\neg A \to B) \to (((B \to A) \to B) \to B)$ *(G3)*

Remark 1.

*(1) By eliminating FP, RM3(1), and RM3(2) from **IUML**$_3$, we obtain the famous relevance system **RM^{T}**; by omitting RM3(1) and RM3(2) from **IUML**$_3$, Ł3 from **Ł**$_3$, and G3 from **G**$_3$, we get the famous fuzzy systems **IUML**, **Ł** (Łukasiewicz infinite-valued logic), and **G** (Dummett-Gödel infinite-valued logic), respectively. These systems are all axiomatic extensions of the uninorm logic **UL** (see [12, 13]).*

*(2) In the systems **Ł**$_3$ and **G**$_3$, the constants **t**, **f** are the same as **T** and **F**, respectively. The system **IUML**$_3$ is the **RM**$_3^0$ expanded with constants **t**, **f**, **T**, **F** and corresponding axioms.*

For easy reference, we let Ls_3 be the set of the three-valued systems introduced in Definition 1.

Definition 2. $Ls_3 = \{\boldsymbol{IUML}_3, \boldsymbol{Ł}_3, \boldsymbol{G}_3\}$.

A *theory* is a set of formulas closed under consequence relation. A *proof* in a theory Γ over L_3 ($\in Ls_3$) is a sequence s of formulas such that each element of s is either an axiom of L_3, a member of Γ, or is derivable from previous elements of s by means of a rule of L_3. $\Gamma \vdash A$, more exactly $\Gamma \vdash_{L_3} A$, means that A is *provable* in Γ with respect to (w.r.t.) L_3, i.e., there is an L_3-proof of A in Γ. A theory Γ is *trivial* if $\Gamma \vdash \mathbf{F}$; otherwise, it is *non-trivial*.

The deduction theorems for L_3 are as follows:

Proposition 1. *Let Γ be a theory over L_3 and A, B be formulas.*

(i) $\Gamma \cup \{A\} \vdash_{\boldsymbol{IUML}_3} B$ *iff* $\Gamma \vdash_{\boldsymbol{IUML}_3} A_{\boldsymbol{t}} \to B$.

(ii) $\Gamma \cup \{A\} \vdash_{\boldsymbol{Ł}_3} B$ *iff there is n, a positive integer, such that* $\Gamma \vdash_{\boldsymbol{Ł}_3} A^n \to B$.

(iii) $\Gamma \cup \{A\} \vdash_{\boldsymbol{G}_3} B$ *iff* $\Gamma \vdash_{\boldsymbol{G}_3} A \to B$.

Proof. For (i) to (iii), see [7, 12]. □

The following formulas can be proved straightforwardly.

Proposition 2.

(i) L_3 ($\in Ls_3$) *proves:*
- *(1)* $(A\&(B\&C)) \to ((A\&B)\&C)$ *(associativity, AS)*
- *(2)* $(A \to B) \vee (B \to A)$ *(prelinearity, PL)*
- *(3)* $(A \to B) \to (\neg B \to \neg A)$ *(contraposition, CP)*

(ii) $L_3 \in \{\boldsymbol{IUML}_3, \boldsymbol{Ł}_3\}$ *proves:*
- *(1)* $\neg\neg A \leftrightarrow A$ *(double negation,DN)*

(iii) $L_3 \in \{\boldsymbol{G}_3, \boldsymbol{Ł}_3\}$ *proves:*
- *(1)* $\boldsymbol{t} \leftrightarrow \boldsymbol{T}$ *(INT)*

(iv) $\boldsymbol{Ł}_3$ *proves:*
- *(1)* $\neg(A \to B) \to (A \wedge \neg B)$ *(negated implication, nI)*

2.2. Algebraic semantics

Suitable algebraic structures for L_3 ($\in Ls_3$) are obtained as varieties of residuated lattices in the sense of [10].

Definition 3.

(i) A pointed bounded commutative residuated lattice *is a structure* $(A, \top, \bot, t, f, \wedge, \vee, *, \to)$ *such that:*
- *(I)* $(A, \top, \bot, \wedge, \vee)$ *is a bounded lattice with top element* $\top$ *and bottom element* $\bot$.
- *(II)* $(A, *, t)$ *is a commutative monoid.*
- *(III)* $y \leq x \to z$ *iff* $x * y \leq z$, *for all* $x, y, z \in A$ *(residuation).*

(IV) f is an element of A.

(ii) (*UL-algebra*) *Let $x_t := x \wedge t$. A* UL-algebra *is a pointed bounded commutative residuated lattice satisfying the condition: for all x, $y \in A$,*

$(PL_{\boldsymbol{t}}^{\mathcal{A}})$ $t \leq (x \to y)_t \vee (y \to x)_t$.

(iii) (*MTL-algebra*) *An* MTL-algebra *is a UL-algebra satisfying the condition:*

$(INT^{\mathcal{A}})$ $t = \top$.

A pointed commutative residuated lattice is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y. We define the unary operator $\neg$ as follows: $\neg x := x \to f$.

For convenience, '$\neg$,' '$\to$,' '$\wedge$,' and '$\vee$' are used ambiguously as propositional connectives and as algebraic operators, but context should clarify their meanings.

Definition 4. (*L_3-algebras*) *We call the following algebras* L_3-algebras.

(i) An $IUML_3$-algebra is a UL-algebra satisfying the following conditions:

$(DN^{\mathcal{A}})$	$\neg\neg x = x$
$(ID^{\mathcal{A}})$	$x * x = x$
$(FP^{\mathcal{A}})$	$t = f$
$(RM3(1)^{\mathcal{A}})$	$x \leq \neg x \to x$
$(RM3(2)^{\mathcal{A}})$	$t \leq x \vee (x \to y)$.

(ii) An Ł$_3$-algebra is an MTL-algebra satisfying $(DN^{\mathcal{A}})$ and the the following conditions:

$(DIV^{\mathcal{A}})$	$x \wedge y \leq x * (x \to y)$
$(Ł_3^{\mathcal{A}})$	$(x \to \neg x) \to x \leq x$.

(iii) A G_3-algebra is an MTL-algebra satisfying $(ID^{\mathcal{A}})$ and the following condition:

$(G_3^{\mathcal{A}})$	$\neg x \to y \leq ((y \to x) \to y) \to y$.

