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A SOUNDNESS AND COMPLETENESS PROOF ON DIALOGS AND DYNAMIC EPISTEMIC LOGIC

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ABSTRACT

Since Plaza [12], which is most of the time considered as the inaugural paper on announcement logics in public communication contexts, a lot of papers on dynamic epistemic logics have been published. The most famous dynamic epistemic logic is known by the name of **PAL** (Public Announcement Logic). The logic **PAC** is an extension of **PAL** with the common knowledge operator (C_G). Soundness and completeness proofs of those logics are presented in van Ditmarsch *et al.* [18], in Balbiani *et al.* [1] and in de Boer [3]. Each of them used either a model-theoretic approach or a tableaux calculus. In the present paper, we propose an alternative approach to **PAC** based on the dialogical framework.

1. Introduction

Public announcement logic (**PAL**) has been proposed at the end of the 80s by Plaza [12] and then developed further, for instance, by Baltag *et al.* [2], van Benthem *et al.* [17] and van Ditmarsch *et al.* [18]. It belongs to a family of logics called dynamic epistemic logics, which aim at modeling the dynamics of knowledge and belief in multi-agent systems. These logics are extensions of multi-agent epistemic logic with dynamic operators representing several kinds of communications among agents. In the case of **PAL**, the extension is public announcements. In this logic, one can write a formula of the form $[\varphi]\psi$, which means "if φ is publicly announced then, after that announcement, ψ is true". In the standard model-theoretic interpretation of such formula, the public announcement of φ causes an *update* of the current epistemic state of the agents. The result is a new epistemic state without the possible worlds where φ is false. It intends to model the result of the public announcement φ, which should be that, after it, the agents have learned that φ was true.

In this paper, we propose a *dialogical* approach to public announcement logic, also establishing a formal relation with its standard model-theoretical

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approach. The dialogical approach is not a different logical system but rather a rule-based semantic framework. The basic idea, inspired by Wittgenstein's "meaning as use", is that the meaning of a logical constant (or formula) is given by the norms and rules for its use. The truth of logical constants is defined by an argumentative process between two antagonists. In this process, similar to a game, two players confront each other around a thesis (the initial argument). The proponent starts by uttering the thesis and then try to defend it against the opponent, which tries to construct a counter-argument to it. If the proponent is capable to win the argumentative game whatever arguments are advanced by the opponent, the thesis is considered to be valid, i.e., the thesis is valid if and only if the proponent turns out to have a winning strategy to defend it.

The dialogical approach, originally applied to classical and intuitionistic logic, was first proposed at the end of the 50s by Lorenzen and then worked out by Lorenz [8]. On 90s, the framework has been developed further and applied to various non-classical logics.[1] The present proposal borrows some ideas from the dialogical approach to epistemic logic developed by Rebuschi [16] and Rebuschi & Lihoreau [15], as well as from the approach by Magnier [10].

This new approach to public announcement logic gives some interesting insights in the area at the intersection between logic and law. This is so because the validity of an utterance is defined in terms of the ability of the utterer to defend it during a debate, which plays a crucial role in civil law. As a simple example of such kind of debate, consider the following conditional precedent: "if a ship arrives from Asia, I (Primus) give to you (Secundus) 100 coins". Although this is a conditional sentence, this works more like a public announcement than a standard conditional. In fact, Secundus can (legally) compel Primus to give its due only after the arrival of the ship is publicly known, i.e., only after it is publicly announced. We do not develop more on this in this paper, but the interested reader may find interesting material in Magnier [11] and Magnier and Rahman [9].

The remainder of this paper is organized as follows. In Section 2, we present the formal definitions and some examples of argumentative games. In Sections 3 and 4, we establish the main result of this paper, namely, soundness and completeness for the dialogical approach to **PAL** (and its extension with common knowledge **PAC**) with respect to the model-theoretic approach. More precisely, we show that the thesis uttered by the proponent is valid in (the model-theoretic approach to) **PAC** if and only if the proponent has a wining strategy to defend it. This establishes a formal relation between the dialogical and the standard model-theoretic approach

[1] One can find introductory texts on these approaches in Fontaine & Redmond [5] and in Keiff [7].

to **PAC**. Section 5 concludes the paper and discusses some possible future work.

2. Definitions

In this section, we present the public announcement logic with common knowledge **PAC**. We start with its syntax, i.e., its language and intended meanings, and its model-theoretic semantics, as presented in van Ditmarsch *et al.* [18]. After that, we present an alternative semantics based on the dialogical framework.

2.1. *Syntax*

Definition 1 (Language)**.** Let a non-empty finite set of agent names Ag and a countable set of propositional variables $\mathcal{P}$ be given. The language $\mathcal{L}_{\mathbf{PAC}}(Ag,\mathcal{P})$ is the set of formulas φ inductively defined by the following BNF:

$$\varphi := p \mid \neg\varphi \mid \varphi\wedge\varphi \mid K_a\varphi \mid E_G\varphi \mid C_G\varphi \mid \langle\varphi\rangle\varphi$$

where a ranges over Ag, G ranges over 2^{Ag} and p ranges over $\mathcal{P}$.

In what follows, we also use the common abbreviations for $\vee$, $\rightarrow$ and $[\,]$. The latter one is defined by: $[\varphi]\psi \stackrel{\text{def}}{=} \neg\langle\varphi\rangle\neg\psi$.

The intended meanings for formulas in $\mathcal{L}_{\mathbf{PAC}}(Ag,\mathcal{P})$ are the following:

Individual knowledge

$K_a\varphi$ means "agent a knows that φ is true".

Sharing knowledge

$E_G\varphi$ means "all members of the group G know that φ is true".

Common knowledge

$C_G\varphi$ means "all members of the group G know that all members of the group G know that ... φ is true".

Public announcement

$[\varphi]\psi$ means "if φ is publicly announced then, after that announcement, ψ is true".

Dual of public announcement

$\langle\varphi\rangle\psi$ means "φ is publicly announced and, after that announcement, ψ is true".

2.2. *Model-theoretical Semantics*

Definition 2 (Epistemic Model)**.** An epistemic model is a triple $\mathcal{M} = \langle \mathcal{W}, \{\mathcal{R}_a\}_{a\in Ag}, \mathcal{V}\rangle$ such that $\mathcal{W}$ is a non-empty set of possible worlds, each $\mathcal{R}_a$ is a binary equivalence relation over $\mathcal{W}$, and $\mathcal{V}$ is a valuation function such that, for every $p \in \mathcal{P}$, it yields a set $\mathcal{V}_p \subseteq \mathcal{W}$ of possible worlds.

Definition 3 (Pointed Epistemic Model)**.** A pointed epistemic model is a pair $(\mathcal{M}, w)$ such that $\mathcal{M} = \langle \mathcal{W}, \{\mathcal{R}_a\}_{a\in Ag}, \mathcal{V}\rangle$ is an epistemic model as defined above and $w \in \mathcal{W}$.

