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# COMMON KNOWLEDGE: A FINITARY CALCULUS WITH A SYNTACTIC CUT-ELIMINATION PROCEDURE

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ABSTRACT

In this paper we present a finitary sequent calculus for the **S5** multi-modal system with common knowledge. The sequent calculus is based on indexed hypersequents which are standard hypersequents refined with indices that serve to show the multi-agent feature of the system **S5**. The calculus has a non-analytic right introduction rule. We prove that the calculus is contraction- and weakening-free, that (almost all) its logical rules are invertible, and finally that it enjoys a syntactic cut-elimination procedure. Moreover, the use of the non-analytic rule can be restricted so that the calculus can be considered as suitable for proof search.

## 1. Introduction

Common knowledge is a key feature of multi-agent systems of knowledge which was first discussed by [14] and [5]. The books [10] and [15] provide an excellent introduction to logics of knowledge in general and of common knowledge in particular.

The common knowledge operator is standardly interpreted as the infinite conjunction "all agents know A, and all agents know that all agents know A and so on". From a syntactic point of view, the traditional way to capture common knowledge is by means of Hilbert-style systems comprising of a fixed point axiom, which states that common knowledge is a fixed point, and an induction rule that states that this fixed point is the greatest fixed point. From a semantic point of view, the common knowledge operator is formally defined as the modality of reachability that uses accessibility edges corresponding to any of the knowledge operators for the agents.

In this paper we consider common knowledge from the perspective of Gentzen-style sequent calculi. Whilst considerable progress has been made in developing other sorts of calculi for common knowledge, such as tableaux systems [1, 12], the situation regarding Gentzen-calculi is not entirely satisfactory. Two sorts of calculi have been explored: finitary calculi, for example in [4, 13] and infinitary calculi, for example [2, 22, 7]. (Whilst we

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concentrate on the literature on common knowledge, there have been related developments in the study of proof systems for the modal mu-calculus and fixed points logics more generally, for example [3, 17, 16, 8, 9].) None of the finitary systems presents a syntactic cut-elimination procedure; cut-elimination, if it is established, is proved indirectly by showing completeness of the cut-free system. Among the cited infinitary systems, only [7, 17, 16, 8] propose a cut-elimination procedure.

The aim of this paper is to develop a Gentzen-style calculus for common knowledge that is composed of a finite set of finitary rules, but that nevertheless admits a syntactic cut-elimination procedure. The proposed calculus has other desirable structural properties, in particular the admissibility of all the structural rules and the invertibility of (all but one) logical rules. However, in the light of the difficulties in finding Gentzen-style sequent calculi for common knowledge [2], these advantages come at a price: the rule that introduces the common knowledge operator on the right side of the sequent is non-analytic, i.e. there is a principal formula $B$ in the premises of the rule that does not occur in the conclusion.[1] In order to mitigate this shortcoming, we note that one can always identify a pair of possibly appropriate principal formulas $B$ for any given application of the $\boxtimes R$ rule; as such, the calculus retains much of the interest of a fully analytic calculus as regards proof search.

The calculus proposed in this paper is for the modal logic **S5** plus common knowledge. Since the system **S5** is used to formalise knowledge, this logic is the most appropriate for possible applications in the domain of common knowledge. However, we underline that many of the main results in this paper (and specifically, the central results in Sections 3-5) are not **S5**-dependent, i.e. they could be straightforwardly adapted to other normal modal systems, by exploiting the sequent calculi for these systems introduced in [21].

The calculus introduced in this paper is based on *indexed hypersequents*. Hypersequents were used in [19] in order to construct a cut-free sequent calculus for the system **S5**. Then hypersequents were refined by adding indices in order to build a cut-free sequent calculus for the multi-agent version of the system **S5** [20]. We exploit this last result as a base for building a sequent calculus for **S5** plus common knowledge. In the papers [19] and [20], the intuitive ideas that are behind hypersequents and indexed hypersequents are fully explained, and shall not be repeated here; instead, we focus on their formal interpretation.

The paper is organised as follows. In the next section we present the calculus for **S5** with common knowledge, while in Section 3. we show the

[1] For this reason, the cut-elimination result may be considered to be ‘partial’; see Section 6. for further discussion.

admissibility of the structural rules and the invertibility of (almost all) logical rules. In Section 4. we prove that the calculus is sound and complete with respect to the Hilbert system for common knowledge. In Section 5., we present a syntactic cut-elimination procedure for our calculus, and in Section 6. we show that for any given application of the $\boxplus R$ rule, the principal formula can be restricted to one of two formulas.

## 2. The calculus HS5C

**Definition 1.** *We consider a language $\mathcal{L}_h^{\Box}$ with h agents for some (finite) $h > 0$. The set of agents is denoted by $\Phi$; in order to denote agents, we will use the letters a, b, c, d. Propositions S are atoms. The set of atoms is denoted by $\Psi$. Formulas are denoted by capital letters A, B, C, D. They are given by the following grammar:*

$$A ::= S \mid \neg A \mid (A \wedge A) \mid \Box_z A \mid \boxplus A$$

*where $z \in \Phi$, the formula $\Box_z A$ is read as "agent z knows A" and the formula $\boxplus A$ is read as "A is common knowledge". The other propositional connectives, as well as the (dual) modal operators are defined as usual. We will use the formula $\Box A$ as an abbreviation for "everybody knows A":*

$$\Box A = \Box_1 A \wedge \ldots \wedge \Box_h A$$

**Definition 2.** *In what follows we will use the following syntactic conventions:*

- *$M$, $N$,...: finite multisets of formulas,*
- *$\Gamma$, $\Delta$, ...: classical sequents,*
- *$G$, $H$, ...: indexed hypersequents,...*
- *$\alpha$, $\beta$, ...: finite (perhaps empty) sets of indices of the form nz, where $n \in \mathbb{N}$ and $z \in \Phi$, and, for each set $\alpha$ and for each $z \in \Phi$, there exists at most one index $nz \in \alpha$. So, for instance, $\alpha$ could be the set $\{1a, 1b, 2c\}$, but $\{1a, 2a\}$ is not a legal set of indices.*

We use $\alpha^{nz}$ to denote the set of indices (understood to satisfy the property just mentioned) formed by adding the index *nz* to $\alpha$. This notation serves to draw the reader's attention to the index *nz*. We use $\|H\|$ to denote the union of all the sets of indices contained in the hypersequent *H*. Classical sequents are defined in the standard way (i.e. they are objects of the form $M \Rightarrow N$); indexed hypersequents are defined as follows.

**Definition 3.** *An* indexed hypersequent *is a syntactic object of the form:*

$$\alpha_1 : M_1 \Rightarrow N_1 \mid \alpha_2 : M_2 \Rightarrow N_2 \mid \ldots \mid \alpha_n : M_n \Rightarrow N_n$$

*where* $M_i \Rightarrow N_i$ *(*$i = 1, \ldots, n$*) is a classical sequent,* $\alpha_i$ *is a finite set of indexes as defined above, and, for all m p,* $1 \leq m,\ p \leq n$ *and* $m \neq p$*,*

1. $\alpha_m \cap \alpha_p$ *contains at most one element;*
2. *there exists a sequence* $k_1, \ldots, k_q$ *with* $k_1 = m$ *and* $k_q = p$*, and for all r,* $1 \leq r < q$*,* $\alpha_{k_r} \cap \alpha_{k_{r+1}} \neq \varnothing$*.*
3. *there does not exist a sequence of indexed sequents* $\beta_1 : P_1 \Rightarrow Q_1 \mid \beta_2 : P_2 \Rightarrow Q_2 \mid \ldots \mid \beta_q : P_q \Rightarrow Q_q$ *such that:*
   - *for each pair of indexed sequents* $\beta_r : P_r \Rightarrow Q_r$*,* $\beta_{r+1} : P_{r+1} \Rightarrow Q_{r+1}$*, with* $1 \leq r < q$*,* $\beta_r \cap \beta_{r+1}$ *contains one element;*
   - $\beta_1 : P_1 \Rightarrow Q_1$ *is the same sequent as* $\beta_q : P_q \Rightarrow Q_q$*.*

*Let us call* disconnected indexed hypersequent*, for short* DIH*, an indexed hypersequent that satisfies 1 and 3, but not necessarily 2. We use the same syntactic notation for DIH as for indexed hypersequents, without risk of confusion.*

As a point of notation, empty sets of indices may be omitted (e.g. we write $\Gamma$ rather than $\varnothing : \Gamma$). Moreover, with slight abuse of notation for a indexed sequent $\alpha : \Gamma$ and an indexed hypersequent $H$, we write $\alpha : \Gamma \in H$ to express the statement that $\alpha : \Gamma$ appears in $H$.

**Definition 4.** *For* $\alpha_i : \Gamma_i$ *an indexed sequent belonging to an indexed hypersequent H, define the set of all the indexed sequents belonging to H that have at least one common index with* $\alpha_i : \Gamma_i$ *as follows:*

$$\Sigma_{\alpha_i : \Gamma_i} = \{\alpha_j : \Gamma_j \in H \mid \alpha_i \cap \alpha_j \neq \varnothing\}$$

**Definition 5.** *Given an indexed hypersequent H containing a sequent* $\alpha_i : \Gamma_i$*, we define:*

$$H \setminus \alpha_i : \Gamma_i = \alpha'_1 : \Gamma_1 \mid \ldots \mid \alpha'_{i-1} : \Gamma_{i-1} \mid \alpha'_{i+1} : \Gamma_{i+1} \mid \ldots \mid \alpha'_n : \Gamma_n$$

*where* $\alpha'_j = \alpha_j \setminus \alpha_i$*. That is* $H \setminus \alpha_i : \Gamma_i$ *is the result of dropping, from H, the sequent* $\alpha_i : \Gamma_i$ *and each of the indices belonging to* $\alpha_i$ *that occur in other indexed sequents of H. Note that* $H \setminus \alpha_i : \Gamma_i$ *is a DIH.*

For any $\alpha_i, \alpha_j$ with a single common element $nz$, we use $f(\alpha_i, \alpha_j)$ to denote the agent $z$.

**Definition 6.** *The* interpretation $\tau$ of a DIH $H$ *rooted at* $\alpha_i : \Gamma_i$, $(H)^{\tau}_{\alpha_i : \Gamma_i}$ *is inductively defined as follows:*

- *if* $H = \Gamma_i$ *or* $H = \Gamma_i \mid G$*, and* $\Gamma_i = M \Rightarrow N$*, then* $(H)^{\tau}_{\Gamma_i} = \bigwedge M \rightarrow \bigvee N$

- *if* $H = \alpha_1 : \Gamma_1 \mid \ldots \mid \alpha_i : \Gamma_i \mid \ldots \mid \alpha_n : \Gamma_n$*, then* $(H)^{\tau}_{\alpha_i:\Gamma_i} =$

$$(\Gamma_i)^{\tau}_{\Gamma_i} \vee \bigvee_{\alpha_j:\Gamma_j \in \Sigma_{\alpha_i:\Gamma_i}} \Box_{f(\alpha_j,\alpha_i)} (H \setminus \alpha_i : \Gamma_i)^{\tau}_{\alpha_j:\Gamma_j}$$

**Definition 7.** *The* interpretation of an indexed hypersequent *H is defined in the following way:*

$$(H)^{\tau} = \bigwedge_{\alpha_i:\Gamma_i \in H} (H)^{\tau}_{\alpha_i:\Gamma_i}$$

We have thus introduced the notion of indexed hypersequent and its syntactic interpretation. In order to introduce the calculus **HS5C** which exploits indexed hypersequents, we require the following definitions.