Definition 5. (*Evaluation*) *Let $\mathcal{A}$ be an L_3-algebra. An* $\mathcal{A}$-evaluation *is a function $v : Fm \to \mathcal{A}$ satisfying: $v(A \to B) = v(A) \to v(B)$, $v(A \wedge B) = v(A) \wedge v(B)$, $v(A \vee B) = v(A) \vee v(B)$, $v(A \& B) = v(A) * v(B)$, $v(\boldsymbol{F}) = \bot$, $v(\boldsymbol{f}) = f$, (and hence $v(\boldsymbol{t}) = t$, $v(\boldsymbol{T}) = \top$ and $v(\neg A) = \neg v(A)$).*

Definition 6. ([4]) *Let $\mathcal{A}$ be an L_3-algebra, Γ a theory, A a formula, and K a class of L_3-algebras.*

(i) (*Tautology*) *A is a* t-tautology *in $\mathcal{A}$, briefly an* $\mathcal{A}$-tautology *(or $\mathcal{A}$-valid), if $v(A) \geq t$ for each $\mathcal{A}$-evaluation v.*

(ii) (*Model*) *An $\mathcal{A}$-evaluation v is an $\mathcal{A}$-model of Γ if $v(A) \geq t$ for each $A \in \Gamma$. By $Mod(\Gamma, \mathcal{A})$, we denote the class of $\mathcal{A}$-models of Γ.*

(iii) (*Semantic consequence*) *A is a* semantic consequence *of* Γ *w.r.t.* K, *denoted by* $\Gamma \models_{\mathsf{K}} A$, *if* $Mod(\Gamma, \mathcal{A}) = Mod(\Gamma \cup \{A\}, \mathcal{A})$ *for each* $\mathcal{A} \in \mathsf{K}$.

Definition 7. (*$\mathbf{L}_3$-algebra,* [4]) *Let $\mathcal{A}$, Γ, and A be as in Definition 6. $\mathcal{A}$ is an* $\mathbf{L}_3$-algebra *iff whenever A is L_3-provable in any Γ (i.e. $\Gamma \vdash_{L_3} A$, L_3 an L_3 logic), it is a semantic consequence of Γ w.r.t. $\{\mathcal{A}\}$ (i.e. $\Gamma \models_{\{\mathcal{A}\}} A$, $\mathcal{A}$ a corresponding L_3-algebra). By* $\mathrm{MOD}(\mathrm{L}_3)$, *we denote the class of $\mathbf{L}_3$-algebras; by* $\mathrm{MOD}^l(\mathrm{L}_3)$, *the class of linearly ordered $\mathbf{L}_3$-algebras. Finally, we write $\Gamma \models_{L_3} A$ and $\Gamma \models^l_{L_3} A$ in place of $\Gamma \models_{MOD(L_3)} A$ and $\Gamma \models_{MOD^l(L_3)} A$, respectively.*

Note that since each condition for an L_3-algebra has the form of an equation or can be defined in an equation, it can be ensured that the classes of all L_3-algebras are varieties.

We first show that classes of provably equivalent formulas form an $\mathbf{L}_3$-algebra. Let Γ be a fixed theory over L_3. For each formula A, let $[A]_\Gamma$ be the set of all formulas B such that $\Gamma \vdash_{L_3} A \leftrightarrow B$ (formulas Γ-provably equivalent to A). A_Γ is the set of all the classes $[A]_\Gamma$. We define that $[A]_\Gamma \to [B]_\Gamma = [A \to B]_\Gamma$, $[A]_\Gamma * [B]_\Gamma = [A\&B]_\Gamma$, $[A]_\Gamma \wedge [B]_\Gamma = [A \wedge B]_\Gamma$, $[A]_\Gamma \vee [B]_\Gamma = [A \vee B]_\Gamma$, $\bot = [\mathbf{F}]_\Gamma$, $f = [\mathbf{f}]_\Gamma$, (and so $t = [\mathbf{t}]_\Gamma$, $\top = [\mathbf{T}]_\Gamma$ and $\neg[A]_\Gamma = [\neg A]_\Gamma$). By $\mathcal{A}_\Gamma$, we denote this algebra, i.e., Lindenbaum-Tarski algebra.

Proposition 3. *For Γ, a theory over L_3, $\mathcal{A}_\Gamma$ is an $\mathbf{L}_3$-algebra.*

Proof. Note that SI, $\wedge$-E, $\wedge$-I, $\vee$-I, $\vee$-E, EF, and VE ensure that $\wedge$ and $\vee$ satisfy (I) in Definition 3; that &-C, PP, and AS ensure that (II) holds; that RE and $\mathrm{PL}_{\mathbf{t}}$ ensure that (III) and ($PL^{\mathcal{A}}_{\mathbf{t}}$) hold; that the constant $\mathbf{f}$ ensures (IV) holds. The additional axioms for L_3 ensure that the corresponding algebraic conditions hold. It is obvious that $[A]_\Gamma \leq [B]_\Gamma$ iff $\Gamma \vdash_{L_3} A \leftrightarrow (A \wedge B)$ iff $\Gamma \vdash_{L_3} A \to B$. Finally, recall that $\mathcal{A}_\Gamma$ is an $\mathbf{L}_3$-algebra iff $\Gamma \vdash_{L_3} B$ implies $\Gamma \models_{L_3} B$, and observe that, for A in Γ, since $\Gamma \vdash_{L_3} \mathbf{t} \to A$, it follows that $[\mathbf{t}]_\Gamma \leq [A]_\Gamma$. Thus, $\mathcal{A}_\Gamma$ is an $\mathbf{L}_3$-algebra. □

Proposition 4. (*Cf.* [17]) *Each L_3-algebra is a subdirect product of linearly ordered L_3-algebras.*

Theorem 1. (*Strong completeness*) *Let Γ be a theory over L_3 and A a formula. $\Gamma \vdash_{L_3} A$ iff $\Gamma \models_{L_3} A$ iff $\Gamma \models^l_{L_3} A$.*

Proof. We first prove that $\Gamma \vdash_{L_3} A$ iff $\Gamma \models_{L_3} A$. The left-to-right direction follows from the Definition 7 and Proposition 3. The right-to-left direction is as follows: From Proposition 3, it follows that $\mathcal{A}_\Gamma \in MOD(L_3)$, and so for $\mathcal{A}_\Gamma$-evaluation v defined as $v(B) = [B]_\Gamma$, $v \in Mod(\Gamma, \mathcal{A}_\Gamma)$. Thus, since from $\Gamma \models_{L_3} A$, we can obtain $[A]_\Gamma = v(A) \geq t$, it holds that $\Gamma \vdash_{L_3} \mathbf{t} \to A$. Then, since $\Gamma \vdash_{L_3} \mathbf{t}$, we have $\Gamma \vdash_{L_3} A$, as required. That $\Gamma \models_{L_3} A$ iff $\Gamma \models^l_{L_3} A$ follows from Proposition 4. (Note that w.r.t. any many-valued logic with a set of finite values, the compactness theorem holds (see Theorem 3.2.5 in [11]). Thus, in $\mathbf{Ł}_3$, we do not need to restrict a theory to be finite.) □

Remark 2. *Let Ls = {**IUML, Ł, G**}. The system L (∈ Ls) is obtained from L_3 by eliminating the corresponding three-valued axiom scheme(s). Then, analogously, we can define L-algebras and then establish algebraic completeness for L. Note that systems in Ls are famous fuzzy logics.*

3. Kripke-style semantics (I)

3.1. Algebraic and binary relational semantics

Here, we consider a particular kind of binary relational Kripke-style semantics, which we shall call *algebraic* Kripke-style semantics, for L_3.

Definition 8.