Definition 4 (Satisfaction Relation)**.** The satisfaction relation $\models$ between pointed epistemic models and formulas in $\mathcal{L}_{\mathbf{PAC}}(Ag,\mathcal{P})$ is inductively defined as follows:

$$
\begin{array}{lll}
(\mathcal{M},w) \models p & \text{iff} & w \in \mathcal{V}_p \\
(\mathcal{M},w) \models \neg\varphi & \text{iff} & (\mathcal{M},w) \not\models \varphi \\
(\mathcal{M},w) \models \varphi \wedge \psi & \text{iff} & (\mathcal{M},w) \models \varphi \text{ and } (\mathcal{M},w) \models \psi \\
(\mathcal{M},w) \models K_a\varphi & \text{iff} & (\mathcal{M},w') \models \varphi, \text{ for all } w' \text{ such that } (w,w') \in \mathcal{R}_a \\
(\mathcal{M},w) \models E_G\varphi & \text{iff} & (\mathcal{M},w') \models \varphi, \text{ for all } w' \text{ such that } (w,w') \in \bigcup_{a\in G}\mathcal{R}_a \\
(\mathcal{M},w) \models C_G\varphi & \text{iff} & (\mathcal{M},w') \models \varphi, \text{ for all } w' \text{ such that } (w,w') \in \left(\bigcup_{a\in G}\mathcal{R}_a\right)^* \\
(\mathcal{M},w) \models \langle\varphi\rangle\psi & \text{iff} & (\mathcal{M},w) \models \varphi \text{ and } (\mathcal{M}^{\varphi},w) \models \psi
\end{array}
$$

where $\left(\bigcup_{a\in G}\mathcal{R}_a\right)^*$ is the reflexive, symmetric and transitive closure of $\bigcup_{a\in G}\mathcal{R}_a$,[2] and $\mathcal{M}^{\varphi} = \langle \mathcal{W}^{\varphi}, \{\mathcal{R}_a^{\varphi}\}_{a\in Ag}, \mathcal{V}^{\varphi}\rangle$ is the update of $\mathcal{M}$ by the announcement of φ, which is defined as follows:

$$
\begin{aligned}
\mathcal{W}^{\varphi} &= \{w' \in \mathcal{W} : (\mathcal{M},w') \models \varphi\} \\
\mathcal{R}_a^{\varphi} &= \mathcal{R}_a \cap (\mathcal{W}^{\varphi} \times \mathcal{W}^{\varphi}) \\
\mathcal{V}_p^{\varphi} &= \mathcal{V}_p \cap \mathcal{W}^{\varphi}
\end{aligned}
$$

Let $\varphi_1\varphi_2\cdots\varphi_n$ be a sequence of formulas from $\mathcal{L}_{\mathbf{PAC}}(Ag,\mathcal{P})$. For the sake of readability, we use $\mathcal{M}^{\varphi_1\varphi_2\cdots\varphi_n}$ to denote $(((\mathcal{M}^{\varphi_1})^{\varphi_2})\ldots)^{\varphi_n}$; and we also use $(\mathcal{M},w) \models \varphi_1\varphi_1\cdots\varphi_n$ to denote $(\mathcal{M},w) \models \varphi_1$ and $(\mathcal{M}_1^{\varphi},w) \models \varphi_2$ and and $(\mathcal{M}^{\varphi_1\varphi_2\cdots\varphi_{n-1}},w) \models \varphi_n$.

Definition 5 (Validity)**.** A formula $\varphi \in \mathcal{L}_{\mathbf{PAC}}(Ag,\mathcal{P})$ is valid in (the model-theoretical approach of) **PAC**, which is noted $\models \varphi$, if and only if $(\mathcal{M},w) \models \varphi$, for all pointed epistemic models $(\mathcal{M},w)$.

[2] We note that the different semantic definitions for common knowledge, for instance in Gerbrandy [6, chapter 3] and Fagin *et al.* [4, chapter 11] are equivalent to ours.

All tautologies of classical propositional logic	
$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$	(distribution of K_a over $\to$)
$K_a\varphi \to \varphi$	(truth)
$K_a\varphi \to K_aK_a\varphi$	(positive introspection)
$\neg K_a\varphi \to K_a\neg K_a\varphi$	(negative introspection)
$C_G(\varphi \to \psi) \to (C_G\varphi \to C_G\psi)$	(distribution of C_G over $\to$)
$C_G\varphi \to (\varphi \wedge E_GC_G\varphi)$	(mix of common knowledge)
$C_G(\varphi \to E_G\varphi) \to (\varphi \to C_G\varphi)$	(intro. of common knowledge)
$[\varphi]p \leftrightarrow (\varphi \to p)$	(atomic permanence)
$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$	(announcement and negation)
$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$	(announcement and conjunction)
$[\varphi]K_a\psi \leftrightarrow (\varphi \to K_a[\varphi]\psi)$	(announcement and knowledge)
$[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$	(announcement composition)
From φ and $\varphi \to \psi$ infer ψ	(modus ponens)
From φ infer $K_a\varphi$	(necessitation of K_a)
From φ infer $C_G\varphi$	(necessitation of C_G)
From φ infer $[\psi]\varphi$	(necessitation of $[\varphi]$)
From $\chi \to [\varphi]\psi$ and $\chi \wedge \varphi \to E_G\chi$ infer $\chi \to [\varphi]C_G\psi$	(announcement and common knowledge)

Table 1: Axiomatization of **PAC**

Definition 6 (Satisfiability)**.** A formula $\varphi \in \mathcal{L}_{\mathbf{PAC}}(Ag, \mathcal{P})$ is satisfiable in (the model-theoretical approach of) **PAC** if and only if, $(\mathcal{M}, w) \models \varphi$, for some pointed epistemic model $(\mathcal{M}, w)$.

Note that the latter means that φ is satisfiable if and only if $\not\models \neg\varphi$.

At this point, it is perhaps worthwhile to recall the axiomatization of **PAC**. This is displayed in Table 1. Its soundness and completeness is proved in van Ditmarsch *et al.* [18].

2.3. *Dialogical Semantics*

As mentioned earlier, in the dialogical semantics, the validity of a formula is defined via an argumentative game between two adversaries, the proponent and the opponent. The proponent starts the game by uttering a thesis (a formula in $\mathcal{L}_{\mathbf{PAC}}(Ag, \mathcal{P})$) and then tries to defend it against the opponent. The moves in this game are either challenges on adversary's moves or defenses against adversary's challenges. The players alternate their moves and must respect a number of rules. In the sequel, we define this argumentative game formally.

Definition 7 (Dialog)**.** A dialog is a (possibly infinite) sequence of moves $d = \mu_0\mu_1\mu_2\ldots$, which is allowed by the rules of the dialogical game (given in several definitions to come).

Definition 8 (Move)**.** A move is a quintuple $(\mathbf{X}, \omega, \lambda, t, \varphi)$, where:

- $\mathbf{X} \in \{\mathbf{P}, \mathbf{O}\}$ is the player who authors of the move: **P** represents the proponent whereas **O** represents the opponent;
- $\omega \in (Ag\mathbb{N})^*$ is a finite (possibly empty) sequence $a_0 n_0 a_1 n_1 \ldots n_k a_k$ such that for each $0 \leq i \leq k$, $a_i \in Ag$ and $n_i \in \mathbb{N}$;
- $\lambda \in \mathcal{L}_{\mathbf{PAC}}(Ag, \mathcal{P})^*$ is a finite (possibly empty) sequence of public announcements, i.e., a finite sequence $\varphi_0\varphi_1\ldots\varphi_k$ such that for each $0 \leq i \leq k$, $\varphi_i \in \mathcal{L}_{\mathbf{PAC}}(Ag, \mathcal{P})$;
- $t \in \{\mathbf{!}, \mathbf{?}\neg, \mathbf{?}\wedge, \mathbf{?}K, \mathbf{?}RK, \mathbf{?}E, \mathbf{?}RE, \mathbf{?}C, \mathbf{?}RC, \mathbf{?}\langle\,\rangle, \mathbf{?}U, \mathbf{?}P\}$ is the type of the move: '**!**' represents a defense and the other ones represent several different challenges (which are explained latter);
- $\varphi \in \mathcal{L}_{\mathbf{PAC}}(Ag, \mathcal{P})$ is the argument advanced in the move.

We use $()$ to denote the empty sequence. Moreover, **Y** denotes the adversary of player **X**, i.e., if $\mathbf{X} = \mathbf{P}$ then $\mathbf{Y} = \mathbf{O}$ and if $\mathbf{X} = \mathbf{O}$ then $\mathbf{Y} = \mathbf{P}$.