**Definition 8.** *For any pair of sets of indices* $\alpha$*,* $\beta$*,*

$$\overline{\beta}_{\alpha} = \{nz \in \beta \mid \exists m \in \mathbb{N}, mz \in \alpha\}$$

*Moreover, for any* $nz \in \overline{\beta}_{\alpha}$*, call the corresponding element in* $\alpha$ *(if it exists),* $n_{\alpha}z$*.*

*Finally,*

$$\alpha + \beta = (\alpha \cup \beta)[n_{1\alpha}z_1 \ldots n_{l\alpha}z_l / n_1 z_1 \ldots n_l z_l]$$

*where* $\overline{\beta}_{\alpha} = \{n_1 z_1, \ldots, n_l z_l\}$*.*

**Definition 9.** *Let H be a DIH, and let* $\alpha$ *and* $\beta$ *be sets of indices. We define* $H_{\alpha/\beta}$ *as follows:*

$$H_{\alpha/\beta} = H[m_{1\alpha}w_1 \ldots m_{l\alpha}w_l / m_1 w_1 \ldots m_l w_l]$$

*where* $\overline{\beta}_{\alpha} = \{m_1 w_1, \ldots, m_l w_l\}$*. For a set of indices* $\gamma$*,* $\gamma_{\alpha/\beta}$ *is defined similarly.*

In the previous definitions, the substitution of indices for indices in an indexed hypersequent is defined in the standard way, and the standard notation is used.

The rules of the calculus **HS5C** are given in Figure 1. Note that, despite the restriction, the cut rule is indeed general as standard, due to the possibility of renaming indices which will be shown in Lemma 1 below.

As remarked in the Introduction, the rule $\boxplus R$ is non-analytic: $B$ does not appear in the conclusion. We shall discuss some consequences of this in Section 6. Note that a similar rule has been studied in the literature on temporal logics [18], using semantic techniques.

**Initial Indexed Hypersequents**

$$G \mid \alpha : M, S \Rightarrow N, S \qquad\qquad G \mid \alpha : M, \boxplus A \Rightarrow N, \boxplus A$$

**Propositional Rules**

$$\frac{G \mid \alpha : M \Rightarrow N, A}{G \mid \alpha : \neg A, M \Rightarrow N}\ \neg L \qquad\qquad \frac{G \mid \alpha : A, M \Rightarrow N}{G \mid \alpha : M \Rightarrow N, \neg A}\ \neg R$$

$$\frac{G \mid \alpha : A, B, M \Rightarrow N}{G \mid \alpha : A \wedge B, M \Rightarrow N}\ \wedge L \qquad \frac{G \mid \alpha : M \Rightarrow N, A \qquad G \mid \alpha : M \Rightarrow N, B}{G \mid \alpha : M \Rightarrow N, A \wedge B}\ \wedge R$$

**Modal Rules**

$$\frac{G \mid \alpha : \Box_z A, A, M \Rightarrow N}{G \mid \alpha : \Box_z A, M \Rightarrow N}\ \Box_z L_1$$

$$\frac{G \mid \alpha^{nz} : \Box_z A, M \Rightarrow N \mid \beta^{nz} : A, P \Rightarrow Q}{G \mid \alpha^{nz} : \Box_z A, M \Rightarrow N \mid \beta^{nz} : P \Rightarrow Q}\ \Box_z L_2 \qquad \frac{G \mid \alpha^{nz} : M \Rightarrow N \mid nz :\Rightarrow A}{G \mid \beta : M \Rightarrow N, \Box_z A}\ \Box_z R$$

where in the rule $\Box_z R$, $\beta = \begin{cases} \alpha^{nz}, & if \quad nz \in \|G\| \\ \alpha, & otherwise \end{cases}$

**Common Knowledge Rules**

$$\frac{G \mid \alpha : \boxplus A, \Box A, M \Rightarrow N}{G \mid \alpha : \boxplus A, M \Rightarrow N}\ \boxplus L_1$$

$$\frac{G \mid \alpha : \boxplus A, M \Rightarrow N \mid \beta : \boxplus A, P \Rightarrow Q}{G \mid \alpha : \boxplus A, M \Rightarrow N \mid \beta : P \Rightarrow Q}\ \boxplus L_2 \qquad \frac{G \mid \alpha : M \Rightarrow N, B \quad B \Rightarrow \Box A \quad B \Rightarrow \Box B}{G \mid \alpha : M \Rightarrow N, \boxplus A}\ \boxplus R$$

**Cut Rule**

$$\frac{G \mid \alpha : M \Rightarrow N, A \quad H \mid \beta : A, P \Rightarrow Q}{G \mid H_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q}\ cut_A$$

where $\|G\| \cup \alpha$ and $\|H\| \cup \beta$ are disjoint.

Figure 1: The calculus **HS5C**

## 3. Admissibility of the Structural Rules

In this section we show which structural rules are admissible in the calculus **HS5C**. Moreover, we prove that the propositional rules, the modal rules and the rules $\boxtimes L_1$ and $\boxtimes L_2$ are invertible. The cut-elimination proof is given in the Section 5.

**Definition 10.** *For a formula $A$, we define its complexity, $dg(A)$, as follows:*

- $dg(S) = 0$
- $dg(\Box_z A) = dg(\neg A) = dg(A)+1$
- $dg(A \wedge B) = max(dg(A), dg(B))+1$
- $dg(\boxtimes A) = dg(A)+h+1$

**Definition 11.** *We associate to each derivation $d$ in* **HS5C** *three natural numbers $h(d)$ (the height of $d$), $crk(d)$ (the cut-rank of $d$), and $prk(d)$ (the pr-rank of $d$). The height corresponds to the length of the longest branch in a tree-derivation $d$, minus one. The cut-rank corresponds to the complexity of the cut-formulas in $d$. $crk(d)$ is the smallest $n \in \mathbb{N}$ such that each cut-formula $A$ occurring in $d$ is such that $dg(A) < n$. If $crk(d) = 0$, then $d$ is a cut-free derivation. Finally the pr-rank corresponds to the maximal number of applications of the rule $\boxtimes R$ in any branch of a tree-derivation $d$. We omit the standard inductive definitions of height and cut-rank of a derivation [23].*

**Definition 12.** *$d \vdash^n_{p,q} G$ means that $d$ is a derivation of $G$ in* **HS5C***, with $h(d) \leq n$, $crk(d) \leq p$ and $prk(d) \leq q$. We write $^{\langle n \rangle}_{\langle p,q \rangle} G$, for: "there exists a derivation $d$ in* **HS5C** *such that $d \vdash^n_{p,q} G$."*

**Definition 13.** *An inference rule $\mathcal{R}$ with premises $G_1, ..., G_n$ and conclusion $H$ is height-, cut-rank- and pr-rank-preserving admissible in the calculus* **HS5C** *if, whenever* **HS5C** $\vdash^n_{p,q} G_i$*, for each premise $G_i$, then* **HS5C** $\vdash^n_{p,q} H$*. For each rule $\mathcal{R}$, we denote its inverse, which has the conclusion of $\mathcal{R}$ as its only premise and any premise of $\mathcal{R}$ as its conclusion, by $\overline{\mathcal{R}}$. An inference rule is height-, cut-rank- and pr-rank-preserving invertible in the calculus* **HS5C** *if $\overline{\mathcal{R}}$ is height-, cut-rank- and pr-rank-preserving admissible in* **HS5C***.*

**Lemma 1.** *For any indexed hypersequent $G$, if $G$ is derivable in* **HS5C***, then $G[n'_1 z_1 \ldots n'_k z_k / n_1 z_1 \ldots n_k z_k]$ is also derivable with the same height and the same cut- and pr-rank, provided that $G[n'_1 z_1 \ldots n'_k z_k / n_1 z_1 \ldots n_k z_k]$ is an indexed hypersequent (i.e. that it respects the conditions 1. and 3. of Definition 3).*

*Proof.* By straightforward induction on the height of the derivation. □

$$\frac{G \mid \alpha : M \Rightarrow N}{G \mid \alpha : M, P \Rightarrow N, Q}\ IW \qquad \frac{G \mid \alpha : M \Rightarrow N \mid \beta : P \Rightarrow Q}{G \mid \alpha^{nz} : M \Rightarrow N \mid \beta^{nz} : P \Rightarrow Q}\ IndW$$

Figure 2: Internal Weakening and Indices Weakening

$$\frac{G \mid \alpha : M \Rightarrow N}{G \mid \beta : M \Rightarrow N \mid nz : P \Rightarrow Q}\ EW \qquad \frac{G \mid \alpha : M \Rightarrow N \mid \beta : P \Rightarrow Q}{G^{-} \mid \gamma : M, P \Rightarrow N, Q}\ me$$

$\beta = \alpha$, if $nz \in ||G||$,

$\beta = \alpha^{nz}$, otherwise

$\alpha \cap \beta = nz$, $G^{-} = G[n_{1\alpha}z_1 \dots n_{l\alpha}z_l / n_1 z_1 \dots n_l z_l]$ for $\overline{\beta}_\alpha = \{n_1 z_1, \dots, n_l z_l\}$, and

if $nz \in ||G||$, $\gamma = \alpha + \beta$

if $nz \notin ||G||$, $\gamma = \alpha + \beta \setminus \{nz\}$

Figure 3: External Weakening and Merge

**Lemma 2.** *Indexed hypersequents of the form* $G \mid \alpha : A, M \Rightarrow N, A$*, with* $A$ *an arbitrary formula, are derivable in* **HS5C***.*

*Proof.* By straightforward induction on $A$. □

**Lemma 3.** *In the calculus* ***HS5C*** *the following holds:*

1. *The rules of internal weakening and indices weakening (Figure 2) are height-, cut-rank- and pr-rank- admissible.*
2. *The rules of external weakening and merge (Figure 3) are height-, cut-rank- and pr-rank- admissible.*
3. *The propositional and modal rules, as well as the rules* $⊞L_1$ *and* $⊞L_2$ *are height-, cut-rank- and pr-rank- invertible.*

*Proof.* (i) and (ii) follow from a standard induction on the height of the proof. The same works for the propositional rules, and the rule $\Box_z R$ in (iii). As an illustrative example of this, let us consider the height-, cut-rank- and pr-rank- invertibility of the rule $\Box_z R$ in case the premise has been derived by the rule ⊞. We have the following situation:[2]

[2] The symbol $\rightsquigarrow$ means: the premise of the right side is obtained by induction hypothesis on the premise of the left side.

$$\frac{{}^{\langle n-1\rangle}G \mid \alpha^{nz} : M \Rightarrow N, C \mid nz :\Rightarrow A \quad C \Rightarrow \Box B \quad C \Rightarrow \Box C}{{}^{\langle n\rangle}G \mid \alpha^{nz} : M \Rightarrow N, \boxast B \mid nz :\Rightarrow A}\ {\boxast R}$$

$$\rightsquigarrow$$

$$\frac{{}^{\langle n-1\rangle}G \mid \beta : M \Rightarrow N, C, \Box_z A \qquad C \Rightarrow \Box B \qquad C \Rightarrow \Box C}{{}^{\langle n\rangle}G \mid \beta : M \Rightarrow N, \boxast B, \Box_z A}\ {\boxast R}$$

The inverses of the rules $\Box_z L_1$, $\Box_z L_2$, $\boxast L_1$ and $\boxast L_2$ are just internal weakenings. □

Note that, for the rule of indices weakening, since the conclusion is an indexed hypersequent, there is an implicit restriction on the application of the rule to cases where the conditions 1.-3. in Definition 3 are respected.