- *(i) (Kripke frame) A* Kripke frame *is a structure* $\mathcal{X} = (X, \leq)$ *such that* $(X, \leq)$ *is a partially ordered set. The elements of* $\mathcal{X}$ *are called* nodes.
- *(ii) (Algebraic Kripke frame) An* algebraic Kripke frame *is a Kripke frame* $\mathcal{X} = (X, \top, \bot, t, f, \leq, *)$ *such that* $(X, \top, \bot, \leq)$ *is a bounded linearly ordered set with top and bottom elements* $\top$, $\bot$, *and* $(X, t, f, \leq, *)$ *is a linearly ordered pointed commutative monoid satisfying that for all* $x, y \in X$, *the set* $\{z : z * x \leq y\}$ *has a supremum, denoted by* $x \to y$. *This monoid is called* residuated.
- *(iii) (L frame) An* IUML *frame is an algebraic Kripke frame satisfying* $(DN^{\mathcal{A}})$, $(ID^{\mathcal{A}})$ *and* $(FP^{\mathcal{A}})$; *An* Ł *frame is an algebraic Kripke frame satisfying* $(INT^{\mathcal{A}})$ $t = \top$, $(DN^{\mathcal{A}})$, *and* $(DIV^{\mathcal{A}})$; *A* G frame *is an algebraic Kripke frame satisfying* $(INT^{\mathcal{A}})$ *and* $(ID^{\mathcal{A}})$. *By an* L *frame, we ambiguously denote any of these frames.*
- *(iv) (L_3 frame) An* L_3 *frame is an L frame where X consists of three elements, i.e.,* $X = \{\top, x, \bot\}$. *By* X_3, *we denote such X.*

It may be useful to point out that Kripke's semantics for modal logics were not defined on ordered frames with further operators. In the case of the modal system **S4**, it is the order relation itself from which the modal operator is defined.

An *evaluation* on an algebraic Kripke frame is a relation $\Vdash$ between nodes and propositional variables, and arbitrary formulas subject to the conditions below: For every propositional variable p,

(Atomic Hereditary Condition, AHC)	if $x \Vdash p$ and $y \leq x$, then $y \Vdash p$;
(min)	$\bot \Vdash p$; and

for arbitrary formulas,

(t) $x \Vdash \mathbf{t}$ iff $x \le t$;
(f) $x \Vdash \mathbf{f}$ iff $x \le f$;
($\bot$) $x \Vdash \mathbf{F}$ iff $x = \bot$;
($\wedge$) $x \Vdash A \wedge B$ iff $x \Vdash A$ and $x \Vdash B$;
($\vee$) $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$;
(&) $x \Vdash A\&B$ iff there are $y, z \in X$ such that $y \Vdash A$, $z \Vdash B$, and $x \le y * z$;
($\to$) $x \Vdash A \to B$ iff for all $y \in X$, if $y \Vdash A$, then $x * y \Vdash B$.

Definition 9.

(i) (*Algebraic Kripke model*) *An* algebraic Kripke model *is a pair* $(\mathcal{X}, \Vdash)$, *where* $\mathcal{X}$ *is an algebraic Kripke frame and* $\Vdash$ *is an evaluation on* $\mathcal{X}$.
(ii) (*L model*) *An* L model *is a pair* $(\mathcal{X}, \Vdash)$, *where* $\mathcal{X}$ *is an L frame and* $\Vdash$ *is an evaluation on* $\mathcal{X}$.
(iii) (*L_3 model*) *An* L_3 model *is a pair* $(\mathcal{X}, \Vdash)$, *where* $\mathcal{X}$ *is an* L_3 *frame and* $\Vdash$ *is an evaluation on* $\mathcal{X}$.

Definition 10. (*Cf.* [16]) *Given an* L_3 *model* $(\mathcal{X}, \Vdash)$, *a node* x *of* $\mathcal{X}$ *and a formula* A, *we say that* x makes A true *to express* $x \Vdash A$. *We say that* A *is* true *in* $(\mathcal{X}, \Vdash)$ *if* $t \Vdash A$, *and that* A *is* valid *in the frame* $\mathcal{X}$ (*expressed by* $\mathcal{X} \vDash A$) *if* A *is true in* $(\mathcal{X}, \Vdash)$ *for every evaluation* $\Vdash$ *on* $\mathcal{X}$.

Definition 11. *An* L_3 *frame* $\mathcal{X}$ *is an* $\mathbf{L}_3$ *frame iff all axioms of* L_3 *are valid in* $\mathcal{X}$. *We say that an* L_3 *model* $(\mathcal{X}, \Vdash)$ *is an* $\mathbf{L}_3$ model *if* $\mathcal{X}$ *is an* $\mathbf{L}_3$ *frame.*

3.2. Soundness and completeness for $\mathbf{L}_3$

First, we introduce the following lemma.

Lemma 1.

(i) (*Hereditary Lemma, HL*) *Let* $\mathcal{X}$ *be an* L_3 *frame. For any sentence* A *and for all nodes* $x, y \in \mathcal{X}$, *if* $x \Vdash A$ *and* $y \le x$, *then* $y \Vdash A$.
(ii) *Let* $\Vdash$ *be an evaluation on an* L_3 *frame and* A *a sentence. Then the set* $\{x \in X : x \Vdash A\}$ *has a maximum.*
(iii) $\top \Vdash A \to B$ *iff for all* $x \in X$, *if* $x \Vdash A$, *then* $x \Vdash B$.

Proof. *Easy.* □

Proposition 5. *(Soundness) If* $\vdash_{L_3} A$, *then* A *is valid in every* L_3 *frame.*

Proof. Since X_3 in $\mathcal{X}$ is $\{1, \frac{1}{2}, 0\}$ (up to isomorphism), We henceforth regard X_3 as the set $\{1, \frac{1}{2}, 0\}$. We prove (Ł3) as an example: It suffices to show that, for all $x \in X_3$ such that $x \Vdash (A \to \neg A) \to A$, $x \Vdash A$. Let $x \Vdash (A \to \neg A) \to A$. By ($\to$), we have, for all $y \in X_3$ such that $y \Vdash A \to \neg A$, $x * y \Vdash A$. Suppose toward contradiction that $x \nVdash A$. Then $x = 1$ or $x = \frac{1}{2}$.

Let x be 1. By the supposition, we get $1 \not\Vdash A \to \neg A$. But this cannot be the case because if $1 \not\Vdash A$ and $\frac{1}{2} \Vdash A$, then $\frac{1}{2} \Vdash \neg A$ and so $1 \Vdash A \to \neg A$, a contradiction. Let x be $\frac{1}{2}$. By the supposition, we have $\frac{1}{2} \not\Vdash A \to \neg A$. But since $\frac{1}{2} \not\Vdash A$, it holds that $0 \Vdash A$ and so $\frac{1}{2} \Vdash A \to \neg A$, a contradiction.

The proof for the other cases is left to the interested reader. □

This proposition ensures that L_3 frames are $\mathbf{L}_3$ frames. Moreover, the next proposition connects L_3 semantics and algebraic semantics (cf. see [16]).