In a move $\mu = (\mathbf{X}, \omega, \lambda, t, \varphi)$, the pair formed by ω and λ is also called the context of μ. In the model-theoretical semantics, ω would correspond to a possible world in the epistemic model updated by the sequence of public announcements λ. The role of such contexts in dialogs will become clearer after the explanations of the examples in the next pages.

For the sake of readability, we sometimes drop parentheses, comas and empty sequences when displaying moves. For example, we may simply write $\mathbf{P}\,!\varphi$ to mean that 'proponent defends argument φ in the empty context' instead of $(\mathbf{P}, (), (), !, \varphi)$.

In the sequel, we define the rules that govern the dialogical game. There are two kinds of rules. The first kind, called structural rules, define how the game starts, how it ends, how the players execute moves and how to determine the winner of a dialog.

Definition 9 (Structural Rules)**.**

- **Starting:** The dialog starts with the defense of the thesis in the empty context by the proponent. That is, the first move of a dialog is a move of the form $(\mathbf{P}, (), (), !, \varphi)$, where $\varphi \in \mathcal{L}_{\mathbf{PAC}}(Ag, \mathcal{P})$.

- **Game-playing:** Players act alternatively. That is, each move immediately following a proponent's move is an opponent's move and each move immediately following an opponent's move is a proponent's move. Moves cannot be repeated.
- **Proponent's Restrictions:**
  - A propositional variable $p \in \mathcal{P}$ can be the argument of a proponent's move in context $(\omega, ())$ only if it has already been the argument of an opponent's move in this same context in the dialog. That is, a move of the form $(\mathbf{P}, \omega, (), t, p)$ can only appear in a dialog if a move of the form $(\mathbf{O}, \omega, (), t', p)$ has already appeared in the dialog.
  - A non-empty sequence ω can be used in a proponent's move only if it has already been used in an opponent's move in the dialog. That is, a move of the form $(\mathbf{P}, \omega \cdot an, \lambda, t, \varphi)$ can only appear in a dialog if a move of the form $(\mathbf{O}, \omega \cdot an, \lambda', t', \varphi')$ has already appeared in the dialog, where $\omega \cdot an$ stand for the concatenation of *an* at the end of ω.
- **Opponent's Restriction:** The opponent can react to the same move only once.
- **Terminal:** A dialog is terminal if and only if there are no more allowed moves for it.
- **Winning:** The proponent wins the game if and only if there is no new allowed moves for the opponent.

We remark that Proponent's Restrictions forbids the proponent to start the dialog with a formula consisting of a propositional variable $p \in \mathcal{P}$.

The second kind of rules, called particle rules, define which moves are allowed during a dialog. They depend on the arguments advanced by the players. For the sake of readability, we split them in several categories. The first category concerns boolean operators.[3]

Definition 10 (Particle Rules for Boolean Operators)**.**

- **Challenge of Negation ($?\neg$):**
  If $(\mathbf{X}, \omega, \lambda, t, \neg\varphi)$ is in d, for $t \in \{\mathbf{!}, \mathbf{?}\neg\}$,
  then $(\mathbf{Y}, \omega, \lambda, \mathbf{?}, \varphi)$ is allowed.
- **Challenge of Conjunction ($?\wedge$):**
  If $(\mathbf{X}, \omega, \lambda, t, \varphi \wedge \psi)$ is in d, for $t \in \{\mathbf{!}, \mathbf{?}\neg\}$,
  then both $(\mathbf{Y}, \omega, \lambda, \mathbf{?}\wedge, \neg\varphi)$ and $(\mathbf{Y}, \omega, \lambda, \mathbf{?}\wedge, \neg\psi)$ are allowed.
- **Defense of Conjunction ($!\wedge$):**
  - If $(\mathbf{Y}, \omega, \lambda, \mathbf{?}\wedge, \neg\varphi)$ is in d,
    then $(\mathbf{X}, \omega, \lambda, \mathbf{!}, \varphi)$ is allowed.

[3] For the sake of simplicity, we drop the rules for disjunction and implication.

– If $(\mathbf{Y}, \omega, \lambda, \mathbf{?}\wedge, \neg\psi)$ is in d,
then $(\mathbf{X}, \omega, \lambda, \mathbf{!}, \psi)$ is allowed.

In words, if player **X** advances argument φ, it can be challenged by **Y**. If it is a negation, i.e., $\varphi = \neg\psi$, then **Y** can challenge it with ψ. If it is a conjunction, i.e., $\varphi = \psi_1 \wedge \psi_2$, then **Y** can either challenge the first conjunct, advancing argument $\neg\psi_1$, or the second one, advancing argument $\neg\psi_2$. The defense of **X** will have to be made advancing ψ_1 or ψ_2, respectively.

Note that defenses can be challenged with rules $\mathbf{?}\neg$ and $\mathbf{?}\wedge$, and challenges of negation can also be counter-challenged with these rules. On the other hand, challenges of conjunction cannot be counter-challenged and can only be defended using $\mathbf{!}\wedge$.[4] An exception to this is when the argument is a literal, i.e., a propositional variable $p \in \mathcal{P}$ or its negation $\neg p$. Note that there are no possible challenges or defenses against a move with argument p and also, due to the structural rule Proponent's Restrictions, the proponent cannot challenge or defend a move with argument $\neg p$ if the opponent has not yet advanced argument p.

Also note that boolean particle rules do not use the context of moves. They are used only when epistemic operators or public announcements come to the picture (this will be the case latter).

At this point, we still do not have all the particle rules defined. But, to grasp the idea behind the dialogical approach, it may be worthwhile to see an example using only boolean particle rules now. Then, let us consider the dialog in Table 2. There, each line corresponds to a move. The numbers on the right identify the move and the numbers in parenthesis refer to the move(s) allowing the corresponding defense or challenge. In move 0, the proponent starts by defending the thesis $\neg(\neg p \wedge p)$. Since it is a valid formula, the proponent should be able to win the dialog. Indeed. In move 1, the opponent challenges the negation in the argument of move 0. In move 2, the proponent counter-attacks by challenging the conjunction in the argument of move 1. The opponent then defends the challenge in move 3. Note that the proponent cannot challenge the negation in the argument of move 3 because of the structural rule Proponent's Restrictions. Therefore, in move 4, the proponent decides to challenge the conjunction in the argument of move 1 again, but now advancing argument $\neg p$. The opponent defends it with p in move 5. Now, since argument p have been advanced by the opponent, the proponent can challenge the negation in the argument of move 3. It is done in move 6. Note that move 6 cannot be challenged or defended. In addition, the opponent has already defended all other proponent's challenges and has challenged all other proponent's moves that could

[4] In fact, we will define latter the rule $\mathbf{?}U$, that will be the only one allowed to counter-challenge conjunctions.

0.	**P**	**!**	$\neg(\neg p \wedge p)$	
1.	**O**	**?**$\neg$	$\neg p \wedge p$	(0)
2.	**P**	**?**$\wedge$	$\neg\neg p$	(1)
3.	**O**	**!**	$\neg p$	(2)
4.	**P**	**?**$\wedge$	$\neg p$	(3)
5.	**O**	**!**	p	(4)
6.	**P**	**?**$\neg$	p	(3,5)

Table 2: Dialog about a tautology

be challenged. Thus, the opponent does not have any move allowed. This means that the proponent is the winner.

The second category of particle rules concerns the knowledge operators K_a. These are normal modal box operators. Thus, we decided to use rules that are close to the ones proposed in Rahman & Rückert [14] and in Rahman & Keiff [13] to deal with them.