In order to show the admissibility of the contraction rules, we firstly need to prove the following lemma.

**Lemma 4.** *The rule $\boxast R$ permutes down with respect to all the other rules of the calculus **HS5C**.*

*Proof.* The proof is straightforward. Let us anyway make an example of permutation with the one-premise logical rule $\neg R$, we have:

$$\dfrac{\dfrac{G \mid \alpha : C, M \Rightarrow N, B \quad B \Rightarrow \Box A \quad B \Rightarrow \Box B}{G \mid \alpha : C, M \Rightarrow N, \boxast A}\ {\boxast R}}{G \mid \alpha : M \Rightarrow N, \boxast A, \neg C}\ {\neg R}$$

$$\downarrow$$

$$\dfrac{\dfrac{G \mid \alpha : C, M \Rightarrow N, B}{G \mid \alpha : M \Rightarrow N, B, \neg C}\ {\neg R} \quad B \Rightarrow \Box A \quad B \Rightarrow \Box B}{G \mid \alpha : M \Rightarrow N, \boxast A, \neg C}\ {\boxast R}$$

□

**Lemma 5.** *In the calculus **HS5C** the contraction rules*

$$\frac{G \mid \alpha : A, A, M \Rightarrow N}{G \mid \alpha : A, M \Rightarrow N}\ {CL} \qquad \frac{G \mid \alpha : M \Rightarrow N, A, A}{G \mid \alpha : M \Rightarrow N, A}\ {CR}$$

*are cut- and pr-rank admissible.*

*Proof.* The proof is by induction on the height of the derivation of the premise. The cases of the propositional rules and the rules $\Box_z L_1$, $\Box_z L_2$, $\boxast L_1$ and $\boxast L_2$ are straightforward. The case of the rule $\Box_z R$ is also straightforward, using the rule of merge. We analyse the following critical case:

$$\dfrac{\begin{matrix} d_1 \\ \vdots \\ G \mid \alpha : M \Rightarrow N, B, \boxplus A \end{matrix} \quad \begin{matrix} d_2 \\ \vdots \\ B \Rightarrow \Box A \end{matrix} \quad \begin{matrix} d_3 \\ \vdots \\ B \Rightarrow \Box B \end{matrix}}{G \mid \alpha : M \Rightarrow N, \boxplus A, \boxplus A} \,\boxplus R$$

We go up the derivation $d_1$ to the point where the formula $\boxplus A$ has been introduced. There we have several possibilities.

Case 1. The formula $\boxplus A$ comes from an initial indexed hypersequent. Case 1a. The initial indexed hypersequent is of the form $G' \mid \alpha : S, M' \Rightarrow N'$, $B'$, $S$, $\boxplus A$. We take the initial indexed hypersequent obtained by removing the occurrence of the formula $\boxplus A$, and continue the derivation $d_1 + \boxplus R$ as before. Case 1b. The initial indexed hypersequent is of the form $G' \mid \alpha : \boxplus A, M' \Rightarrow N', B', \boxplus A$. Let us denote this initial indexed hypersequent by $H$. Case 1b1. $B' = B$. We consider the initial indexed hypersequent $H'$ obtained from $H$ by removing the occurrence of the formula $B$, and continue the derivation $d_1$ as before, without applying the rule $\boxplus R$ at the end. Case 1b2. $B' \neq B$, so $B$ has been constructed in the course of the derivation $d_1$. We consider the initial indexed hypersequent $H''$ obtained from $H$ by removing all formulas, indices and indexed sequents that are used only to construct $B$, and develop the derivation $d_1$ as before omitting those inference rules that gave rise to the formula $B$. We no longer need to apply the rule $\boxplus R$.

Case 1c. The initial indexed hypersequent is of the form $G' \mid \alpha : \boxplus C, M' \Rightarrow N', \boxplus C, B', \boxplus A$. The case can be solved as Case 1a.

Case 2. The formula $\boxplus A$ comes from the rule $\boxplus R$, so we have:

$$\begin{matrix} \dfrac{\begin{matrix} d_1^2 \\ \vdots \\ G' \mid \alpha : M' \Rightarrow N', B', D \end{matrix} \quad \begin{matrix} d_1^3 \\ \vdots \\ D \Rightarrow \Box A \end{matrix} \quad \begin{matrix} d_1^4 \\ \vdots \\ D \Rightarrow \Box D \end{matrix}}{G' \mid \alpha : M' \Rightarrow N', B', \boxplus A} \,\boxplus R \\ d_1^1 \\ \vdots \\ G \mid \alpha : M \Rightarrow N, B, \boxplus A \end{matrix}$$

Using Lemma 4, we permute down the application of the rule $\boxplus R$ to obtain a derivation of $G \mid \alpha : M \Rightarrow N, B, D$. Applying the rule $\vee R$[3] on this indexed hypersequent we obtain (i) $G \mid \alpha : M \Rightarrow N, B \vee D$. From $D \Rightarrow \Box A$ and $B \Rightarrow \Box A$, by application of the rule $\vee L$, we obtain (ii) $D \vee B \Rightarrow \Box A$. From $D \Rightarrow \Box D$ and $D \Rightarrow \Box B$, by weakening and $\vee L$, we get $D \vee B \Rightarrow \Box D, \Box B$. From $D \vee B \Rightarrow \Box D, \Box B$, we can derive (iii) $D \vee B \Rightarrow \Box(D \vee B)$. We use

[3] The rule $\vee R$, as well as the rule $\vee L$, can be straightforwardly formulated on the basis of the other propositional rules.

(i), (ii) and (iii) to obtain, by means of the rule $\boxplus R$, the conclusion $G \mid \alpha : M \Rightarrow N, \boxplus A$. □

## 4. Adequateness Theorem

In this section we show that the calculus **HS5C** proves exactly the same formulas as its corresponding Hilbert-style system **S5C**. The Hilbert system **S5C** is fully described in [10, Ch 3].

**Theorem 4.1.** *For all indexed hypersequents $G$ and for all formulas $A$,*

1. *if* $\vdash G$ *in* **HS5C**, *then* $\vdash (G)^{\tau}$ *in* **S5C**.
2. *if* $\vdash A$ *in* **S5C**, *then* $\vdash \Rightarrow A$ *in* **HS5C**.

*Proof.* The proof of (i) is relatively standard (it is similar to [21, Lemma 5.1]). In order to acquaint the reader with the calculus **HS5C**, we give as examples the proofs of the fixed point axiom and the induction rule; rest of (ii) is similar.

– fixed point axiom[4]

$$
\dfrac{\vdots \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{1z : \boxplus A, \cdots \Box_z A \cdots \Rightarrow\ \mid 1z : A \Rightarrow A}{1z : \boxplus A, \cdots \Box_z A \cdots \Rightarrow\ \mid 1z :\Rightarrow A}\,\Box_z L}{1z : \boxplus A, \Box A \Rightarrow\ \mid 1z :\Rightarrow A}\,\wedge L^*}{1z : \boxplus A \Rightarrow\ \mid 1z :\Rightarrow A}\,\boxplus L_1 \qquad \dfrac{1z : \boxplus A \Rightarrow\ \mid 1z : \boxplus A \Rightarrow \boxplus A}{1z : \boxplus A \Rightarrow\ \mid 1z :\Rightarrow \boxplus A}\,\boxplus L_2}{1z : \boxplus A \Rightarrow\ \mid 1z :\Rightarrow A \wedge \boxplus A}\,\wedge R}{\boxplus A \Rightarrow \Box_z (A \wedge \boxplus A)}\,\Box_z R \qquad \vdots}{\dfrac{\boxplus A \Rightarrow \Box(A \wedge \boxplus A)}{\Rightarrow \boxplus A \rightarrow \Box(A \wedge \boxplus A)}\,\rightarrow R}\,\wedge R^*
$$

– induction rule

$$
\dfrac{B \Rightarrow B \qquad \dfrac{B \Rightarrow \Box A \wedge \Box B}{B \Rightarrow \Box A \quad B \Rightarrow \Box B}\,\overline{\wedge R}}{B \Rightarrow \boxplus A}\,\boxplus R
$$

□

## 5. Cut-elimination

In this section we prove that the cut-rule is eliminable in the calculus **HS5C**. In the next section we discuss the non-analyticity of rule $\boxplus R$.

[4] We use the notation $\mathcal{R}_1^* + ... + \mathcal{R}_n^*$ to mean repeated applications of the rules $\mathcal{R}_1, \ldots, \mathcal{R}_n$. We take this notation for granted in what follows.

**Lemma 6.** *If*

$$\dfrac{\begin{array}{c} d_1 \\ \vdots \\ G \mid \alpha : M \Rightarrow N, A \end{array} \quad \begin{array}{c} d_2 \\ \vdots \\ H \mid \beta : A, P \Rightarrow Q \end{array}}{G \mid H_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q} \; cut_A$$

*and $d_1$ and $d_2$ do not contain any application of the cut-rule, then we can construct a derivation of $G \mid H_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q$ with no application of the cut-rule.*

*Proof.* The proof is developed by induction on the pr-rank of the derivation, with subinduction on the complexity of the cut-formula, and with a third subinduction on the sum of the heights of the derivations of the premises of the cut-rule. We distinguish cases according to the last rule applied on the left premise.

CASE 1. $G \mid \alpha : M \Rightarrow N, A$ is an initial indexed hypersequent. Then either the conclusion is also an initial indexed tree-hypersequent, or the cut can be replaced by various applications of the rules *IW*, *IndW* and *EW* on the right premise $H \mid \beta : A, P \Rightarrow Q$, and renaming of indices (Lemma 1).

CASE 2. $G \mid \alpha : M \Rightarrow N, A$ is inferred by a rule $\mathcal{R}$ in which $A$ is not principal. This case can be standardly solved by induction on the sum of the heights of the derivations $d_1$ and $d_2$.

CASE 3. $G \mid \alpha : M \Rightarrow N, A$ is inferred by a rule $\mathcal{R}$ in which $A$ is the principal formula. We distinguish three subcases: in the first subcase, 3.1., $\mathcal{R}$ is a propositional rule, in the second subcase, 3.2., $\mathcal{R}$ is a modal rule, in the third subcase, 3.3., $\mathcal{R}$ is a common knowledge rule.

CASE 3.1. This case can be solved by applying Lemma 3 on the right premise, and by replacing the previous cut with one (or two, in case of the rule $\wedge R$) which is (are) eliminable by induction on the complexity.