Proposition 6.

(i) *The* $\{\top, \bot, \leq, *, \to\}$ *reduct of a linearly ordered* L_3*-algebra* $\mathcal{A}$ *is an* L_3 *frame.*

(ii) *Let* $\mathcal{X} = (X, \top, \bot, \leq, *, \to)$ *be an* L_3 *frame. Then the structure* $\mathcal{A} = (X, \top, \bot, max, min, *, \to)$ *is an* L_3*-algebra (where* max *and* min *are meant w.r.t.* $\leq$*).*

(iii) *Let* $\mathcal{X}$ *be the* $\{\top, \bot, \leq, *, \to\}$ *reduct of a linearly ordered* L_3*-algebra* $\mathcal{A}$*, and let* v *be an evaluation in* $\mathcal{A}$*. Let, for every atomic formula* p *and for every* $x \in \mathcal{A}$*,* $x \Vdash p$ *iff* $x \leq v(p)$*. Then* $(\mathcal{X}, \Vdash)$ *is an* L_3 *model, and for every formula* A *and for every* $x \in \mathcal{A}$*, we obtain* $x \Vdash A$ *iff* $x \leq v(A)$*.*

(iv) *Let* $(\mathcal{X}, \Vdash)$ *be an* L_3 *model, and let* $\mathcal{A}$ *be the* L_3*-algebra defined as in (ii). Define, for every atomic formula* p*,* $v(p) = max\{x \in X : x \Vdash p\}$*. Then, for every formula* A*,* $v(A) = max\{x \in X : x \Vdash A\}$*.*

Proof. The proofs for (i) and (ii) are easy. Since (iv) follows almost directly from (iii) and Lemma 1 (ii), we prove (iii). With regard to claim (iii), we consider the induction steps corresponding to the cases where $A = B\&C$ and $A = B \to C$:

Consider the case where $A = B\&C$. By the condition (&), $x \Vdash B\&C$ iff there exist y, z such that $y \Vdash B$, $z \Vdash C$, and $x \leq y * z$. Then, by the induction hypothesis, $y \Vdash B$ and $z \Vdash C$ iff $y \leq v(B)$ and $z \leq v(C)$. This implies that $x \leq y * z \leq v(B) * v(C) = v(B\&C)$. Conversely, if $x \leq v(B) * v(C) = v(B\&C)$, then, letting $y = v(B)$ and $z = v(C)$, we obtain $x \leq y * z$, $y \Vdash B$, and $z \Vdash C$, therefore $x \Vdash B\&C$.

Consider the case where $A = B \to C$. By the condition ($\to$), we have that $x \Vdash B \to C$ iff, for all $y \in X$ such that $y \Vdash B$, $x * y \Vdash C$. Then, since by the induction hypothesis $y \Vdash B$ only if $x * y \Vdash C$ iff $y \leq v(B)$ only if $x * y \leq v(C)$, it holds true that, for any $y \in X$ such that $y \Vdash B$, $x * y \Vdash C$ iff $x * v(B) \leq v(C)$, therefore iff $x \leq v(B) \to v(C) = v(B \to C)$, as desired. □

Proposition 7. *Let* $\mathcal{X} = (X, \top, \bot, \leq, *, \to)$ *be an L frame, and let (*L_3*) be the corresponding axiom for three-valuedness in Definition 1 and (*L_3*)*$_{\mathcal{F}}$ *the corresponding property of an* L_3 *frame. Then,* $(\diamond)$ $\mathcal{X} \models (L_3)$ *iff* $\mathcal{X}$ *satisfies (*L_3*)*$_{\mathcal{F}}$*.*

Proof. By Proposition 6, in order to prove ($\diamond$), it suffices to show that a linearly ordered L-algebra is an L_3-algebra iff it satisfies $(L_3)_{\mathcal{F}}$. As an example, we prove that $\mathcal{X} \models$ (Ł3) iff $\mathcal{X}$ satisfies $(Ł3)_{\mathcal{F}}$. The right-to-left direction is obvious. For the left-to-right direction, let $(x \to \neg x) \to x > x$. Then, it is obvious that an Ł-algebra $\mathcal{A}$ is not an $Ł_3$-algebra, as desired.

The proof for the other cases is left to the interested reader. □

Theorem 2. (*Strong completeness*) *L_3 is strongly complete w.r.t. the class of all L_3 frames.*

Proof. It follows from Propositions 6 and 7 and Theorem 1. □

Remark 3. *The introduction of L_3 semantics (as algebraic Kripke-style semantics) for L_3 is a generalization of that of L semantics (as algebraic Kripke-style semantics) for L in [15, 16]. Note that here we do not establish completeness for L using the class of all L frames since algebraic and binary relational Kripke-style semantics for* **Ł** *and* **G** *were already introduced in [15, 16], and that for* ***IUML*** *is almost immediate from that for* ***UL*** *in [19].*

4. Kripke-style semantics (II)

4.1. Non-algebraic and binary relational semantics

4.1.1. *Semantics for* $\mathbf{L}_N$

Here we consider non-algebraic and binary relational Kripke-style semantics for $L_N \in \{\mathbf{G}_3, \mathbf{Ł}_3\}$. Let us regard an 'evaluation' to be a function from sentences to sets of two truth values, including the empty set of truth values

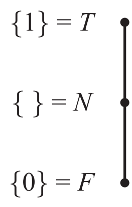

Figure 1: The lattice $\mathbf{3}_N$

$\neg$	
T^+	F
N	F
F	T

$\sim$	
T^+	F
N	N
F	T

$\wedge$	T^+	N	F
T^+	T	N	F
N	N	N	F
F	F	F	F

$\vee$	T^+	N	F
T^+	T	T	T
N	T	N	N
F	T	N	F

$\to_{G_3}$	T^+	N	F
T^+	T	N	F
N	T	T	F
F	T	T	T

$\to_{Ł_3}$	T^+	N	F
T^+	T	N	F
N	T	T	N
F	T	T	T

Table 1: Three-valued matrices for evaluations of L_N

to account for underdetermination. We regard a three-valued matrix as a lattice and call it the *lattice $\mathbf{3}_N$*; we denote each set of value(s) $\{0\}$, $\{1\}$, and $\{\ \}$ by *F*, *T*, and *N*, respectively (see Figure 1). In order to distinguish the negation of $\mathbf{Ł}_3$ from that of $\mathbf{G}_3$, we express the former negation as $\sim$ and the latter one as $\neg$. Each matrix for $\neg$, $\sim$, $\wedge$, $\vee$, and $\to$ can be defined as in Table 1 (+ indicates the designated value).[2] Note that, in Table 1, we take $\to_{G_3}$ and $\to_{Ł_3}$ for $\mathbf{G}_3$ and $\mathbf{Ł}_3$, respectively.