Definition 11 (Particle Rules for Normal Modal Operators)**.**

- **Challenge of Knowledge Operator (?K):**
  If $(\mathbf{X},\omega,\lambda,t,K_a\varphi)$ is in d, for $t \in \{\mathbf{!},\mathbf{?}\neg\}$,
  then $(\mathbf{Y},\omega\cdot an,\lambda,\mathbf{?}K,\neg\varphi)$ is allowed, for a fresh $n \in \mathbb{N}$.
- **Defense of Knowledge Operator (!K):**
  If $(\mathbf{Y},\omega,\lambda,\mathbf{?}K,\neg\varphi)$ is in d,
  then $(\mathbf{X},\omega,\lambda,\mathbf{!},\varphi)$ is allowed.
- **Challenge of Accessibility Relation (?R):**
  If $(\mathbf{X},\omega\cdot an,\lambda',t',\psi)$, $(\mathbf{X},\omega,\lambda,t,K_a\varphi)$ is in d, for $t \in \{\mathbf{!},\mathbf{?}\neg\}$, any t', λ' and ψ,
  then $(\mathbf{Y},\omega\cdot an,\lambda,\mathbf{?}R,\neg\varphi)$ is allowed.
- **Defense of Accessibility Relation (!R):**
  If $(\mathbf{Y},\omega,\lambda,\mathbf{?}R,\neg\varphi)$ is in d,
  then $(\mathbf{X},\omega,\lambda,\mathbf{!},\varphi)$ is allowed.

As explained before, the context (ω,λ) can be seen as the possible world ω in the epistemic model updated by the sequence of public announcements λ. Now, assume that formula $K_a\varphi$ is true in (ω,λ). Then, the particle rules for normal modal operators correspond to the propagation of formula φ from (ω,λ) to $(\omega\cdot an,\lambda)$. The latter corresponds to another possible world in the same epistemic model that is indistinguishable from ω by agent a. The rules **?**K and **!**K create new possible worlds whereas rules **?**R and **!**R

0.	**P**		**!**	$\neg(K_a\neg(p\wedge\neg q)\wedge(K_a p\wedge\neg K_a q))$	
1.	**O**		**?**$\neg$	$K_a\neg(p\wedge\neg q)\wedge(K_a p\wedge\neg K_a q)$	(0)
2.	**P**		**?**$\wedge$	$\neg(K_a p\wedge\neg K_a q)$	(1)
3.	**O**		**!**	$K_a p\wedge\neg K_a q$	(2)
4.	**P**		**?**$\wedge$	$\neg\neg K_a q$	(3)
5.	**O**		**!**	$\neg K_a q$	(4)
6.	**P**		**?**$\neg$	$K_a q$	(5)
7.	**O**	$a1$	**?**K	$\neg q$	(6)
8.	**P**		**?**$\wedge$	$\neg K_a p$	(3)
9.	**O**		**!**	$K_a p$	(8)
10.	**P**	$a1$	**?**R	$\neg p$	(9,7)
11.	**O**	$a1$	**!**	p	(10)
12.	**P**		**?**$\wedge$	$\neg K_a\neg(p\wedge\neg q)$	(1)
13.	**O**		**?**$\neg$	$K_a\neg(p\wedge\neg q)$	(12)
14.	**P**	$a1$	**?**R	$\neg\neg(p\wedge\neg q)$	(13,7)
15.	**O**	$a1$	**!**	$\neg(p\wedge\neg q)$	(14)
16.	**P**	$a1$	**?**$\neg$	$p\wedge\neg q$	(15)
17.	**O**	$a1$	**?**$\wedge$	$\neg\neg q$	(16)
18.	**P**	$a1$	**!**	$\neg q$	(17)
19.	**O**	$a1$	**?**$\neg$	q	(18)
20.	**P**	$a1$	**!**	q	(7,19)

Table 3: Dialog about an instance of distribution of K_a over $\rightarrow$

propagate formulas to already existing possible worlds. Thus, in the dialog, if player **X** advances the argument $K_a\varphi$ in context (ω,λ) then **Y** can challenge it by choosing a context $(\omega\cdot an,\lambda)$, and then obliges **X** to defend φ in it.

Let us use another example to illustrate the use of the particle rules for normal modal operators. This time, consider the dialog in Table 3. The thesis correspond to an instance of distribution of K_a over $\rightarrow$ (also known as Axiom K, see Table 3). Since it is a valid formula, the proponent should be able to win the dialog. Indeed. The opponent uses rule **?**K in move 7 to challenge the knowledge operator in the argument of move 6. The opponent introduces the new context $a1$ and advances argument $\neg q$ in it. The proponent defends it only in move 20, because is necessary to wait until the opponent advances argument q in context $a1$, due to the structural rule

Proponent's Restrictions. In this dialog, the proponent also uses rule $\mathbf{?}R$ in move 10. Note that the proponent must use the same context $a1$ used in move 7 by the opponent, again, because of the structural rule Proponent's Restrictions. A similar situation happens in move 14. Finally, also note that the opponent cannot challenge move 16 again, due to the structural rule Opponent's Restriction.

The rules $\mathbf{?}K$, $\mathbf{!}K$, $\mathbf{?}R$ and $\mathbf{!}R$ introduced above work fine for normal modal operators. That is, operators respecting Axiom K and Necessitation. In our case, however, we want that these operators respect, in addition, the common properties of knowledge, i.e., truth, positive and negative introspection.[5] We deal with this by redefining the rules $\mathbf{?}R$ and $\mathbf{!}R$ below. Note that these two particle rules subsume the old ones of Definition 11.

Definition 12 (Particle Rules for Knowledge Properties)**.**

- **Challenge of Accessibility Relation for Knowledge (?RK):**
  If $(\mathbf{X},\omega\cdot an_1\cdots an_k,\lambda,t,K_a\varphi)$, $(\mathbf{X},\omega\cdot an'_1\cdots an'_{k'},\lambda',t',\psi)$ is in d,
  for any $t\in\{\mathbf{!},\mathbf{?}\neg\}$, any t', λ', ψ and any finite (possibly empty) sequences $n_1\cdots n_k$ and $n'_1\cdots n'_{k'}$,
  then $(\mathbf{Y},\omega\cdot an'_1\cdots an'_{k'},\lambda,\mathbf{?}RK,\neg\varphi)$ is allowed.
- **Defense of Accessibility Relation for Knowledge (!RK):**
  If $(\mathbf{Y},\omega,\lambda,\mathbf{?}RK,\neg\varphi)$ is in d,
  then $(\mathbf{X},\omega,\lambda,\mathbf{!},\varphi)$ is allowed.

We show examples of the use of rules $\mathbf{?}RK$ and $\mathbf{!}RK$ in tables 3, 4 and 5.

0.	**P**	**!**	$\neg(K_a p\wedge\neg p)$	
1.	**O**	**?**$\neg$	$K_a p\wedge\neg p$	(0)
2.	**P**	**?**$\wedge$	$\neg K_a p$	(1)
3.	**O**	**!**	$K_a p$	(2)
4.	**P**	**?**RK	$\neg p$	(3)
5.	**O**	**!**	p	(4)
6.	**P**	**?**$\wedge$	$\neg\neg p$	(1)
7.	**O**	**!**	$\neg p$	(6)
8.	**P**	**?**$\neg$	p	(7,5)

Table 4: Dialog about an instance of truth

[5] We are well aware that some works do not consider that knowledge respects all these properties. We do not intend to discuss such properties here and prefer to use this well accepted definition of knowledge. For more on this, please see the third paragraph of Section 5.