CASE 3.2. $\mathcal{R}$ is $\Box_z R$ and $A = \Box_z B$. Consider the last rule $\mathcal{R}'$ of $d_2$. If no rule $\mathcal{R}'$ introduces $H \mid \beta : \Box_z B, P \Rightarrow Q$ because $H \mid \beta : \Box_z B, P \Rightarrow Q$ is an initial indexed hypersequent, then we can solve the case as in the case 1. If $\Box_z B$ is not principal in the rule $\mathcal{R}'$, then we can solve the case as in the case 2. If $\Box_z B$ is the principal formula of the rule $\mathcal{R}'$, then there are two cases: 3.2.1. $\mathcal{R}'$ is $\Box_z L_1$, and 3.2.2. $\mathcal{R}'$ is $\Box_z L_2$. We consider first 3.2.1. We have[5]

$$\dfrac{\dfrac{G \mid \alpha^{nz} : M \Rightarrow N \mid nz : \Rightarrow B}{G \mid \alpha : M \Rightarrow N, \Box_z B} \; \Box_z R \qquad \dfrac{H \mid \beta : \Box_z B, B, P \Rightarrow Q}{H \mid \beta : \Box_z B, P \Rightarrow Q} \; \Box_z L_1}{G \mid H_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q} \; cut_{\Box_z B}$$

[5] Note that we analyse the case where the index $nz$ only appears in the displayed sequents $M \Rightarrow N$ and $\Rightarrow B$ in the premise of $\mathcal{R}$. The case where $nz \in \|G\|$ is dealt with analogously.

which we reduce to

$$\dfrac{\dfrac{\dfrac{G \mid \alpha^{nz} : M \Rightarrow N \mid nz :\Rightarrow B}{G \mid \alpha : M \Rightarrow N, B}\,me \qquad \dfrac{G^* \mid \alpha^* : M \Rightarrow N, \Box_z B \quad H \mid \beta : \Box_z B, B, P \Rightarrow Q}{G^* \mid H_{\alpha^*/\beta} \mid \alpha^* + \beta : B, M, P \Rightarrow N, Q}\,cut_{\Box_z B}}{G \mid G^*_{\alpha/(\alpha^*+\beta)} \mid (H_{\alpha^*/\beta})_{\alpha/(\alpha^*+\beta)} \mid \alpha + (\alpha^* + \beta) : M, M, P \Rightarrow N, N, Q}\,cut_B}{G \mid H_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q}\,C^*{+}me^*$$

where $G^* \mid \alpha^* : M \Rightarrow N, \Box_z B$ is the result of renaming the indexed hypersequent $G \mid \alpha : M \Rightarrow N, \Box_z B$ so that $\|G\| \cup \alpha$, $\|G^*\| \cup \alpha^*$ and $\|H\| \cup \beta$ are mutually disjoint. We assume this notation in all the cases below.

The first cut is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule, while the second cut is eliminable by induction on the complexity of the cut-formula. Moreover, since $\alpha + (\alpha^* + \beta) = \alpha + \beta$ and $(H_{\alpha^*/\beta})_{\alpha/(\alpha^*+\beta)} = H_{\alpha/\beta}$, only repeated applications of merge and contraction to $G$ and $G^*_{\alpha/(\alpha^*+\beta)}$ are required to obtain the conclusion.

As concerns case 3.2.2 ($\mathcal{R}'$ is $\Box_z L_2$), we have:

$$\dfrac{\dfrac{G \mid \alpha^{nz} : M \Rightarrow N \mid nz :\Rightarrow B}{G \mid \alpha : M \Rightarrow N, \Box_z B}\,\Box_z R \qquad \dfrac{H' \mid \beta^{mz} : \Box_z B, P \Rightarrow Q \mid \gamma^{mz} : B, Z \Rightarrow W}{H' \mid \beta^{mz} : \Box_z B, P \Rightarrow Q \mid \gamma^{mz} : Z \Rightarrow W}\,\Box_z L_2}{G \mid H'_{\alpha/\beta^{mz}} \mid \alpha + (\beta^{mz}) : M, P \Rightarrow N, Q \mid \gamma^{mz} : Z \Rightarrow W}\,cut_{\Box_z B}$$

which we reduce to

$$\dfrac{G \mid \alpha^{nz} : M \Rightarrow N \mid nz :\Rightarrow B \qquad \dfrac{G^* \mid \alpha^* : M \Rightarrow N, \Box_z B \quad H' \mid \beta^{mz} : \Box_z B, P \Rightarrow Q \mid \gamma^{mz} : Z \Rightarrow W}{G^* \mid H'_{\alpha^*/\beta^{mz}} \mid \alpha^* + (\beta^{mz}) : M, P \Rightarrow N, Q \mid \gamma^{mz} : B, Z \Rightarrow W}\,cut_{\Box_z B}}{G \mid G^*_{nz/mz} \mid (H'_{\alpha^*/\beta^{mz}})_{nz/mz} \mid \alpha^{nz} : M \Rightarrow N \mid \alpha^* + (\beta^{nz}) : M, P \Rightarrow N, Q \mid \gamma^{nz} : Z \Rightarrow W}\,cut_B$$

By repeated applications of merge and contraction, an observation similar to that in the previous case, and an application of Lemma 1, we obtain the desired conclusion.

The first cut is eliminable by induction of the sum of the heights of the derivations of the premises of the cut-rule, while the second cut is eliminable by induction on the complexity of the cut-formula.

Case 3.3. $\mathcal{R}$ is $\boxplus R$ and $A = \boxplus B$. Let us suppose that $\boxplus B$ is the principal formula of the rule $\mathcal{R}'$; the other cases are treated as in 3.2. There are two subcases: 3.3.1. $\mathcal{R}'$ is $\boxplus L_1$, and 3.3.2. $\mathcal{R}'$ is $\boxplus L_2$. In the former case, we have:

$$\dfrac{\dfrac{G \mid \alpha : M \Rightarrow N, C \quad C \Rightarrow \Box B \quad C \Rightarrow \Box C}{G \mid \alpha : M \Rightarrow N, \boxplus B}\,\boxplus R \qquad \dfrac{H \mid \beta : \boxplus B, \Box B, P \Rightarrow Q}{H \mid \beta : \boxplus B, P \Rightarrow Q}\,\boxplus L_1}{G \mid H_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q}\,cut_{\boxplus B}$$

which we reduce to

$$\dfrac{\dfrac{\dfrac{G \mid \alpha : M \Rightarrow N, C \quad C \Rightarrow \Box B}{G \mid \alpha : M \Rightarrow N, \Box B}\,{\scriptstyle cut_C} \qquad \dfrac{G^* \mid \alpha^* : M \Rightarrow N, \boxast B \quad H \mid \beta : \boxast B, \Box B, P \Rightarrow Q}{G^* \mid H_{\alpha^*/\beta} \mid \alpha^* + \beta : \Box B, M, P \Rightarrow N, Q}\,{\scriptstyle cut_{\boxast B}}}{G \mid G^*_{\alpha/\alpha^*+\beta} \mid (H_{\alpha^*/\beta})_{\alpha/\alpha^*+\beta} \mid \alpha + (\alpha^* + \beta) : M, M, P \Rightarrow N, N, Q}\,{\scriptstyle cut_{\Box B}}}{G \mid H_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q}\,{\scriptstyle C^* + merge^*}$$

where the conclusion is obtained in a similar way to case 3.2 above. The $cut_C$ is eliminable by induction on the pr-rank, the $cut_{\boxast B}$ is eliminable by induction on the sum of the heights of the derivations of the premises of the cut-rule, and the $cut_{\Box B}$ is eliminable by induction on the complexity of the cut-formula.

We now consider case 3.3.2 ($\mathcal{R}'$ is $\boxast L_2$), where we have:

$$\dfrac{\dfrac{G \mid \alpha : M \Rightarrow N, C \quad C \Rightarrow \Box B \quad C \Rightarrow \Box C}{G \mid \alpha : M \Rightarrow N, \boxast B}\,{\scriptstyle \boxast R} \qquad \dfrac{H' \mid \beta : \boxast B, P \Rightarrow Q \mid \gamma : \boxast B, Z \Rightarrow W}{H' \mid \beta : \boxast B, P \Rightarrow Q \mid \gamma : Z \Rightarrow W}\,{\scriptstyle \boxast L_2}}{G \mid H'_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q \mid \gamma_{\alpha/\beta} : Z \Rightarrow W}\,{\scriptstyle cut_{\boxast B}}$$

We go up the derivation $d_2$ to the first rule $\mathcal{R}''$ that is not a $\boxast L_2$ rule applied to some of the $\boxast B$'s. We distinguish three cases.

- The premise of $\mathcal{R}''$ is an initial indexed hypersequent, call it $I$. If the formula $\boxast B$ is not the principal formula in $I$, then even the conclusion of the cut is an initial indexed hypersequent and the case is solved. If the formula $\boxast B$ is the principal formula, then $I$ contains an indexed sequent $\delta : Z', \boxast B \Rightarrow W', \boxast B$.[6] So the conclusion of the cut has the following form $G \mid H''_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q \mid \gamma_{\alpha/\beta} : Z \Rightarrow W \mid \delta_{\alpha/\beta} : Z' \Rightarrow W', \boxast B$.

  By the condition 2 of Definition 3 we know that the set of indices $\beta$ and $\delta$ in $I$ are linked by a chain of indices $n_1 i_1, \ldots, n_m i_m$. We now build the following derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \Rightarrow \Box C}{C \Rightarrow \Box_{i_1} C}\,{\scriptstyle \overline{\wedge R}}}{n_1 i_1 : C \Rightarrow \ \mid n_1 i_1 :\Rightarrow C}\,{\scriptstyle \overline{\Box_z R}} \qquad C \Rightarrow \Box C}{n_1 i_1 : C \Rightarrow \ \mid n_1 i_1 :\Rightarrow \Box C}\,{\scriptstyle cut_C}}{n_1 i_1 : C \Rightarrow \ \mid n_1 i_1 :\Rightarrow \Box_{i_2} C}\,{\scriptstyle \overline{\wedge R}}}{n_1 i_1 : C \Rightarrow \ \mid n_1 i_1, n_2 i_2 :\Rightarrow \ \mid n_2 i_2 :\Rightarrow C}\,{\scriptstyle \overline{\Box_z R}}$$

$$\vdots$$

[6] We consider the case where this sequent is in $H'$, the case where it is $\gamma : \boxast B, Z \Rightarrow W$ is treated similarly.

where the derivation is continued with the same succession of inferences to obtain as conclusion the indexed hypersequent $n_1 i_1 : C \Rightarrow | ... | n_m i_m : \Rightarrow C$, where $n_1 i_1, ..., n_m i_m$ are exactly those indices that link the sets $\beta$ and $\delta$ in $I$. The cuts in this derivation are eliminable by induction on the pr-rank. We finish solving the case with the following derivation; the cut is also eliminable by induction on the pr-rank.

$$\dfrac{\dfrac{\dfrac{G \mid \alpha : M \Rightarrow N, C \quad n_1 i_1 : C \Rightarrow \ | ... | \, n_m i_m : \Rightarrow C}{G \mid \alpha + n_1 i_1 : M \Rightarrow N \mid ... \mid n_m i_m : \Rightarrow C}\, cut_C \qquad \dfrac{C \Rightarrow \Box B \quad C \Rightarrow \Box C}{}}{G \mid \alpha + n_1 i_1 : M \Rightarrow N \mid ... \mid n_m i_m : \Rightarrow \boxtimes B}\, \boxtimes R}{G \mid H''_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q \mid \gamma_{\alpha/\beta} : Z \Rightarrow W \mid \delta_{\alpha/\beta} : Z' \Rightarrow W', \boxtimes B}\, {}^{IW^*,\ EW^*}_{IndW^*,\ Lem\ 1}$$