Next, as in [8], let us define evaluations. An *evaluation* into $\mathbf{3}_N$ is a function v from sentences into $\mathbf{3}_N$ such that $v(\neg A) = \neg v(A)$, $v(\sim A) = \sim v(A)$, $v(A \wedge B) = v(A) \wedge v(B)$, $v(A \vee B) = v(A) \vee v(B)$, and $v(A \to B) = v(A) \to v(B)$. As the labeling of Figure 1 reveals, we can view $\mathbf{3}_N$ as consisting of subsets of the usual two truth values. Thus, equivalently, an evaluation can be regarded as a map v from sentences into the powerset of $\{1, 0\}$ (see below). For a *functional evaluation*, we never have both $0, 1 \in v(A)$. We write $\Vdash_1^v A$ for $1 \in v(A)$ and $\Vdash_0^v A$ for $0 \in v(A)$. Like the two-valued matrix for classical logic CL, we call a matrix *characteristic* for a calculus $\mathbf{L}$ when a formula A is provable if it assumes a designated value for every assignment of values to its variables, i.e., if $\mathbf{L}$ is weak complete w.r.t. the matrix (see e.g. [8, 9]).

Definition 12. ([8]) *A* binary relational Kripke frame *(briefly a* frame*) is a structure $\boldsymbol{S} = (U, \zeta, \sqsubseteq)$, where $\zeta \in U$ and $\sqsubseteq$ is a partial order (p.o.) on U.*

As X in Section 3, we regard U as a set of nodes. Then, ζ is the base state of information, and it further does not hurt to require that ζ be the least element of U under $\sqsubseteq$. By Σ, we denote the class of all frames. For L_N, we need to consider frames where $\sqsubseteq$ is *connected* in the sense that, for any α, $\beta \in U$, either $\alpha \sqsubseteq \beta$ or $\beta \sqsubseteq \alpha$. A *linear order* (l.o.) is a connected partial order. Then a *linear frame* is a structure $\mathbf{S} = (U, \zeta, \sqsubseteq)$, where $\zeta \in U$ and $\sqsubseteq$ is an l.o. on U.

We assume that there are denumerably many atomic sentences, and that the class of formulas *Fm* is defined inductively from these in the usual manner, utilizing the connectives $\neg$, $\sim$, $\wedge$, $\vee$, and $\to$. A (*parameterized*) *L_N-evaluation* on a linear frame $\mathbf{S}$ is a function $v(A, \alpha)$ from $Fm \times U$ into $\mathbf{3}_N$ subject to the conditions below. We denote the set of these evaluations as $\mathbf{Val}_{L_N}$, and we write $\alpha \Vdash_1^v A$ for $1 \in v(A, \alpha)$ and $\alpha \Vdash_0^v A$ for $0 \in v(A, \alpha)$. In context, we often leave the superscript v implicit.

Atomic Hereditary Conditions (AHC) for any atomic sentence p,

(HC$_1$) $\alpha \Vdash_1^v p$ and $\alpha \sqsubseteq \beta \Longrightarrow \beta \Vdash_1^v p$;

(HC$_0$) $\alpha \Vdash_0^v p$ and $\alpha \sqsubseteq \beta \Longrightarrow \beta \Vdash_0^v p$.

[2] We do not have to introduce the matrix for & because & is $\wedge$ in $\mathbf{G}_3$, and & is definable in $\mathbf{Ł}_3$ (and $\mathbf{IUML}_3$ in Section 4.1.2) using their respective $\sim$ and $\to$ connectives.

Truth and falsity conditions for compound sentences are then given by the following clauses:

($\neg_1$) $\alpha \Vdash_1 \neg A \Longleftrightarrow \alpha \Vdash_0 A$;
($\neg_0$) $\alpha \Vdash_0 \neg A \Longleftrightarrow$ for all $\beta \sqsupseteq \alpha$, $\beta \nVdash_0 A$;
($\sim_1$) $\alpha \Vdash_1 {\sim}A \Longleftrightarrow \alpha \Vdash_0 A$;
($\sim_0$) $\alpha \Vdash_0 {\sim}A \Longleftrightarrow \alpha \Vdash_1 A$;
($\wedge_1$) $\alpha \Vdash_1 A \wedge B \Longleftrightarrow \alpha \Vdash_1 A$ and $\alpha \Vdash_1 B$;
($\wedge_0$) $\alpha \Vdash_0 A \wedge B \Longleftrightarrow \alpha \Vdash_0 A$ or $\alpha \Vdash_0 B$;
($\vee_1$) $\alpha \Vdash_1 A \vee B \Longleftrightarrow \alpha \Vdash_1 A$ or $\alpha \Vdash_1 B$;
($\vee_0$) $\alpha \Vdash_0 A \vee B \Longleftrightarrow \alpha \Vdash_0 A$ and $\alpha \Vdash_0 B$;
($\rightarrow_1$) $\alpha \Vdash_1 \mathrm{A} \rightarrow \mathrm{B} \Longleftrightarrow$ (i) for all $\beta \sqsupseteq \alpha$, ($\beta \Vdash_1 \mathrm{A} \Longrightarrow \beta \Vdash_1 \mathrm{B}$), and
(ii) for all $\beta \sqsupseteq \alpha$, ($\beta \Vdash_0 \mathrm{B} \Longrightarrow \beta \Vdash_0 \mathrm{A}$);
($\rightarrow_{0G_3}$) $\alpha \Vdash_0 A \rightarrow B \Longleftrightarrow \alpha \Vdash_0 \neg A$, i.e., for all $\beta \sqsupseteq \alpha$, $\beta \nVdash_0 A$, and $\alpha \Vdash_0 B$;
($\rightarrow_{0Ł_3}$) $\alpha \Vdash_0 A \rightarrow B \Longleftrightarrow$ (i) $\alpha \Vdash_1 A$ and $\alpha \Vdash_0 B$, and
(ii) $\alpha \Vdash_1 \sim (A \rightarrow B)$.

Note that, w.r.t. the truth condition of implication, we take ($\rightarrow_1$) for L_N, but w.r.t. the falsity condition of implication, we take ($\rightarrow_{0G_3}$) and ($\rightarrow_{0Ł_3}$) for $\mathbf{G}_3$ and $\mathbf{Ł}_3$, respectively. More exactly, the $\mathbf{G}_3$-evaluation has the conditions ($\neg_1$), ($\neg_0$), ($\wedge_1$), ($\wedge_0$), ($\vee_1$), ($\vee_0$), ($\rightarrow_1$), and ($\rightarrow_{0G_3}$); the $\mathbf{Ł}_3$-evaluation has the conditions ($\sim_1$), ($\sim_0$), ($\wedge_1$), ($\wedge_0$), ($\vee_1$), ($\vee_0$), ($\rightarrow_1$), and ($\rightarrow_{0Ł_3}$).

A sentence A is L_N-*valid* in a frame $\mathbf{S} = (U, \zeta, \sqsubseteq)$ iff, for all $v \in \mathbf{Val}_{L_N}$, $\zeta \Vdash_1^v A$. Let Θ be the class of linear frames. A sentence A is L_N-*valid*, in symbols $\vDash_{L_N} A$, iff, for all $\mathbf{S} \in \Theta$, A is L_N-*valid* in $\mathbf{S}$.