0.	**P**		!	$\neg(K_a p \wedge \neg K_a K_a p)$	
1.	**O**		?$\neg$	$K_a p \wedge \neg K_a K_a p$	(0)
2.	**P**		?$\wedge$	$\neg\neg K_a K_a p$	(1)
3.	**O**		!	$\neg K_a K_a p$	(2)
4.	**P**		?$\neg$	$K_a K_a p$	(3)
5.	**O**	$a1$	?K	$\neg K_a p$	(4)
6.	**P**	$a1$	!	$K_a p$	(5)
7.	**O**	$a1a2$	?K	$\neg p$	(6)
8.	**P**		?$\wedge$	$\neg K_a p$	(1)
9.	**O**		!	$K_a p$	(8)
10.	**P**	$a1a2$	?RK	$\neg p$	(9,7)
11.	**O**	$a1a2$	!	p	(10)
12.	**P**	$a1a2$	!	p	(7,11)

Table 5: Dialog about an instance of positive introspection

The dialog in Table 4 starts with an instance of truth (also known as Axiom T, see Table 1). The rule **?**RK is used in move 4 instantiating both sequences $an_1 \cdots an_k$ and $an'_1 \cdots an'_{k'}$ with the empty sequence. Thus, both sequences are considered as being the same. Intuitively, this corresponds to the appeal to reflexivity of accessibility relations in epistemic models. Reflexivity is the property responsible for the validity of Axiom T in epistemic logic.

The dialog in Table 5 starts with an instance of positive introspection (also known as Axiom 4, see Table 1). The rule **?**RK is used in move 10 instantiating sequence $an_1 \cdots an_k$ with the empty sequence and $an'_1 \cdots an'_{k'}$ with the sequence $a1a2$. Thus, the first sequence is a proper subsequence of the second one with all agents in it being the same agent a. Intuitively, this corresponds to the appeal to transitivity of accessibility relations in epistemic models. Transitivity is the property responsible for the validity of Axiom 4 in epistemic logic.

The dialog in Table 6 starts with an instance of negative introspection (also known as Axiom 5, see Table 1). The rule **?**RK is used in move 8 instantiating both sequences $an_1 \cdots an_k$ and $an'_1 \cdots an'_{k'}$ with $a1$. This is exactly as if the rule **?**R (from Definition 11) have been used. Here, the interesting use of rule **?**RK is the one made in move 14, instantiating the sequence $an_1 \cdots an_k$ with $a1$ and the sequence $an'_1 \cdots an'_{k'}$ with $a2$. Thus, both sequences are different but all agents in them are the same agent a. Intuitively, this corresponds to the appeal to euclidicity of accessibility relations in epistemic models. Euclidicity is the property responsible for the validity of Axiom 5 in epistemic logic.

0.	**P**		**!**	$\neg(\neg K_a p \wedge \neg K_a \neg K_a p)$	
1.	**O**		**?**$\neg$	$\neg K_a p \wedge \neg K_a \neg K_a p$	(0)
2.	**P**		**?**$\wedge$	$\neg\neg K_a \neg K_a p$	(1)
3.	**O**		**!**	$\neg K_a \neg K_a p$	(2)
4.	**P**		**?**$\neg$	$K_a \neg K_a p$	(3)
5.	**O**	$a1$	**?**K	$\neg\neg K_a p$	(4)
6.	**P**	$a1$	**!**	$\neg K_a p$	(5)
7.	**O**	$a1$	**?**$\neg$	$K_a p$	(6)
8.	**P**	$a1$	**?**RK	$\neg p$	(7)
9.	**O**	$a1$	**!**	p	(8)
10.	**P**		**?**$\wedge$	$\neg\neg K_a p$	(1)
11.	**O**		**!**	$\neg K_a p$	(10)
12.	**P**		**?**$\neg$	$K_a p$	(11)
13.	**O**	$a2$	**?**K	$\neg p$	(12)
14.	**P**	$a2$	**?**RK	$\neg p$	(7,13)
15.	**O**	$a2$	**!**	p	(14)
16.	**P**	$a2$	**!**	p	(13,15)

Table 6: Dialog about an instance of negative introspection

The third category of particle rules contains the ones for shared and common knowledge operators.

Definition 13 (Particle Rules for Shared and Common Knowledge Operators)**.**

- **Challenge of Shared Knowledge Operator (?E):**
  If $(\mathbf{X},\omega,\lambda,t,E_G\varphi)$ is in d, for $t\in\{\mathbf{!},\mathbf{?}\neg\}$,
  then $(\mathbf{Y},\omega\cdot an,\lambda,\mathbf{?}E,\neg\varphi)$, for a fresh n and any $a\in G$.
- **Defense of Shared Knowledge Operator (!E):**
  If $(\mathbf{Y},\omega,\lambda,\mathbf{?}E,\neg\varphi)$ is in d,
  then $(\mathbf{X},\omega,\lambda,\mathbf{!},\varphi)$ is allowed.
- **Challenge of Accessibility Relation for Shared Knowledge Operator (?RE):**
  If $(\mathbf{X},\omega\cdot an_1\cdots an_k,\lambda,t,E_G\varphi),(\mathbf{X},\omega\cdot an'_1\cdots an'_{k'},\lambda',t',\psi)$ is in d,
  for any $t\in\{\mathbf{!},\mathbf{?}\neg\}$, any t', λ', ψ, any $a\in G$, and any finite (possibly empty) sequence $n_1\cdots n_k$ and $n'_1\cdots n'_{k'}$,
  then $(\mathbf{Y},\omega\cdot an'_1\cdots an'_{k'},\lambda,\mathbf{?}RE,\neg\varphi)$ is allowed.

- **Defense of Accessibility Relation for Shared Knowledge Operator (!RE):**
  If $(\mathbf{Y}, \omega, \lambda, \mathbf{?}RE, \neg\varphi)$ is in d,
  then $(\mathbf{X}, \omega, \lambda, \mathbf{!}, \varphi)$ is allowed.

- **Challenge of Common Knowledge Operator (?C):**
  If $(\mathbf{X}, \omega, \lambda, t, C_G\varphi)$ is in d, for any $t \in \{\mathbf{!}, \mathbf{?}\neg\}$,
  then $(\mathbf{Y}, \omega \cdot a_1 n_1 \cdots a_k n_k, \lambda, \mathbf{?}C, \neg\varphi)$ for fresh $n_1, \ldots, n_k \in \mathbb{N}$, and any $a_1, \ldots, a_k \in G$.

- **Defense of Common Knowledge Operator (!C):**
  If $(\mathbf{Y}, \omega, \lambda, \mathbf{?}C, \neg\varphi)$ is in d
  then $(\mathbf{X}, \omega, \lambda, \mathbf{!}, \varphi)$.

0.	**P**		**!**	$\neg(C_{ab}p \wedge \neg(K_a p \wedge K_b K_a p))$	
1.	**O**		**?**$\neg$	$C_{ab}p \wedge \neg(K_a p \wedge K_b K_a p)$	(0)
2.	**P**		**?**$\wedge$	$\neg\neg(K_a p \wedge K_b K_a p)$	(1)
3.	**O**		**!**	$\neg(K_a p \wedge K_b K_a p)$	(2)
4.	**P**		**?**$\neg$	$K_a p \wedge K_b K_a p$	(3)
5.	**O**		**?**$\wedge$	$\neg K_a p$	(4)
6.	**P**		**!**	$K_a p$	(5)
7.	**O**	$a1$	**?**K	$\neg p$	(6)
8.	**P**		**?**$\wedge$	$\neg C_{ab}p$	(1)
9.	**O**		**!**	$C_{ab}p$	(8)
10.	**P**	$a1$	**?**RC	$\neg p$	(9,7)
11.	**O**	$a1$	**!**	p	(10)
12.	**P**	$a1$	**!**	p	(7,11)
13.	**O**		**?**$\wedge$	$\neg K_a K_b p$	(4)
14.	**P**		**!**	$K_a K_b p$	(13)
15.	**O**	$a1$	**?**K	$\neg K_b p$	(14)
16.	**P**	$a1$	**!**	$K_b p$	(15)
17.	**O**	$a1b2$	**?**K	$\neg p$	(16)
18.	**P**	$a1b2$	**?**RC	$\neg p$	(9,17)
19.	**O**	$a1b2$	**!**	p	(18)
20.	**P**	$a1b2$	**!**	p	(17,19)

Table 7: Dialog about common knowledge

- **Challenge of Accessibility Relation for Common Knowledge Operator (?RC):**
  If $(\mathbf{X}, \omega \cdot a_1 n_1 \cdots a_k n_k, \lambda, t, C_G\varphi)$ and $(\mathbf{X}, \omega \cdot a_1' n_1' \cdots a_{k'}' n_{k'}', \lambda', t', \psi)$ are in d, for any $t \in \{\mathbf{!}, \mathbf{?}\neg\}$, any t', λ', ψ, any finite (possibly empty) sequences $n_1 \cdots n_k \in \mathbb{N}$ and $n_1' \cdots n_{k'}' \in \mathbb{N}$, and $a_1, \ldots, a_k, a_1', \ldots a_{k'}' \in G$,
  then $(\mathbf{Y}, \omega \cdot a_1 n_1' \cdots a_1 n_{k'}', \lambda, \mathbf{?}RC, \neg\varphi)$ is allowed.