- None of the $\boxtimes B$ are principal formulas of $\mathcal{R}''$. This case is treated similarly to case 2.
- $\mathcal{R}'' = \boxtimes L_1$ and has (any of the) $\boxtimes B$ as principal formula. If the principal formula $\boxtimes B$ of the rule belongs to the indexed sequent $\beta : \boxtimes B, P \Rightarrow Q$, then we apply the rule $\boxtimes L_2$ $n$ times on the premise of the $\boxtimes L_1$ and then operate as in case 3.3.1. Now consider the case where the principal formula does not belong to this indexed sequent. First, in a way analogous to the previous item, we construct a derivation of the indexed hypersequent $n_1 i_1 : C \Rightarrow | ... | \, n_m i_m : \Rightarrow \Box B$. Then we apply the rule $\boxtimes L_2$ $n$ times on the premise of the rule $\boxtimes L_1$ to obtain the indexed hypersequent $H'' \mid \beta : \boxtimes B, P \Rightarrow Q \mid \gamma : Z \Rightarrow W \mid \delta : \Box B, Z' \Rightarrow W'$.[7] We proceed with the following cuts:

$$\dfrac{G \mid \alpha : M \Rightarrow N, C \quad n_1 i_1 : C \Rightarrow \ | ... | \, n_m i_m : \Rightarrow \Box B}{G \mid \alpha + n_1 i_1 : M \Rightarrow N \mid ... \mid n_m i_m : \Rightarrow \Box B}\, cut_C$$

$$\dfrac{G \mid \alpha : M \Rightarrow N, \boxtimes B \quad H'' \mid \beta : \boxtimes B, P \Rightarrow Q \mid \gamma : Z \Rightarrow W \mid \delta : \Box B, Z' \Rightarrow W'}{G \mid H''_{\alpha/\beta} \mid \alpha + \beta : M, P \Rightarrow N, Q \mid \gamma_{\alpha/\beta} : Z \Rightarrow W \mid \delta_{\alpha/\beta} : \Box B, Z' \Rightarrow W'}\, cut_{\boxtimes B}$$

where the former cut is eliminable by induction on the pr-rank, and the latter by induction on the sum of the heights of the derivations of the premises of the cut-rule.

Renaming and applying the cut-rule on the conclusions of these cuts, with principal formula $\Box B$, we obtain the indexed hypersequent:

$$G^* \mid G_{(n_m i_m)^*/\delta_{\alpha/\beta}} \mid (H''_{\alpha/\beta})_{(n_m i_m)^*/\delta_{\alpha/\beta}} \mid (\alpha + \beta)_{(n_m i_m)^*/\delta_{\alpha/\beta}} : M, P \Rightarrow N, Q \mid \alpha +$$

$$n_1 i_1 : M \Rightarrow N \mid ... \mid (\gamma_{\alpha^*/\beta})_{(n_m i_m)^*/\delta_{\alpha/\beta}} : Z \Rightarrow W \mid (n_m i_m)^* + \delta_{\alpha/\beta} : Z' \Rightarrow W'$$

[7] The $\Box B$ could belong to the indexed sequent $\gamma^* : Z \Rightarrow W$, instead of to the sequent $\delta^* : Z' \Rightarrow W', \boxtimes B$. The case is solved in the same way.

This cut is eliminable by induction on the complexity of the cut-formula. We obtain the desired conclusion by renaming indices and several applications of the rules of merge and contraction.

□

The following theorem follows immediately from Lemma 6 by induction on the number of cuts.

**Theorem 5.1.** *Every derivation $d$ in* **HS5C** *can be effectively transformed into a derivation $d'$ where there is no application of the cut-rule.*

## 6. Discussion and refinements

The calculus thus admits a syntactic procedure for eliminating cuts. However, given the non-analyticity of the $\boxtimes R$ rule, cut-elimination does not imply the subformula property. Moreover, one might consider this to be a partial cut-elimination result,[8] insofar as some "cut-like" elements are "built into" the $\boxtimes R$ rule.

In reply to this worry, we show that all applications of the $\boxtimes R$ rule may be restricted.[9] To this end, we shall first define several notions of disjunctive normal form, as follows:

$$\begin{array}{lll}
Form & ::= & S \mid \neg Form \mid Form \wedge Form \mid \Box_z Form \mid \boxtimes Form \\
MForm & ::= & S \mid \neg MForm \mid MForm \wedge MForm \mid \Box_z MForm \\
Lit & ::= & S \mid \neg S \mid \neg\Box_z \neg Term \mid \Box_z \neg Term \\
Term & ::= & Lit \mid Term \wedge Term \\
MClause & ::= & Term \mid \boxtimes Form \mid \neg\boxtimes Form \mid MClause \wedge MClause \\
MDNF & ::= & MClause \mid MClause \vee MClause \\
CKClause & ::= & Term \mid \boxtimes MForm \mid \neg\boxtimes MForm \mid CKClause \wedge CKClause \\
CKDNF & ::= & CKClause \mid CKClause \vee CKClause
\end{array}$$

Form are just the formula of the language (we recall this definition as a reminder to the reader), and MForm (for Modal Formulas) are the formulas containing no occurrences of $\boxtimes$. Formulas in Modal Disjunctive Normal Form (MDNF) are essentially formulas such that all modal subformulas

[8] Partial cut-elimination results (eg. [2]) show that, as concerns the derivation of a given formula, all except a certain class of cuts can be eliminated.

[9] The proof of this fact is, contrary to all the others, of a semantic nature. Given the difficulty of the problem, we already are satisfied of our result. An alternative syntactic proof of the same fact will be subject of future work.

(i.e. not containing occurrences of $\boxtimes$) outside the scope of $\boxtimes$ are in normal form, but any formulas are allowed inside the scope of $\boxtimes$. It is essentially the standard notion of normal form for modal logic, applied only to the operators $\Box_z$ (ie. formula of the form $\boxtimes A$ that do not occur in the scope of a $\boxtimes$ are treated as propositional atoms). Formulas in Common Knowledge Disjunctive Normal Form (CKDNF) are formulas of MDNF with the added restriction that there are no embedded occurrences of $\boxtimes$: inside every occurrence of $\boxtimes$ are only formulas not containing $\boxtimes$, which are themselves disjuncts of a MDNF. It is straightforward to show that, for any formula $A$, there is an equivalent modal disjunctive normal form – call it $A^{MDNF}$ – and an equivalent common knowledge disjunctive normal form – call it $A^{CKDNF}$.

**Proposition 1.** *Any formula $A$ is equivalent to a formula $A^{MDNF}$ which is in MDNF and to a formula $A^{CKDNF}$ which is in CKDNF.*

*Proof.* To prove both clauses together, we shall employ the same technique: we essentially "take out" the CK formulas that are required, and then apply the normal form theorem for modal logic, treating the formula of the form $\boxtimes A$ as propositional atoms. To this end, we introduce the following definitions.

The modal depth of an occurrence of $\boxtimes$ is the number of occurrences of $\Box_z$ in whose scope it is, and the CK depth of an occurrence of $\boxtimes$ is the number of occurrences of $\boxtimes$ in whose scope it is, plus one. (So the CK depth of the occurrence of $\boxtimes$ in $\boxtimes p$ is 1.) The modal depth of a formula is the sum of the modal depths of all occurrences of $\boxtimes$ that are of CK depth one in the formula. The CK depth of a formula is the sum of the CK depths of all occurrences of $\boxtimes$ of depth strictly greater than one.

For the case of $A^{MDNF}$ we operate by induction on the modal depth of the formula. If the modal depth is $n > 0$, then there exists an occurrence of $\boxtimes$ in the scope of a $\Box_z$; without loss of generality, we can take occurrences such that we have a subformula $\Box_z B$ where the occurrence of $\boxtimes$ in $B$ is not in the scope of any occurrence of $\Box_z$ or $\boxtimes$. Hence, applying recursively the following standard equivalences of **S5C** – $\Box_z \boxtimes C \equiv \boxtimes C$, $\Box_z \neg \boxtimes C \equiv \neg \boxtimes C$, $\Box_z(C \wedge D) \equiv \Box_z C \wedge \Box_z D$, $\Box_z(\boxtimes C \vee D) \equiv \boxtimes C \vee \Box_z D$, $\Box_z(\neg \boxtimes C \vee D) \equiv \neg \boxtimes C \vee \Box_z D$ – one obtains an equivalent formula where the occurrence of $\boxtimes$ is not in the scope of the $\Box_z$. Replacing the initial subformula by this formula, one obtains a formula of modal depth less than $n$, as required. By this procedure one obtains a formula $A'$ equivalent to $A$ of modal depth zero. We now apply the normal form theorem for modal logic [11], and treating subformulas of the form $\boxtimes B$ where the occurrence of $\boxtimes$ is of CK depth one as propositional atoms; note that, since $A'$ is of modal depth zero, no such formulas occur in the scope of a $\Box_z$. It is straightforward to see that the formula obtained, which is equivalent to $A$, is in MDNF, as required.

For the case of $A^{CKDNF}$, we begin with the MDNF formula $A^{MDNF}$ and operate by induction on the CK depth of the formula. If the CK depth is $n > 0$, then there exists an occurrence of $\boxast$ of CK depth one with an occurrence of $\boxast$ in its scope; take any ‘outermost’ occurrence of $\boxast$ in its scope. Applying recursively the equivalences cited above inside the scope of $\boxast$ as well as the following standard equivalences of **S5C** – $\boxast\boxast C \equiv \boxast C$, $\boxast\neg\boxast C \equiv \neg\boxast C$, $\boxast(C \wedge D) \equiv \boxast C \wedge \boxast D$, $\boxast(\boxast C \vee D) \equiv \boxast C \vee \boxast D$, $\boxast(\neg\boxast C \vee D) \equiv \neg\boxast C \vee \boxast D$ – one obtains an equivalent formula where the inner occurrence of $\boxast$ is eliminated. Replacing the initial subformula by this formula, one obtains a formula of CK depth less than $n$, as required. By this procedure one obtains a formula $A'$ equivalent to $A$ of CK depth zero. Since $A'$ is of modal depth zero, there are no occurrences of $\boxast$ in the scope of a $\boxast$; hence, applying once again the normal form theorem for modal logic [11] (and treating subformulas with $\boxast B$ as propositional atoms), one obtains a formula obtained equivalent to $A$ that is in CKDNF, as required. $\square$

For a formula $A$, $A^{MDNF}$ is the disjunction of clauses $D$ of the form $D_{prop\square} \wedge D_{-\boxast} \wedge D_{+\boxast}$, where $D_{prop\square}$ contains no occurrences of $\boxast$, and is in normal form for modal logic and $D_{-\boxast}$ and $D_{+\boxast}$ are conjunctions of formulas of the form $\neg\boxast C$ and $\boxast C$ respectively. To fix notation, let $D_{prop\square} = \bigwedge_{i\in I} E_i$. Define $\overline{D_{prop\square}} = \bigwedge_{\substack{i\in I\ s.t.\ D_{-\boxast}\wedge D_{+\boxast} \not\Rightarrow \square E_i \\ and\ D_{-\boxast}\wedge D_{+\boxast}\wedge E_i \Rightarrow \square E_i}} E_i$. For any set of propositional atoms $\mathcal{P}$, let $\widehat{D_{prop\square}}^{\mathcal{P}}$ be the result of removing from $D_{prop\square}$ any propositional atom not belonging to $\mathcal{P}$.[10] For each clause $D$ and any set of propositional atoms $\mathcal{P}$, we define the *common knowledge core of D*, $D^{CK-core} = D_{-\boxast} \wedge D_{+\boxast}$, the *common knowledge reduction of D*, $D^{CK} = \overline{D_{prop\square}} \wedge \neg\boxast\neg D_{prop\square} \wedge D_{-\boxast} \wedge D_{+\boxast}$, and the *common knowledge reduction of D restricted to* $\mathcal{P}$, $D^{CK}_{\mathcal{P}} = \overline{D_{prop\square}} \wedge \neg\boxast\neg\widehat{D_{prop\square}}^{\mathcal{P}} \wedge D_{-\boxast} \wedge D_{+\boxast}$. Similarly, for any formula $A$ with modal disjunctive normal form $A^{MDNF} = \bigvee D_i$, the *common knowledge core of A* is defined to be $A^{CK-core} = \bigvee D^{CK-core}$, the *common knowledge reduction of A* is defined to be $A^{CK} = \bigvee D^{CK}$, and the *common knowledge reduction of A restricted to* $\mathcal{P}$ is defined to be $A^{CK}_{\mathcal{P}} = \bigvee_{D^{CK}_{i\mathcal{P}} \not\equiv \top} D^{CK}_{i\mathcal{P}}$.