Given a class of models $\mathbf{M}_{L_N}$ for L_N, we can define (simple truth preserving, corresponding to $\vDash_1$,) consequence as follows:

Definition 13. $\Gamma \vDash_{L_N} A$ *iff, for all models* $\mathfrak{M} = (U, \zeta, \sqsubseteq, v) \in \boldsymbol{M}_{L_N}$, *if* $\zeta \Vdash_1^v B$ *for all* $B \in \Gamma$, *then* $\zeta \Vdash_1^v A$.

4.1.2. *Semantics for* $\boldsymbol{L}_B$

Here we consider a non-algebraic and binary relational Kripke-style semantics for $\mathbf{IUML}_3$. In order to contrast $\mathbf{IUML}_3$ with L_N, we denote $\mathbf{IUML}_3$ as L_B.

$\rightarrow_{RM3}$	T^+	B^+	F
T^+	T	F	F
B^+	T	B	F
F	T	T	T

Table 2: Three-valued implication matrix for evaluation of L_B

Let us regard an evaluation to be a function from sentences to non-empty sets of two truth values, including the set having both truth values to account for overdetermination. We regard a three-valued matrix as a lattice and call it the *lattice* $\mathbf{3}_B$; we denote each set of value(s) $\{0\}$, $\{1\}$, and $\{0, 1\}$ by *F*, *T*, and *B*, respectively. Note that $\mathbf{3}_B$ is the same as $\mathbf{3}_N$ in Section 4.1.1 except that the former takes *B* in place of *N* (of the latter) as the intermediate or third value (cf. see Figure 1).

First, note that, as the negation of Ł_3 in Section 4.1.1, we express the negation of L_B as $\sim$ because these two systems have involutive negation different from the negation of $\mathbf{G}_3$. Each matrix for $\sim$, $\wedge$, and $\vee$ is the same as $\sim$, $\wedge$, and $\vee$, respectively, in Table 1, (but with *B* in place of *N*); the matrix for $\rightarrow_{RM3}$, i.e., $\rightarrow$ for $\mathbf{IUML}_3$, can be defined as in Table 2 (+ indicates the designated values). Note that L_B has two designated values, i.e., *T* and *B* (and thus we must put B^+ (in place of *N*) in the tables for $\sim$, $\wedge$, and $\vee$).

Next, as in Section 4.1.1, we can define evaluations. An *evaluation* into $\mathbf{3}_B$ is a function v from sentences into $\mathbf{3}_B$ such that $v(\mathbf{f}) = f$, $v(\mathbf{F}) = \bot$, $v(\sim A) = \sim v(A)$, $v(A \wedge B) = v(A) \wedge v(B)$, $v(A \vee B) = v(A) \vee v(B)$, and $v(A \rightarrow B) = v(A) \rightarrow v(B)$. This definition is almost the same as the evaluation into $\mathbf{3}_N$ in Section 4.1.1. Note, however, that L_B has a total evaluation (in place of a functional evaluation of L_N). For a *total evaluation*, we always have at least one of $0, 1 \in v(A)$. In an analogy to the definitions of a frame for L_N, we can define a frame of Kripke-style semantics for L_B.

An *L_B-evaluation* on a linear frame $\mathbf{S}$ is the same as an L_N-evaluation except the truth and falsity conditions for propositional constants $\mathbf{t}$, $\mathbf{f}$, $\mathbf{T}$, $\mathbf{F}$, and the falsity condition $(\rightarrow_{0RM3})$ for L_B.

(tf_1)	$\alpha \Vdash_1 \mathbf{t} \Longleftrightarrow \alpha \Vdash_1 \mathbf{f}$;
(tf_0)	$\alpha \Vdash_0 \mathbf{t} \Longleftrightarrow \alpha \Vdash_0 \mathbf{f}$;
$(\top_1)$	$\alpha \Vdash_1 \mathbf{T}$ always;
$(\top_0)$	$\alpha \Vdash_0 \mathbf{T}$ never;
$(\bot_1)$	$\alpha \Vdash_1 \mathbf{F}$ never;
$(\bot_0)$	$\alpha \Vdash_0 \mathbf{F}$ always;
$(\rightarrow_{0RM3})$	$\alpha \Vdash_0 \mathrm{A} \rightarrow \mathrm{B} \Longleftrightarrow$ (i) $\alpha \Vdash_1 \mathrm{A}$ and $\alpha \Vdash_0 \mathrm{B}$, or (ii) $\alpha \nVdash_1 \mathrm{A} \rightarrow \mathrm{B}$.

The other definitions of validity (in a frame $\mathbf{S}$) and consequence relation for L_B are the same as in L_N with obvious modifications.

Remark 4. *Note that, while L_3 frames as algebraic binary relational Kripke frames are defined as (reducts of) L_3-algebras, non-algebraic binary relational Kripke frames for L_N and L_B are not. The latter frames require just linear orderedness of frames for fuzziness. This is the basic difference between the* *two sorts of Kripke-style semantics and the reason that we call the former frames* algebraic *and the latter* non-algebraic.

4.2. *Soundness and completeness for* L_3

First we note the following lemma, which is useful for the verification of each instance of the axiom schemes in Proposition 8 below:

Lemma 2. *(Hereditary Lemma) For any sentence A,*

(i) if $\alpha \Vdash_1^v A$ *and* $\alpha \sqsubseteq \beta$*, then* $\beta \Vdash_1^v A$*, and*
(ii) if $\alpha \Vdash_0^v A$ *and* $\alpha \sqsubseteq \beta$*, then* $\beta \Vdash_0^v A$*.*

Proof. See Hereditary Lemma in [6] and Lemmas 1 and 5 in [18]. □

Proposition 8. *(Soundness) If* $\vdash_{L_3} A$*, then* $\models_{L_3} A$*.*

Proof. The rules of L_3 are (mp) and (adj). Both of these obviously preserve truth, i.e., L_3-validity. (For the former, look at $(\to_1)$ and recall that $\sqsubseteq$ is reflexive; for the latter, look at $(\wedge_1)$.) Thus, the proof reduces to verification of axioms for L_3. We consider $\mathbf{Ł}_3$ and $\mathbf{IUML}_3$ here. For $\mathbf{G}_3$, see Proposition 3 in [18].