- **Defense of Accessibility Relation for Common Knowledge (!RC):**
  If $(\mathbf{Y}, \omega, \lambda, \mathbf{?}RC, \neg\varphi)$ is in d,
  then $(\mathbf{X}, \omega, \lambda, \mathbf{!}, \varphi)$ is allowed.

The intuitive meaning of these rules should be obvious. If **X** advances argument $E_G\varphi$, then **Y** can challenge it by choosing an agent $a \in G$ and then oblige **X** to defend φ. Similarly, if **X** advances argument $C_G\varphi$, then **Y** can challenge it by choosing an arbitrarily long sequence $a_1 n_1, \ldots, a_k n_k$ and then oblige **X** to defend φ in that context. The rules for common knowledge are used in moves 10 and 18.

Finally, we turn our attention to the rules for public announcements. They are given in the sequel.

Definition 14 (Particle Rules for Public Announcements)**.**

- **Challenge of Public Announcement Operator (?$\langle\,\rangle$):**
  If $(\mathbf{X}, \omega, \lambda, t, \langle\varphi\rangle\psi)$ is in d, for $t \in \{\mathbf{!}, \mathbf{?}\neg\}$,
  then both $(\mathbf{Y}, \omega, \lambda, \mathbf{?}\langle\,\rangle, \neg\varphi)$ and $(\mathbf{Y}, \omega, \lambda \cdot \varphi, \mathbf{?}\langle\,\rangle, \neg\psi)$ are allowed.

- **Defense of Public Announcement Operator (!$\langle\,\rangle$):**
  - If $(\mathbf{Y}, \omega, \lambda, \mathbf{?}\langle\,\rangle, \neg\varphi)$ is in d,
    then $(\mathbf{X}, \omega, \lambda, \mathbf{!}, \varphi)$ is allowed.
  - If $(\mathbf{Y}, \omega, \lambda \cdot \varphi, \mathbf{?}\langle\,\rangle, \neg\psi)$ is in d,
    then $(\mathbf{X}, \omega, \lambda \cdot \varphi, \mathbf{!}, \psi)$ is allowed.

- **Challenge of Propositional Permanence (?P):**
  If $(\mathbf{X}, \omega, \lambda \cdot \varphi, t, p)$ is in d, for $t \in \{\mathbf{!}, \mathbf{?}\neg\}$, and any t', ψ,
  then $(\mathbf{Y}, \omega, \lambda, \mathbf{?}P, \neg p)$ is allowed.

- **Defense of Propositional Permanence (!P):**
  If $(\mathbf{Y}, \omega, \lambda, \mathbf{?}P, \neg p)$ is in d, then $(\mathbf{X}, \omega, \lambda, \mathbf{!}, p)$ is allowed.

- **Challenge of Update (?U):**
  If $(\mathbf{X}, \omega \cdot an, \lambda \cdot \varphi, t, \psi)$ is in d, for $t \in \{\mathbf{!}, \mathbf{?}\neg, \mathbf{?}\wedge, \mathbf{?}K, \mathbf{?}RK, \mathbf{?}E, \mathbf{?}RE, \mathbf{?}C, \mathbf{?}RC, \mathbf{?}\langle\,\rangle, \mathbf{?}P\}$ and any ψ,
  then $(\mathbf{Y}, \omega \cdot an, \lambda, \mathbf{?}U, \neg\varphi)$ is allowed.

- **Defense of Update (!U):**
  If $(\mathbf{Y}, \omega \cdot an, \lambda, \mathbf{?}U, \neg\varphi)$ is in d,
  then $(\mathbf{X}, \omega \cdot an, \lambda, \mathbf{!}, \varphi)$ is allowed.

0.	**P**		**!**	$\neg\langle p\rangle\neg p$	
1.	**O**		**?**$\neg$	$\langle p\rangle\neg p$	(0)
2.	**P**		**?**$\langle\rangle$	$\neg p$	(1)
3.	**O**		**!**	p	(2)
4.	**P**	p	**?**$\langle\rangle$	$\neg\neg p$	(1)
5.	**O**	p	**!**	$\neg p$	(4)
6.	**P**	p	**?**$\neg$	p	(5)

Table 8: Dialog about the formula $\neg\langle p\rangle\neg p$

The intuitive meaning of rules **?**$\langle\rangle$ and **!**$\langle\rangle$ is the following. If **X** advances argument $\langle\varphi\rangle\psi$ in context (ω,λ), then **Y** can challenge it either by advancing $\neg\varphi$ in (ω,λ), meaning that the possible world ω "does not survive" the public announcement of φ, or by advancing $\neg\psi$ in $(\omega,\lambda\cdot\varphi)$, meaning that ψ is not true in the possible world ω of the epistemic model updated by $\lambda\cdot\varphi$. Then, **X** must either defend $\neg\varphi$ in (ω,λ) or ψ in $(\omega,\lambda\cdot\varphi)$, respectively.

The use of rules **?**$\langle\rangle$ and **!**$\langle\rangle$ is illustrated in the dialog on Table 8. It starts with thesis $\neg\langle p\rangle\neg p$. This is a valid formula, since public announcements do not change truth values of propositional variables $p\in\mathcal{P}$. Therefore, the proponent should be able to win the dialog. Indeed. Note that the rule **?**$\langle\rangle$ is used by the proponent in move 2 to force the opponent to defend p in the empty context. This argument is used against the opponent in move 6. The proponent also uses rule **?**$\langle\rangle$ in move 4.

The intuitive meaning of rules **?**P and **!**P is the following. If **X** advances argument p in $(\omega,\lambda\cdot\varphi)$ then **Y** can challenge it by advancing argument $\neg p$ in (ω,λ). It then obliges **X** to defend p in (ω,λ). Technically, this appeals to the fact that the truth values of propositional variables $p\in\mathcal{P}$ are not changed by public announcements. That is, if p is true in the possible world ω in the epistemic model updated by $\lambda\cdot\varphi$, then p is also true in ω in the epistemic model updated by λ.

The intuitive meaning of rules **?**U and **!**U is more subtle. If **X** advances some argument in $(\omega\cdot an,\lambda\cdot\varphi)$ then **Y** can challenge it by advancing argument $\neg\varphi$ in $(\omega\cdot an,\lambda)$, which then obliges **X** to defend φ in $(\omega\cdot an,\lambda)$. Technically, this appeals to the fact that $\omega\cdot an$ must "survive" the update by φ to be able to be used by **X** in a context with $\lambda\cdot\varphi$.

The use of rules **?**P, **!**P, **?**U and **!**U is illustrated in the dialog of Table 9.

This ends the set of particle rules. We have no less than 21 of such rules defined! For the comfort of the reader, we schematically summarize them in Appendix A (page 249).

Now, we are finally at the point to define the semantics of **PAC** according to the dialogical approach.