As standard, an occurrence of the $\boxast$ is said to be *positive* (respectively *negative*) if it is in the scope of a even (resp. odd) number of negations. For a formula $A$, define $\mathcal{P}_{-A}$ to be the set of propositional atoms occurring in the scope of a negative occurrence of the $\boxast$ operator in $A$.

An application of the $\boxast R$ rule yielding the conclusion $G \,|\, \alpha : M \Rightarrow N, \boxast A$ is said to be *canonical* if the principal formula is either $(\neg(G \,|\, \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-A}}$ or $(\neg(G \,|\, \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK-core}$.[11] A deri-

[10] Formally, removing corresponds to replacing a positive occurrence of $p$ by $\top$ and any negative occurrence of $p$ by $\perp$.

[11] Recall the notation from Definition 6.

vation is *canonical* if every application of the $\boxplus R$ rule in the derivation is canonical.

We shall show that for any derivation involving the application of the $\boxplus R$ rule, there is a canonical derivation of the same indexed hypersequent. Before coming to this result, we state three preparatory lemmas.

**Lemma 7.** *For any indexed hypersequent $G \mid \alpha : M \Rightarrow N$ and any set of propositional letters $\mathcal{P}$, there exists a canonical derivation of $G \mid \alpha : M \Rightarrow N, (\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}}$.*

*Proof.* This is a consequence of the observation that, since the definition of $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}}$ does not interfere with occurrences $\boxplus$, there is straightforward derivation of $G \mid \alpha : M \Rightarrow N, (\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}}$ that involves only applications of propositional, modal rules and $\boxplus L_1$ and $\boxplus L_2$. Since there are no applications of the $\boxplus R$ rule, this derivation is canonical, as required. $\square$

We shall say that a rule is *canonical-admissible* if, whenever there exists a canonical derivation(s) of the premise(s) of the rule, there exists a canonical derivation of its conclusion.

**Lemma 8.** *The Common Knowledge Rules in Figure 2 are canonical-admissible.*

$$\frac{G \mid \alpha : M, \boxplus A, \boxplus B \Rightarrow N}{G \mid \alpha : M, \boxplus(A \wedge B) \Rightarrow N}\ \wedge^{CK}{}_L \qquad \frac{G \mid \alpha : M \Rightarrow \boxplus A \wedge \boxplus B, N}{G \mid \alpha : M \Rightarrow \boxplus(A \wedge B), N}\ \wedge^{CK}{}_R$$

$$\frac{G \mid \alpha : M, \boxplus A \vee \boxplus B \Rightarrow N}{G \mid \alpha : M, \boxplus(\boxplus A \vee B) \Rightarrow N}\ \vee^{CK+}{}_L \qquad \frac{G \mid \alpha : M \Rightarrow \boxplus A, \boxplus B, N}{G \mid \alpha : M \Rightarrow \boxplus(\boxplus A \vee B), N}\ \vee^{CK+}{}_R$$

$$\frac{G \mid \alpha : M, \neg \boxplus A \vee \boxplus B \Rightarrow N}{G \mid \alpha : M, \boxplus(\neg \boxplus A \vee B) \Rightarrow N}\ \vee^{CK-}{}_L \qquad \frac{G \mid \alpha : M \Rightarrow \neg \boxplus A, \boxplus B, N}{G \mid \alpha : M \Rightarrow \boxplus(\neg \boxplus A \vee B), N}\ \vee^{CK-}{}_R$$

$$\frac{G \mid \alpha : M, \boxplus A \Rightarrow N}{G \mid \alpha : M, \boxplus \boxplus A \Rightarrow N}\ 4^{CK}{}_L \qquad \frac{G \mid \alpha : M \Rightarrow \boxplus A, N}{G \mid \alpha : M \Rightarrow \boxplus \boxplus A, N}\ 4^{CK}{}_R$$

$$\frac{G \mid \alpha : M, \neg \boxplus A \Rightarrow N}{G \mid \alpha : M, \boxplus\neg \boxplus A \Rightarrow N}\ 5^{CK}{}_L \qquad \frac{G \mid \alpha : M \Rightarrow \neg \boxplus A, N}{G \mid \alpha : M \Rightarrow \boxplus\neg \boxplus A, N}\ 5^{CK}{}_R$$

$$\frac{G \mid \alpha : M, \boxplus A^{MDNF} \Rightarrow N}{G \mid \alpha : M, \boxplus A \Rightarrow N}\ MDNF^{CK}{}_L \qquad \frac{G \mid \alpha : M \Rightarrow \boxplus A^{MDNF}, N}{G \mid \alpha : M \Rightarrow \boxplus A, N}\ MDNF^{CK}{}_R$$

Figure 4: Common Knowledge Rules

*Proof.* The cases are similar, so we shall only consider the cases of $\wedge^{CK}L$, $\wedge^{CK}R$ and $MDNF^{CK}L$ in detail. First consider the case of $\wedge^{CK}L$; suppose we have a canonical derivation $d$ of $G \,|\, \alpha : M, \boxast A, \boxast B \Rightarrow N$. Go up this derivation to the axioms and consider all formulas from which the $\boxast A$ and $\boxast B$ have been derived. Consider firstly the common knowledge formulas $\boxast A$ and $\boxast B$ in the axioms. For any sequents in the axioms containing only occurrences of $\boxast A$ and $\boxast B$ that are not principal (ie. such that neither $\boxast A$ nor $\boxast B$ have a positive occurrence on the right hand side), replace any pair $\boxast A$, $\boxast B$ by $\boxast(A \wedge B)$.

Now consider a sequent in an axiom $H$ where $\boxast A$ is principal: ie. the sequent has the form $\boxast A, M \Rightarrow N, \boxast A$. (The case of a sequent where $\boxast B$ is principal is treated similarly.) Let $H' = H'' : \boxast(A \wedge B), M \Rightarrow N, \boxast A$ be the result of replacing this sequent in the axiom by $\boxast(A \wedge B), M \Rightarrow N, \boxast A$; we shall show that there is a canonical derivation of $H'$. It is straightforward to construct an $\boxast R$-free derivation of $\boxast(A \wedge B) \Rightarrow \Box A$; moreover, since $\boxast(A \wedge B)$ is a conjunct in every disjunct in $(\neg (H')^{\tau}_{\alpha' : \boxast(A \wedge B), M \Rightarrow N, \boxast A})^{CK-core}$, one obtains using weakening a $\boxast R$-free derivation of $(\neg (H')^{\tau}_{\alpha' : \boxast(A \wedge B), M \Rightarrow N})^{CK-core} \Rightarrow \Box A$. Since $(\neg (H')^{\tau}_{\alpha' : \boxast(A \wedge B), M \Rightarrow N, \boxast A})^{CK-core}$ is a disjunction of conjunctions of common knowledge formulas, it is straightforward, using essentially the rules $\boxast L_2$ and the modal rules, to construct a $\boxast R$-free derivation of $(\neg (H')^{\tau}_{\alpha' : \boxast(A \wedge B), M \Rightarrow N})^{CK-core} \Rightarrow \Box(\neg (H')^{\tau}_{\alpha' : \boxast(A \wedge B), M \Rightarrow N})^{CK-core}$. Finally, by the same reasoning as that used in Lemma 7, there exist $\boxast R$-free derivations of the $H'' \,|\, \alpha' : (A \wedge B), M \Rightarrow N, (\neg (H')^{\tau}_{\alpha : \boxast A, M \Rightarrow N})^{CK-core})$. Hence $\boxast R$ may be applied, yielding a canonical derivation of $H'$. Repeating for all such sequents in the axioms, one obtains a derivation where all the relevant occurrences of $\boxast A$ and $\boxast B$ have been replaced by $\boxast(A \wedge B)$. Proceeding similarly for occurrences of $\Box_z A$, $\Box_z B$ which were involved in the derivation of $\boxast A$, $\boxast B$ via an application of the $\boxast L_1$ rule (replace occurrences in the axioms by $\Box_z (A \wedge B)$ and derivations by derivations of this formula), one obtains a derivation $d'$ of $G \,|\, \alpha : M, \boxast(A \wedge B) \Rightarrow N$. Since $d$ is canonical, the only new applications of the $\boxast R$ rule in $d'$ are canonical, and any applications of the $\boxast R$ rule in $d$ evidently correspond to canonical applications in $d'$, $d'$ is a canonical derivation as required.