W.r.t. $\mathbf{Ł}_3$, we verify **Ł**3: We must show that (i) $\alpha \Vdash_1 (A \to \sim A) \to A$ only if $\alpha \Vdash_1 A$ and (ii) $\alpha \Vdash_0 A$ only if $\alpha \Vdash_0 (A \to \sim A) \to A$. For (i), suppose toward contradiction that $\alpha \Vdash_1 (A \to \sim A) \to A$ and $\alpha \nVdash_1 A$. Either $\alpha \Vdash_0 A$ or $\alpha \nVdash_0 A$. Let $\alpha \Vdash_0 A$. Then, since $\alpha \Vdash_1 (A \to \sim A) \to A$, $\alpha \Vdash_0 A \to \sim A$. Thus, by $(\to_{0Ł_3})$, we have $\alpha \Vdash_1 A$ and $\alpha \Vdash_0 \sim A$, a contradiction. Let $\alpha \nVdash_0 A$. Since $\alpha \nVdash_1 A$ and $\alpha \nVdash_0 A$, by $(\sim_1)$ and $(\sim_0)$, we obtain $\alpha \nVdash_0 \sim A$ and $\alpha \nVdash_1 \sim A$. Then, since $\alpha \nVdash_1 A$ and $\alpha \nVdash_0 \sim A$, by $(\to_1)$, we also have $\alpha \Vdash_1 A \to \sim A$. But, since $\alpha \nVdash_1 A$, $\alpha \nVdash_1 (A \to \sim A) \to A$, a contradiction. For (ii), let $\alpha \Vdash_0 A$. By $(\sim_1)$, we have $\alpha \Vdash_1 \sim A$. Thus, using (i) and CP, we can obtain $\alpha \Vdash_1 \sim ((A \to \sim A) \to A)$; therefore, by $(\sim_1)$, $\alpha \Vdash_0 (A \to \sim A) \to A$, as wished.

W.r.t. $\mathbf{IUML}_3$, we verify VE and RM3(2): For VE, we must show that (i) $\alpha \Vdash_1 A$ only if $\alpha \Vdash_1 \mathbf{T}$ and (ii) $\alpha \Vdash_0 \mathbf{T}$ only if $\alpha \Vdash_0 A$. (i) and (ii) directly follow from the conditions $(\top_1)$ and $(\top_0)$. For RM3(2), we must show $\alpha \Vdash_1 A$ or $\alpha \Vdash_1 A \to B$. We instead show that $\alpha \nVdash_1 A$ only if $\alpha \Vdash_1 A \to B$. Let $\alpha \nVdash_1 A$. Since the evaluation is total, $\alpha \Vdash_0 A$. Thus, since $\alpha \nVdash_1 A$ and $\alpha \Vdash_0 A$, for any formula B, $\alpha \Vdash_1 A \to B$, as required.

The verification of other axiom schemes for L is left to the reader. □

We give completeness results for L_3 by using the well-known Henkin-style proofs for modal logic, but with prime theories in place of maximal theories. We call a theory Γ *prime* if, for each pair A, B of formulas such that $\Gamma \vdash A \vee B$, $\Gamma \vdash A$ or $\Gamma \vdash B$. By an L_3*-theory*, we mean a theory Γ closed under rules of L_3. As in relevance logic, by a *regular* L_3-theory, we mean an

L_3-theory containing all of the theorems of L_3. Since we have no use of irregular theories, from now on, by an L_3-theory, we henceforth mean a regular L_3-theory.

Moreover, where Γ is a prime L_3-theory, we define the *canonical L_3 frame* determined by Γ to be a structure $\mathbf{S} = (U_{can}, \zeta_{can}, \sqsubseteq_{can})$, where ζ_{can} is the Γ, U_{can} is the set of prime L_3 theories extending ζ_{can}, and $\sqsubseteq_{can}$ is $\subseteq$ restricted to U_{can}. Note that the base ζ_{can} is constructed as the prime L_3-theory that excludes nontheorems of L_3, i.e., excludes A such that not $\vdash_{L_3} A$. The partial orderedness and the linear orderedness of the canonical L_3 frame depend on $\subseteq$ restricted on U_{can}. Then, first, the following is obvious.

Proposition 9. *The canonical L_3 frame is linearly ordered.*

Proof. By Proposition 26 in [8]. □

Next, we define a canonical evaluation as follows:

(1) $1 \in v_{can}(A, \alpha) \Longleftrightarrow A \in \alpha$;

(2) $0 \in v_{can}(A, \alpha) \Longleftrightarrow \neg A$ ($\sim A$ resp) $\in \alpha$.

This definition allows us to state the following lemma.

Lemma 3. (*Canonical Evaluation Lemma*) v_{can} *is an evaluation.*

Proof. The Hereditary Conditions (HC_1) and (HC_0) are obvious. Thus, we show that the canonical evaluation v_{can} satisfies the truth and falsity conditions above. We prove here the truth and falsity conditions ($\sim_1$) and ($\sim_0$) and the falsity conditions of implications ($\rightarrow_0$Ł$_3$) and ($\rightarrow_{0RM3}$). For the conditions for $\mathbf{G}_3$, see Lemmas 2 and 6 in [18].

For ($\sim_1$), we must show

$$\alpha \Vdash_1^{V_{can}} \sim A \text{ iff } \alpha \Vdash_0^{V_{can}} A.$$

By (1) and (2), we have that $\alpha \Vdash_1^{V_{can}} \sim A$ iff $\sim A \in \alpha$ iff $\alpha \Vdash_0^{V_{can}} A$.

For ($\sim_0$), we must show

$$\alpha \Vdash_0^{V_{can}} \sim A \text{ iff } \alpha \Vdash_1^{V_{can}} A.$$

By (1), (2) and DN, we have that $\alpha \Vdash_0^{V_{can}} \sim A$ iff $\sim\sim A \in \alpha$ iff $A \in \alpha$ iff $\alpha \Vdash_1^{V_{can}} A$.

For ($\rightarrow_0$Ł$_3$), we must show

$$\alpha \Vdash_0^{V_{can}} A \rightarrow B \text{ iff } \begin{array}{l}\text{(i) } \alpha \Vdash_1^{V_{can}} A \text{ and } \alpha \Vdash_0^{V_{can}} B, \text{ and} \\ \text{(ii) } \alpha \Vdash_1^{V_{can}} \sim (A \rightarrow B).\end{array}$$

For the left-to-right direction, let $\alpha \Vdash_0^{V_{can}} A \rightarrow B$. By (1) and (2), we have $\alpha \Vdash_0^{V_{can}} A \rightarrow B$ iff $\sim (A \rightarrow B) \in \alpha$ iff $\alpha \Vdash_1^{V_{can}} \sim (A \rightarrow B)$. Thus (ii) holds. Furthermore, we have that $\sim (A \rightarrow B) \in \alpha$ only if $A \wedge \sim B \in \alpha$ by nI, i.e., Proposition 2 (iv) (1). Then, by (1) and ($\wedge_1$), we obtain that $A \wedge \sim B \in \alpha$ iff $A \in \alpha$ and $\sim B \in \alpha$; therefore, by (1) and (2), iff $\alpha \Vdash_1^{V_{can}} A$ and $\alpha \Vdash_0^{V_{can}} B$. Hence (i) holds. The right-to-left direction is immediate because $\alpha \Vdash_1^{V_{can}} \sim (A \rightarrow B)$ iff $\alpha \Vdash_0^{V_{can}} A \rightarrow B$.