The $\wedge^{CK}R$ rule is treated similarly, with one extra case to be examined. Consider a canonical derivation $d$ of $G \,|\, \alpha : M \Rightarrow \boxast A \wedge \boxast B, N$. If the occurrences of $\boxast A$ and $\boxast B$ in $\boxast A \wedge \boxast B$ have come from axioms, then a procedure similar to the one above can be straightforwardly applied: if either of them or the conjunction come from non-principal occurrences in an axiom, then replace that occurrence and proceed with the derivation from there; if both of them come from principal occurrences, then one must have a sequent of the form $\boxast A, \boxast B, M \Rightarrow N, \boxast A$ in an axiom, and it is straightforward to supply a canonical derivation of $\boxast A, \boxast B, M \Rightarrow N, \boxast(A \wedge B)$ with which to replace it. The final (novel) case is where $\boxast A$ or $\boxast B$ are introduced in an application of the $\boxplus R$ rule; suppose this is the case for $\boxplus A$. By Lemma 4, we can assume without loss of generality that the relevant application(s) of the $\boxplus R$ rule – to $A$ and to $B$ if it too was derived by a $\boxplus R$ rule – occur(s) just before the $\wedge R$ rule used to derive $\boxplus A \wedge \boxplus B$. Let $H$ be the ‘context’ for this application of the $\wedge R$ rule: ie. $H = H' \mid \alpha : \Gamma$ and the premises of the $\wedge R$ rule are $H' \mid \alpha : \Gamma, \boxplus A$ and $H' \mid \alpha : \Gamma, \boxplus B$. Since $d$ is canonical, the application of the $\boxplus R$ rule introducing $\boxplus A$ involves the derivation of a sequent of the form $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-A}} \Rightarrow \Box A$ or a sequent of the form $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK-core} \Rightarrow \Box A$. As concerns $\boxplus B$, either it has been introduced in the axioms, in which case (given that the context $H$ is the same) there are $\boxplus R$-free derivations of $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-A}} \Rightarrow \Box B$ and $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK-core} \Rightarrow \Box B$, or it has also been introduced in an application of the $\boxplus R$ rule, in which case there is a canonical derivation of a sequent of the form $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-B}} \Rightarrow \Box B$ or a sequent of the form $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK-core} \Rightarrow \Box B$. We distinguish two cases. On the one hand, if the principal formula in the introduction of both formula is $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK-core}$, or there is no application of the $\boxplus R$ rule introducing $\boxplus B$ and the principal formula in the derivation of $\boxplus A$ is $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK-core}$, then combining the derivations described above (and using the invertibility of the modal rules) yields a canonical derivation of $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK-core} \Rightarrow \Box(A \wedge B)$. Moreover, the derivation of $\boxplus A$ contains a canonical derivation of $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK-core} \Rightarrow \Box(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK-core}$, giving the third premise of the $\boxplus R$ rule. On the other hand, if the principal formula in the introduction of both formula is $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-A}}$ and $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-B}}$, or there is no application of the $\boxplus R$ rule introducing $\boxplus B$ and the principal formula in the derivation of $\boxplus A$ is $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-A}}$, then, since $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-(A\wedge B)}}$ contains all the conjuncts in $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-A}}$ and in $(\neg(H')^{\tau}_{\alpha:\Gamma'})^{CK}_{\mathcal{P}_{-B}}$, the derivations described above can be used to obtain a canonical derivation of $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-(A\wedge B)}} \Rightarrow \Box(A \wedge B)$. Moreover, since $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-(A\wedge B)}}$ differs from $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-A}}$ only in that it contains more common knowledge formulas, and for any common knowledge formula $\neg \boxplus C$ it is straightforward to construct a $\boxplus R$-free derivation of $\neg \boxplus C \Rightarrow \Box \neg \boxplus C$, one can use the canonical derivation of $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-A}} \Rightarrow \Box(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-A}}$ in the derivation of $\boxplus A$ to construct a canonical derivation of $(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-(A\wedge B)}} \Rightarrow \Box(\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-(A\wedge B)}}$. By Lemma 7 and similar reasoning, there exists $\boxplus R$-free derivations of $H' \mid \alpha : \Gamma, (\neg(H)^{\tau}_{\alpha:\Gamma})^{CK}_{\mathcal{P}_{-(A\wedge B)}}$ and $H' \mid \alpha : \Gamma, (\neg(H)^{\tau}_{\alpha:\Gamma})^{CK-core}$. Hence there are canonical derivations of the premises of a canonical application of the $\boxplus R$ rule with conclusion $H' \mid \alpha : \Gamma, \boxplus(A \wedge B)$. Repeating, and given the canonicity of $d$, one obtains a canonical derivation of $G \mid \alpha : M \Rightarrow \boxplus(A \wedge B), N$, as required.

Finally consider $MDNF^{CK}L$. This is essentially the same as the previous cases, with the added observation that, to derive the appropriate sequent (in this case, $\boxplus A \Rightarrow \Box A^{MDNF}$), one does not require any applications of the rule $\boxplus R$ (because, essentially, by the definition of $A^{MDNF}$, there is a $\boxplus R$-free derivation of $A \Rightarrow A^{MDNF}$), and hence the derivation is canonical. The reasoning in the first case considered above goes through to yield the desired conclusion. □

**Lemma 9.** *If* $G \mid \alpha : M \Rightarrow N, \boxplus A$ *is derivable, then* $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-A}} \Rightarrow \Box A$ *is derivable.*

*Proof.* By Theorem 4.1 and Definition 7, since $G \mid \alpha : M \Rightarrow N, \boxplus A$ is derivable, $\vdash (G \mid \alpha : M \Rightarrow N, \boxplus A)^{\tau}_{\alpha:M\Rightarrow N,\boxplus A}$ in the system **S5C**. It follows from Definition 6 and propositional logic that $\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N} \vdash \boxplus A$. For brevity, let $\mathcal{C} = \neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N}$. We now show that $\mathcal{C}^{CK}_{\mathcal{P}_{-A}} \vdash \boxplus A$, reasoning semantically, and using the soundness and completeness of the standard Kripke semantics with respect to the system **S5C**.[12] We thus have $\mathcal{C} \vDash \boxplus A$, and we wish to show that $\mathcal{C}^{CK}_{\mathcal{P}_{-A}} \vDash \boxplus A$.

First note that, if $\mathcal{C}$ is true for some state in a *CK*-cell, then $\mathcal{C}^{CK}$ holds for that state in the cell; conversely, if $\mathcal{C}^{CK}$ holds for some state in a *CK*-cell, then there must be a state in the cell which satisfies $\mathcal{C}$. Hence the set of *CK*-cells for which $\mathcal{C}$ is true for some state in the cell coincides with the set of *CK*-cells for which $\mathcal{C}^{CK}$ is true for some state in the cell. Since, by the form of $\boxplus A$, the truth of $\boxplus A$ in a state depends entirely on the *CK*-cell to which the state belongs, and since $\mathcal{C} \vDash \boxplus A$, we have that $\mathcal{C}^{CK} \vDash \boxplus A$.

Now, for any set $P$ of *CK*-cells, let the $\mathcal{P}_{-A}$-closure of $P$ be the largest set of *CK*-cells containing $P$ such that the (states in the) cells all give the same valuation to all formulas of the form $\boxplus C$, and to all formulas of the form $\neg \boxplus C$ containing only propositional atoms in $\mathcal{P}_{-A}$. It is clear that the set of *CK*-cells satisfying $\mathcal{C}^{CK}_{\mathcal{P}_{-A}}$ is contained in the $\mathcal{P}_{-A}$-closure of the set satisfying $\mathcal{C}^{CK}$. Moreover, since the only propositional atoms occurring in the scope of negative occurrences of $\boxplus$ in $\boxplus A$ belong to $\mathcal{P}_{-A}$, the set of *CK*-cells satisfying $\boxplus A$ is the $\mathcal{P}_{-A}$-closure of itself. Since the operation of $\mathcal{P}_{-A}$-closure is evidently monotonic (ie. if $P \subseteq Q$, then the $\mathcal{P}_{-A}$-closure of $P$ is contained in the $\mathcal{P}_{-A}$-closure of $Q$), it follows that $\mathcal{C}^{CK}_{\mathcal{P}_{-A}} \vDash \boxplus A$.

Since $\mathcal{C}^{CK}_{\mathcal{P}_{-A}} \vDash \boxplus A$, it follows that $\mathcal{C}^{CK}_{\mathcal{P}_{-A}} \vDash \Box A$. By the completeness of the standard Hilbert calculus and Theorem 4.1, it follows that $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-A}} \Rightarrow \Box A$ is derivable, as required. □

We finally come to the main result concerning the principal formula in the applications of the $\boxplus R$ rule.

[12] We assume standard Kripke semantics terminology (eg. [6]); moreover, we use the term *CK*-cell for the set of states accessible from a given state by the accessibility relation for the common knowledge operator.

**Proposition 2.** *If $G \mid \alpha : M \Rightarrow N, \boxplus A$ is derivable, then there exists a canonical derivation of it.*

*Proof.* We construct a derivation whose last rule is an application of the $\boxplus R$ rule with principal formula $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-A}}$ (so the application is canonical), and such that all the derivations of the premises only contain canonical applications of the $\boxplus R$ rule. Consider the derivations of these premises.

Lemma 7 guarantees that there exists a canonical derivation of the left premise, $G \mid \alpha : M \Rightarrow N, (\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-A}}$. Now consider the right premise, $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK} \Rightarrow \Box(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-A}}$. By definition, $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-A}} = \bigvee(\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus})$; it suffices to give, for each conjunct, a canonical derivation of $\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus} \Rightarrow \Box(\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus})$ – combination of these derivations into a derivation of $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-A}} \Rightarrow \Box(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-A}}$ is a straightforward application of modal and propositional rules (and their invertibility). It is straightforward, using modal rules, $\boxplus L_1$ and $\boxplus L_2$, to construct (canonical) derivations of $\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge D_k^{+\boxplus} \Rightarrow \Box \bigwedge \neg \boxplus D_j^{-\boxplus}$ and similarly for $\bigwedge \boxplus D_k^{+\boxplus}$. As concerns $\bigwedge D_i^{prop}$, either there is a $\boxplus R$-free derivation of $\bigwedge D_i^{prop} \wedge \bigwedge \boxplus D_k^{+\boxplus} \Rightarrow \Box \bigwedge D_i^{prop}$ or not. In the former case, weakening evidently yields a $\boxplus R$-free – and hence canonical – derivation of the required sequent. In the latter case, note that, by the definition of $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-A}}$ and CKDNF, $\left(\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus}\right)^{CKDNF} \Rightarrow \Box \bigwedge D_i^{prop}$ is derivable; moreover, since $D_i^{prop}$ does not contain any occurrences of $\boxplus$ and there are no embedded occurrences of $\boxplus$ on the left hand side of this sequent, $\left[\left(\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus}\right)^{CKDNF}\right] \Rightarrow \Box \bigwedge D_i^{prop}$ is derivable, where $\left[\left(\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus}\right)^{CKDNF}\right]$ is the result of removing every conjunct of the form $\neg \boxplus E$ from $\left(\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus}\right)^{CKDNF}$. Since there are no negative occurrences of $\boxplus$ on the left hand side of this sequent and no positive occurrences on the right hand side, the derivation of this sequent does not involve any applications of the $\boxplus R$ rule, and hence is canonical. By weakening, one thus obtains a canonical derivation of $\left(\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus}\right)^{CKDNF} \Rightarrow \Box \bigwedge D_i^{prop}$. However, by definition of CKDNF (see in particular the proof of Proposition 1), $\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus}$ can be obtained from $\left(\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus}\right)^{CKDNF}$ by a series of equivalences that correspond to the common knowledge rules in Figure 2; Lemma 8 thus implies that there is a canonical derivation of

$\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus} \Rightarrow \Box \bigwedge D_i^{prop}$. Hence one obtains a canonical derivation of $\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus} \Rightarrow \Box(\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus})$, as required.

Consider now the central premise $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-\mathcal{A}}} \Rightarrow \Box A$. By Lemma 9, there exists a derivation of this premise. It remains to be shown that there exists a derivation in which all applications of the $\boxplus R$ rule are canonical. We shall do this by essentially reasoning by induction on the number of (appropriate) occurrences of $\boxplus$ in the indexed hypersequent. To this end, for each formula $A$, we define the positive (resp. negative) CK degree of $A$, $dgCK^{+}(A)$ (resp. $dgCK^{-}(A)$) to be the number of positive occurrences of $\boxplus$ in $A$. Similarly, the positive (resp. negative) CK degree of a multi-set of formulas $M$, $dgCK^{+}(M)$ (resp. $dgCK^{-}(M)$) is the sum of the positive (resp. negative) CK degrees of the formulas in $A$. Finally, the CK degree of a sequent $M \Rightarrow N$, $dgCK(M \Rightarrow N) = dgCK^{-}(M) + dgCK^{+}(N)$ and the CK degree of an indexed hypersequent is the sum of the CK degrees of the sequents composing it. This notion is important since, as is easily seen on inspection, the CK degree of a indexed hypersequent $H$ all of whose formulas are in CKDNF is the maximum possible number of applications of the rule $\boxplus R$ in a derivation of $H$. In particular, if the CK degree of an indexed hypersequent sequent is zero, then any derivation of it contains no applications of the $\boxplus R$ rule, and hence is canonical.

Since $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-\mathcal{A}}} \Rightarrow \Box A$ is derivable, and since $\left(\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus}\right)^{CKDNF}$ can be obtained from $\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus}$ by a series of equivalences (see the proof of Proposition 1), and likewise for $(\Box A)^{CKDNF}$, there exists a derivation of $\left((\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-\mathcal{A}}}\right)^{CKDNF} \Rightarrow (\Box A)^{CKDNF}$. Moreover, by Lemma 8, it suffices to show that there is a canonical derivation of this sequent to conclude that there is a canonical derivation of $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-\mathcal{A}}} \Rightarrow \Box A$. By definition, $\left((\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-\mathcal{A}}}\right)^{CKDNF} = \bigvee(\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus})$, for some $D^{prop}$, $D_j^{-\boxplus}$ and $D_k^{\boxplus}$ not containing any occurrences of $\boxplus$. By the invertibility of the propositional rules (Lemma 3), there are derivations of sequents of the form $\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus} \Rightarrow (\Box A)^{CKDNF}$. Naturally, it suffices to show that there are canonical derivations of these sequents; applications of the appropriate propositional rules will yield the required canonical derivation of $((\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-\mathcal{A}}})^{CKDNF} \Rightarrow (\Box A)^{CKDNF}$.

Consider any such sequent $\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxplus D_j^{-\boxplus} \wedge \bigwedge \boxplus D_k^{+\boxplus} \Rightarrow (\Box A)^{CKDNF}$ and any derivation of this sequent, $d$. If $d$ is canonical, there is nothing more to show; now suppose that this is not the case, and consider the last application of the $\boxplus R$ rule in $d$. Let the conclusion of this application be $H \mid \beta : P \Rightarrow Q, \boxplus B$. By the same reasoning as applied above to

$G \mid \alpha : M \Rightarrow N, \boxtimes A$, this application can be replaced by a canonical application, whose central premise $(\neg(H \mid \beta : P \Rightarrow Q)^{\tau}_{\beta:P\Rightarrow Q})^{CK}_{\mathcal{P}_{-B}} \Rightarrow \Box B$ is derivable (as shown above, there are canonical derivations of the other premises). Since all the occurrences of the $\boxtimes$ in $\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxtimes D_j^{-\boxtimes} \wedge \bigwedge \boxtimes D_k^{+\boxtimes} \Rightarrow (\Box A)^{CKDNF}$, and hence in $H \mid \beta : P \Rightarrow Q, \boxtimes B$ have CK depth one (and hence do not have any occurrences of $\boxtimes$ in their scope), $\mathcal{P}_{-B} = \varnothing$ and so $dgCK((\neg(H \mid \beta : P \Rightarrow Q)^{\tau}_{\beta:P\Rightarrow Q})^{CK}_{\mathcal{P}_{-B}} \Rightarrow \Box B) = dgCK(H \mid \beta : P \Rightarrow Q) < dgCK(H \mid \beta : P \Rightarrow Q, \boxtimes B) = dgCK(\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxtimes D_j^{-\boxtimes} \wedge \bigwedge \boxtimes D_k^{+\boxtimes} \Rightarrow (\Box A)^{CKDNF})$. Since $(\neg(H \mid \beta : P \Rightarrow Q)^{\tau}_{\beta:P\Rightarrow Q})^{CK}_{\mathcal{P}_{-B}} \Rightarrow \Box B$ is derivable, this procedure can be repeated on any derivation of this premise. Moreover, since, by the argument just used, the CK degree of the central premise of the 'next' application up of the $\boxtimes R$ is strictly less than CK degree of the central premise of the last application of the $\boxtimes R$ rule to be treated, and since, as noted above, when the CK degree of a indexed hypersequent is zero, any derivation of it is canonical, this procedure will eventually halt with a canonical derivation of the central premise of an application of the $\boxtimes R$ rule. Since all the applications of the $\boxtimes R$ rules in the derivation obtained by this procedure are canonical, this yields a canonical derivation of $\left(\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxtimes D_j^{-\boxtimes} \wedge \bigwedge \boxtimes D_k^{+\boxtimes}\right)^{CKDNF} \Rightarrow (\Box A)^{CKDNF}$, and hence, by Lemma 8, a canonical derivation of $\bigwedge D_i^{prop} \wedge \bigwedge \neg \boxtimes D_j^{-\boxtimes} \wedge \bigwedge \boxtimes D_k^{+\boxtimes} \Rightarrow \Box A$. Repeating for the other disjuncts, we obtain a canonical derivation of $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-A}} \Rightarrow \Box A$, as required.

$\Box$

By restricting the form of the principal formula in every application of the $\boxtimes R$ rule, this result limits the non-analyticity of the calculus. On the one hand, it indicates that, to search for a proof of a formula, it suffices at any point to consider at most two possible applications of the $\boxtimes R$ rule (with the formula $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK}_{\mathcal{P}_{-A}}$ or with the formula $(\neg(G \mid \alpha : M \Rightarrow N)^{\tau}_{\alpha:M\Rightarrow N})^{CK-core}$); a major inconvenience of the lack of subformula property, namely the fact that it renders proof search impossible, because one would have to search for 'disappearing' principal formulas, is thus largely overcome. Indeed, it should be noticed that often in practice the two formulas defined in the notion of canonical application of the $\boxtimes R$ in fact coincide, so there is only one possible application of the rule to consider. On the other hand as concerns the 'partialness' of our cut-elimination, they strengthen the cut-elimination result, insofar as they greatly restrict the application of the $\boxtimes R$ rule: to one of two potential principal formulas for each conclusion. Indeed, Proposition 2 could be thought of as a sort of elimination result for all applications of the $\boxtimes R$ rule except, at most, two.

To give an idea of the strength of the restrictions Proposition 2 places on the application of the $\boxtimes R$ rule, to give a comparison with partial cut-elimination results for finitary calculi elsewhere in the literature, as well as to give an example of an application of the calculus, suppose that there are only two agents $a$ and $b$, and consider the following (derivable) sequent, taken from [2]: $\Box_a(P \wedge \boxplus Q), \Box_b(Q \wedge \boxplus P) \Rightarrow \boxplus(P \vee Q)$. This sequent is not derivable in the finitary calculus proposed by [2] without the cut rule, and the partial cut-elimination result they have limits the set of cuts that can be used to derive the formula to (at least) an order of $2^{18}$.[13]

By contrast, straightforward calculation shows that the formula proposed in Proposition 2 for this case is just $\boxplus P \wedge \boxplus Q$. (This is an example where the two canonical principal formula coincide.) To search for a proof involving a final application of the $\boxplus R$ rule, it suffices to search for one where the principal formula is $\boxplus P \wedge \boxplus Q$. And indeed, it is easy to see how to construct such a proof. The derivation of the leftmost premise of the rule is:

$$\dfrac{\dfrac{\dfrac{\Box_a(P \wedge \boxplus Q), P, \boxplus Q, \Box_b(Q \wedge \boxplus P), \Rightarrow \boxplus Q}{\Box_a(P \wedge \boxplus Q), P \wedge \boxplus Q, \Box_b(Q \wedge \boxplus P), \Rightarrow \boxplus Q}\wedge L}{\Box_a(P \wedge \boxplus Q), \Box_b(Q \wedge \boxplus P) \Rightarrow \boxplus Q}\Box L_1 \quad \dfrac{\dfrac{\Box_a(P \wedge \boxplus Q), \Box_b(Q \wedge \boxplus P), Q, \boxplus P \Rightarrow \boxplus P}{\Box_a(P \wedge \boxplus Q), \Box_b(Q \wedge \boxplus P), Q \wedge \boxplus P \Rightarrow \boxplus P}\wedge L}{\Box_a(P \wedge \boxplus Q), \Box_b(Q \wedge \boxplus P) \Rightarrow \boxplus P}\Box L_1}{\Box_a(P \wedge \boxplus Q), \Box_b(Q \wedge \boxplus P) \Rightarrow \boxplus P \wedge \boxplus Q}\wedge R$$

The derivation of the middle premise is:[14]

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{1a : \boxplus P, \Box_a P, \boxplus Q \Rightarrow \ | \ 1a : P \Rightarrow P, Q}{1a : \boxplus P, \Box_a P, \boxplus Q \Rightarrow \ | \ 1a : P \Rightarrow P \vee Q}\vee R}{1a : \boxplus P, \Box_a P, \boxplus Q \Rightarrow \ | \ 1a : \Rightarrow P \vee Q}\Box_a L_2}{\boxplus P, \Box_a P, \boxplus Q \Rightarrow \Box_a(P \vee Q)}\Box_a R}{\boxplus P, \boxplus Q \Rightarrow \Box_a(P \vee Q)}\boxplus_a L_1}{\boxplus P \wedge \boxplus Q \Rightarrow \Box_a(P \vee Q)}\wedge L$$

and similarly for $\boxplus P \wedge \boxplus Q \Rightarrow \Box_b(P \vee Q)$, with a final application of the $\wedge R$ rule. Finally, the derivation of the right premise is:

$$\dfrac{\dfrac{\dfrac{\dfrac{1a : \boxplus P, \boxplus Q \Rightarrow \ | \ 1a : \boxplus P, \boxplus Q \Rightarrow \boxplus P \quad 1a : \boxplus P, \boxplus Q \Rightarrow \ | \ 1a : \boxplus P, \boxplus Q \Rightarrow \boxplus Q}{1a : \boxplus P, \boxplus Q \Rightarrow \ | \ 1a : \boxplus P, \boxplus Q \Rightarrow \boxplus P \wedge \boxplus Q}\wedge R}{1a : \boxplus P, \boxplus Q \Rightarrow \ | \ 1a : \Rightarrow \boxplus P \wedge \boxplus Q}\boxplus L_2^*}{1a : \boxplus P \wedge \boxplus Q \Rightarrow \ | \ 1a : \Rightarrow \boxplus P \wedge \boxplus Q}\wedge L}{\boxplus P \wedge \boxplus Q \Rightarrow \Box_a(\boxplus P \wedge \boxplus Q)}\Box_a R$$

and similarly for $\boxplus P \wedge \boxplus Q \Rightarrow \Box_b(\boxplus P \wedge \boxplus Q)$.

[13] [2] proposes a partial cut-elimination result according to which any derivable sequent can be derived using only cuts on formula in the disjunctive-conjunctive closure of the Fisher-Ladner closure of the sequent to be proven, though they cite stronger results involving only the conjunctive closure of the Fisher-Ladner closure. They state that the size of the Fischer-Ladner closure is of the order of the length of the formula (which in this case is 18), so the set of conjunctions of elements of the Fischer-Ladner closure is of order $2^{18}$.

[14] See footnote 3 concerning the $\vee R$ rule.

We conclude that, though the proposed calculus is not strictly speaking analytic, it is remarkably easy to construct proofs using it, given the difficulty in finding finitary calculi for common knowledge, and in comparison to other proposals.

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