For ($\rightarrow_{0RM3}$), we must show that

$$\alpha \Vdash_0^{V_{can}} \mathrm{A} \rightarrow \mathrm{B} \text{ iff } \begin{array}{l} \text{(i) } \alpha \Vdash_1^{V_{can}} \mathrm{A} \text{ and } \alpha \Vdash_0^{V_{can}} \mathrm{B}, \text{ or} \\ \text{(ii) } \alpha \nVdash_1^{V_{can}} \mathrm{A} \rightarrow \mathrm{B}. \end{array}$$

This is by Lemma 29 in [8]. □

Let us call a model $\mathfrak{M}$, = (U, ζ, $\sqsubseteq$, v), for L_3, an L_3 model. Then, by Lemma 3, the canonically defined (U_{can}, ζ_{can}, $\sqsubseteq_{can}$, v_{can}) is an L_3 model. Thus, since, by construction, ζ_{can} excludes our chosen nontheorem A, and the canonical definition of $\vDash$ agrees with membership, we can state that, for each non-theorem A of L_3, there is an L_3 model in which A is not $\zeta_{can} \vDash A$. It gives us the weak completeness of L_3 as follows.

Theorem 3. *(Weak completeness) If* $\vDash_{L_3} A$*, then* $\vdash_{L_3} A$.

Next, we prove the strong completeness of L_3. As for $\mathbf{R}^+$ in [2], we define A to be an L_3 *consequence* of a theory Γ iff for every L_3 model, whenever $\alpha \vDash B$ for every $B \in \Gamma$, $\alpha \vDash A$, for all $\alpha \in U$. We say that A is L_3 *deducible* from Γ iff A is in every L_3-theory containing Γ. Where Δ is a set of formulas not necessarily a theory, $\Delta \vdash A$ can be thought of as saying that A is deducible from the axioms Δ. The set of $\{A : \Delta \vdash A\}$ is intuitively the smallest theory containing the axioms Δ, and we shall label it as *Th(Δ)*. Then,

Proposition 10. *Let* Γ *be a theory over* L_3*. If* $\Gamma \nvdash_{L_3} A$*, then there is a prime theory* Γ' *such that* $\Gamma \subseteq \Gamma'$ *and* $A \notin \Gamma'$.

Proof. We prove the case of $\mathbf{IUML}_3$ as an example. Let L_3 be $\mathbf{IUML}_3$. Take an enumeration $\{A_n : n \in \omega\}$ of the well-formed formulas of L_3. We define a sequence of sets by induction as follows:

$$\Gamma_0 = \{A' : \Gamma \vdash_{L_3} A'\}.$$

$$\Gamma_{i+1} = \begin{cases} Th(\Gamma_i \cup \{A_{i+1}\}) & \text{if } \Gamma_i, A_{i+1} \nvdash_{L_3} A, \\ \Gamma_i & \text{otherwise.} \end{cases}$$

Let Γ' be the union of all these Γ_n's. The primeness of Γ' can be proved using the deduction theorem for $\mathbf{IUML}_3$, i.e., Proposition 1 (i), along the usual lines.

Analogously for the others. □

Thus, using Lemma 3 and Proposition 10, we can show strong completeness of L_3 as follows.

Theorem 4. *(Strong completeness) Let* Γ *be a theory over* L_3. *If* $\Gamma \models_{L_3} A$, *then* $\Gamma \vdash_{L_3} A$.

5. Concluding remarks

As is known, Kripke-style semantics for many-valued predicate logics (as well as propositional logics) have been introduced (see [14, 15, 16]). A trivial generalization of Kripke-style semantics for such predicate logics in [14, 15, 16] gives us similar Kripke-style semantics for the first-order extensions of L_3. We leave this generalization to the interested reader.

Eynsuk Yang
Department of Philosophy & Institute of Critical Thinking and Writing,
Chonbuk National University, Rm 307,
College of Humanities Blvd. (14-1), Jeonju, 561-756, KOREA
Email: eunsyang@jbnu.ac.kr

References

[1] Anderson, A. R. and Belnap, N. D. *Entailment: The Logic of Relevance and Necessity*, vol. 1, Princeton, Princeton Univ. Press, 1975.
[2] Anderson, A. R., Belnap, N. D., and Dunn, J. M., *Entailment: The Logic of Relevance and Necessity*, vol. 2, A. R. Anderson and N. D. Belnap, Princeton, Princeton Univ. Press, 1992.
[3] Bimbo, K. and Dunn, J. M., *Generalized Galois logics*, Standford, CSLI, 2008.
[4] Cintula, P., "Weakly Implicative (Fuzzy) Logics I: Basic properties", *Archive for Mathematical Logic*, 45 (2006), 673-704.
[5] Brady, R., ed., *Relevant Logics and their Rivals*, vol 2, Burlington, Ashgate, 2003.
[6] Dunn, J. M., "A Kripke-style semantics for R-Mingle using a binary accessibility relation", *Studia Logica*, 35 (1976), 163-172.
[7] Dunn, J. M., "Relevance logic and entailment", *Handbook of Philosophical Logic*, D. Gabbay and F. Guenthner (eds.), Dordrecht, D. Reidel Publ. Co., 1986, pp. 117-224.
[8] Dunn, J. M., "Partiality and its Dual", *Studia Logica*, 66 (2000), 5-40.
[9] Dunn, J. M. and Hardegree, G., *Algebraic Methods in Philosophical Logic*, Oxford, Oxford Univ Press, 2001.
[10] Galatos, N., Jipsen, P., Kowalski, T., and Ono, H., *Residuated lattices: an algebraic glimpse at substructural logics*, Amsterdam, Elsevier, 2007.
[11] Gottwald, S., *A Treatise on Many-valued Logics*, Baldock, Research studies press LTD., 2001.

[12] HÁJEK, P., *Metamathematics of Fuzzy Logic*, Amsterdam, Kluwer, 1998.
[13] METCALFE, G., and MONTAGNA, F., "Substructural Fuzzy Logics", *Journal of Symbolic Logic*, 72 (2007), 834-864.
[14] MONTAGNA, F. and ONO, H., "Kripke semantics, undecidability and standard completeness for Esteva and Godo's Logic MTL∀", *Studia Logica*, 71 (2002), 227-245.
[15] MONTAGNA, F. and SACCHETTI, L., "Kripke-style semantics for many-valued logics", *Mathematical Logic Quarterly*, 49 (2003), 629-641.
[16] MONTAGNA, F. and SACCHETTI, L., "Corrigendum to 'Kripke-style semantics for many-valued logics'", *Mathematical Logic Quarterly*, 50 (2004), 104-107.
[17] TSINAKIS, C. and BLOUNT, K., "The structure of residuated lattices", *International Journal of Algebra and Computation*, 13 (2003), 437-461.
[18] YANG, E., "(Star-based) three-valued Kripke-style semantics for pseudo and weak-Boolean logics", *Logic Journal of the IGPL*, 20 (2012), 187-206.
[19] YANG, E., "Kripke-style semantics for UL", *Korean Journal of Logic*, 15 (2012), 1-15.
[20] YANG, E., "**R** and Relevance principle revisited", *Journal of Philosophical Logic*, 42 (2013), 767-782.