0.	**P**			!	$\neg\langle p\rangle\neg K_a p$	
1.	**O**			?$\neg$	$\langle p\rangle\neg K_a p$	(0)
2.	**P**		p	?$\langle\,\rangle$	$\neg\neg K_a p$	(1)
3.	**O**		p	!	$\neg K_a p$	(2)
4.	**P**		p	?$\neg$	$K_a p$	(3)
5.	**O**	$a1$	p	?K	$\neg p$	(4)
6.	**P**	$a1$	p	!	p	(5)
7.	**O**	$a1$		?P	$\neg p$	(6)
8.	**P**	$a1$		?U	$\neg p$	(5)
9.	**O**	$a1$		!	p	(8)
10.	**P**	$a1$		!	p	(7,9)

Table 9: Dialog about the formula $\neg\langle p\rangle\neg K_a p$

Definition 15 (Validity)**.** A formula $\varphi \in \mathcal{L}_{\mathbf{PAC}}(Ag, \mathcal{P})$ is valid in (the dialogical approach of) **PAC**, which is noted $\vDash_{\mathbf{D}} \varphi$, if and only if the proponent wins all dialogs with thesis φ.

Therefore, when the thesis is not valid in this approach, the opponent may win the dialog. We illustrate such situation in Table 10. There, we choose a formula involving the Moore Sentence $p \wedge \neg K_a p$. In van Ditmarsch [19], an interesting understanding of such sentences is presented. The special feature of Moore Sentences is that they are unsuccessful. That is, such sentences are false after their own announcement. Thus, the proponent may not win the dialog starting with thesis $\neg\langle p \wedge \neg K_a p\rangle\neg(p \wedge \neg K_a p)$ (which states that the formula $p \wedge \neg K_a p$ is successful). It is indeed what happens in Table 10.

Note, however, that if the opponent does not play well, the proponent may win a dialog starting with such thesis. For instance, assume that, in move 9, the opponent does not change the context, i.e., the argument $\neg p$ is also advanced in the empty context. Then, the proponent can reuse the argument of opponent's move 5 and defend itself against move 9 with argument p. In this case, the opponent cannot move anymore.

The remainder of this article is devoted to show that the dialogical approach to **PAC** coincides with the model-theoretical approach.

3. Soundness

In this section, we show that the dialogical approach is sound with respect to the model-theoretical approach. In other words, we show that $\vDash_{\mathbf{D}} \varphi$ implies $\vDash \varphi$. In fact, we use its contrapositive, i.e., we show that $\nvDash \varphi$ implies

0.	**P**			**!**	$\neg\langle p\wedge\neg K_a p\rangle\neg(p\wedge\neg K_a p)$	
1.	**O**			**?**$\neg$	$\langle p\wedge\neg K_a p\rangle\neg(p\wedge\neg K_a p)$	(0)
2.	**P**			**?**$\langle\,\rangle$	$\neg(p\wedge\neg K_a p)$	(1)
3.	**O**			**!**	$p\wedge\neg K_a p$	(2)
4.	**P**			**?**$\wedge$	$\neg p$	(3)
5.	**O**			**!**	p	(4)
6.	**P**			**?**$\wedge$	$\neg\neg K_a p$	(3)
7.	**O**			**!**	$\neg K_a p$	(6)
8.	**P**			**?**$\neg$	$K_a p$	(7)
9.	**O**	$a1$		**?**K	$\neg p$	(8)
10.	**P**		$p\wedge\neg K_a p$	**?**$\langle\,\rangle$	$\neg\neg(p\wedge\neg K_a p)$	(1)
11.	**O**		$p\wedge\neg K_a p$	**!**	$\neg(p\wedge\neg K_a p)$	(10)
12.	**P**		$p\wedge\neg K_a p$	**?**$\neg$	$p\wedge\neg K_a p$	(11)
13.	**O**		$p\wedge\neg K_a p$	**?**$\wedge$	$\neg\neg K_a p$	(12)
14.	**P**		$p\wedge\neg K_a p$	**!**	$\neg K_a p$	(13)
15.	**O**		$p\wedge\neg K_a p$	**?**$\neg$	$K_a p$	(14)
16.	**P**	$a1$	$p\wedge\neg K_a p$	**?**R	$\neg p$	(15,9)
17.	**O**	$a1$		**?**U	$\neg(p\wedge\neg K_a p)$	(16)
18.	**P**	$a1$		**!**	$p\wedge\neg K_a p$	(17)
19.	**O**	$a1$		**?**$\wedge$	$\neg p$	(18)

Table 10: Dialog involving a Moore Sentence

$\not\vDash_{\mathbf{D}}\varphi$. In words, this means that, if there exists a pointed epistemic model satisfying $\neg\varphi$ (i.e., a counter-model for φ) then the proponent does not win all dialogs with thesis φ. First, however, we need some definitions and two lemmas.

Definition 16 (Satisfiable Dialog)**.** The dialog d is satisfiable if and only if there exists an epistemic model $\mathcal{M}=(\mathcal{W},\{\mathcal{R}_a\}_{a\in Ag},\mathcal{V}\rangle$ and a function $f:(Ag\mathbb{N})^*\to\mathcal{W}$ such that $(f(\omega),f(\omega'))\in\mathcal{R}_a$, for all moves $(\mathbf{X},\omega\cdot a\cdot\omega''$, $\lambda,t,\varphi)$ in d, where $\omega'=\omega\cdot a\cdot\omega''$, for any t, and:

- $(\mathcal{M},f(\omega))\not\vDash\lambda$ or $(\mathcal{M}^\lambda,f(\omega))\not\vDash\varphi$, for all moves $(\mathbf{P},\omega,\lambda,t,\varphi)$ in d, for any t;
- $(\mathcal{M},f(\omega))\vDash\lambda$ and $(\mathcal{M}^\lambda,f(\omega))\vDash\varphi$, for all moves $(\mathbf{O},\omega,\lambda,t,\varphi)$ in d, for any t.

Lemma 1. *Let d be a dialog. If d is satisfiable and there is a particle rule that is applicable in d then its application generates a dialog d' that is satisfiable.*

Proof. Assume that the pointed epistemic model $(\mathcal{M}, f(\omega))$ satisfies d and that there is a particle rule that is applicable in d. This means that there is at least one move $\mu = (\mathbf{X}, \omega, \lambda, t, \varphi)$ in d allowing the application of such particle rule. The application of such rule generates the dialog d'. There are several cases to be considered, depending on which particle rule is applicable:

1. Defenses: There are three cases to be considered:
   (a) Assume $(\mathbf{O}, \omega, \lambda, t, \neg\varphi)$ in d, for some challenge t. Then, the application of the defense rule generates d' with $(\mathbf{P}, \omega, \lambda, \mathbf{!}, \varphi)$. By assumption, we have $(\mathcal{M}, f(\omega)) \vDash \lambda$ and $(\mathcal{M}^{\lambda}, f(\omega)) \vDash \neg\varphi$, iff $(\mathcal{M}, f(\omega)) \vDash \lambda$ and $(\mathcal{M}^{\lambda}, f(\omega)) \nvDash \varphi$. Therefore, d' is satisfiable.
   (b) Assume $(\mathbf{P}, \omega, (), t, \neg\varphi)$ in d, for some challenge t. Then, the application of the rule generates d' with $(\mathbf{O}, \omega, (), \mathbf{!}, \varphi)$. By assumption, we have $(\mathcal{M}, f(\omega)) \nvDash \neg\varphi$, iff $(\mathcal{M}, f(\omega)) \vDash \varphi$. Therefore, d' is satisfiable.
   (c) Assume $(\mathbf{P}, \omega, \lambda' \cdot \chi, t, \neg\varphi)$ in d, for some challenge t. In this case, rule $\mathbf{?}U$ is also applicable. Then, the application of one of the two rules generates either d'_1 with $(\mathbf{O}, \omega, \lambda', \mathbf{?}U, \neg\chi)$ or d'_2 with $(\mathbf{O}, \omega, \lambda' \cdot \chi, \mathbf{!}, \varphi)$. By assumption, we have $(\mathcal{M}, f(\omega)) \nvDash \lambda' \cdot \chi$ or $(\mathcal{M}^{\lambda' \cdot \chi}, f(\omega)) \nvDash \neg\varphi$, iff $(\mathcal{M}, f(\omega)) \nvDash \lambda' \cdot \chi$ or $(\mathcal{M}^{\lambda' \cdot \chi}, f(\omega)) \vDash \varphi$. Therefore, d' is satisfiable.
2. Challenge of Negation: It is shown exactly as for the case Defenses above.
3. Challenge of Conjunction: There are three cases to be considered:
   (a) Assume $(\mathbf{O}, \omega, \lambda, t, \varphi \wedge \psi)$ in d, for $t \in \{\mathbf{!}, \mathbf{?}\neg\}$. Then, the application of this rule generates d' with both $(\mathbf{P}, \omega, \lambda, \mathbf{?}\wedge, \neg\varphi)$ and $(\mathbf{P}, \omega, \lambda, \mathbf{?}\wedge, \neg\psi)$. By assumption, we have $(\mathcal{M}, f(\omega)) \vDash \lambda$ and $(\mathcal{M}^{\lambda}, f(\omega)) \vDash \varphi \wedge \psi$, iff $(\mathcal{M}, f(\omega)) \vDash \lambda$ and $(\mathcal{M}^{\lambda}, f(\omega)) \nvDash \neg\varphi$ and $(\mathcal{M}^{\lambda}, f(\omega)) \nvDash \neg\psi$. Therefore, d' is satisfiable.
   (b) Assume $(\mathbf{P}, \omega, (), t, \varphi \wedge \psi)$ in d, for $t \in \{\mathbf{!}, \mathbf{?}\neg\}$. Then, the application of this rule generates either d'_1 with $(\mathbf{O}, \omega, \lambda, \mathbf{?}\wedge, \neg\varphi)$ or d'_2 with $(\mathbf{O}, \omega, \lambda, \mathbf{?}\wedge, \neg\psi)$. By assumption, we have $(\mathcal{M}, f(\omega)) \nvDash \varphi \wedge \psi$, iff $(\mathcal{M}, f(\omega)) \vDash \neg\varphi$ or $(\mathcal{M}, f(\omega)) \vDash \neg\psi$. Therefore, d' is satisfiable.
   (c) Assume $(\mathbf{P}, \omega, \lambda' \cdot \chi, t, \varphi \wedge \psi)$ in d, for $t \in \{\mathbf{!}, \mathbf{?}\neg\}$. In this case, rule $\mathbf{?}U$ is also applicable. Then, the application of one of these rules generates either d'_1 with $(\mathbf{O}, \omega, \lambda', \mathbf{?}U, \neg\chi)$ or d'_2 with

$(\mathbf{O}, \omega, \lambda' \cdot \chi, \mathbf{?}\wedge, \neg\varphi)$ or d_3' with $(\mathbf{O}, \omega, \lambda' \cdot \chi, \mathbf{?}\wedge, \neg\psi)$. By assumption, we have $(\mathcal{M}, f(\omega)) \not\models \lambda' \cdot \chi$ or $(\mathcal{M}^{\lambda' \cdot \chi}, f(\omega)) \not\models \varphi \wedge \psi$, iff $(\mathcal{M}, f(\omega)) \not\models \lambda' \cdot \chi$ or $(\mathcal{M}^{\lambda' \cdot \chi}, f(\omega)) \models \neg\varphi$ or $(\mathcal{M}^{\lambda' \cdot \chi}, f(\omega)) \models \neg\psi$. Therefore, d' is satisfiable.

4. Challenge of Knowledge Operator: Note that this rule is allowed only for the opponent. There are two cases to be considered:
   (a) Assume $(\mathbf{P}, \omega, (), t, K_a\varphi)$ in d, for $t \in \{\mathbf{!}, \mathbf{?}\neg\}$. Then, the application of this rule generates d' with $(\mathbf{O}, \omega \cdot an, (), \mathbf{?}K, \neg\chi)$, for a fresh $n \in \mathbb{N}$. By assumption, we have $(\mathcal{M}, f(\omega)) \not\models K_a\varphi$, iff $(\mathcal{M}, f(\omega \cdot an)) \models \neg\varphi$, for some $\omega \cdot an$ in d. Therefore, d' is satisfiable.
   (b) Assume $(\mathbf{P}, \omega, \lambda' \cdot \chi, t, K_a\varphi)$ in d, for $t \in \{\mathbf{!}, \mathbf{?}\neg\}$. In this case, rule $\mathbf{?}U$ is also applicable. Then, the application of one of these rules generates either d_1' with $(\mathbf{O}, \omega, \lambda', \mathbf{?}U, \neg\chi)$ or d_2' with $(\mathbf{O}, \omega \cdot an, \lambda' \cdot \chi, \mathbf{?}K, \neg\chi)$, By assumption, we have $(\mathcal{M}, f(\omega)) \not\models \lambda' \cdot \chi$ or $(\mathcal{M}^{\lambda' \cdot \chi}, f(\omega)) \not\models K_a\varphi$, iff $(\mathcal{M}, f(\omega)) \not\models \lambda' \cdot \chi$ or $(\mathcal{M}, f(\omega \cdot an)) \models \lambda' \cdot \chi$ and $(\mathcal{M}^{\lambda' \cdot \chi}, f(\omega \cdot an)) \models \neg\varphi$, for some $\omega \cdot an$ in d. Therefore, d' is satisfiable.
5. Challenge of Accessibility Relation for Knowledge: There are three cases to be considered:
   (a) Assume $(\mathbf{O}, \omega \cdot an_1 \cdots an_k, \lambda, t, K_a\varphi)$ in d, for $t \in \{\mathbf{!}, \mathbf{?}\neg\}$. Then, successive applications of this rule generate d' with $(\mathbf{P}, \omega \cdot an_1' \cdots an_{k'}', \lambda, \mathbf{?}RK, \leftarrow\varphi)$, for all $\omega \cdot an_1' \cdots an_{k'}'$ in d. By assumption, we have $(\mathcal{M}, f(\omega \cdot an_1 \cdots an_k)) \models \lambda$ and $(\mathcal{M}^{\lambda}, f(\omega \cdot an_1 \cdots an_k)) \models K_a\varphi$, iff $(\mathcal{M}, f(\omega \cdot an_1' \cdots an_{k'}')) \not\models \lambda$ or $(\mathcal{M}^{\lambda}, f(\omega \cdot an_1' \cdots an_{k'}')) \not\models \neg\varphi$, for all $\omega an_1' \cdots an_{k'}'$ in d. Therefore, d' is satisfiable.
   (b) Assume $(\mathbf{P}, \omega \cdot an_1 \cdots an_k, (), t, K_a\varphi)$ in d, for $t \in \{\mathbf{!}, \mathbf{?}\neg\}$. Then, the application of this rule generates d' with $(\mathbf{O}, \omega \cdot an_1' \cdots an_{k'}', (), \mathbf{?}RK, \neg\varphi)$, for some $\omega \cdot an_1' \cdots an_{k'}'$ in d. (Recall that the opponent can apply it only once, due to the structural rule Opponent's Restriction.) By assumption, we have $(\mathcal{M}, f(\omega \cdot an_1 \cdots an_k)) \not\models K_a\varphi$, iff $(\mathcal{M}, f(\omega \cdot an_1' \cdots an_{k'}')) \models \neg\varphi$, for some $\omega an_1' \cdots an_{k'}'$ in d. Therefore, d' is satisfiable.
   (c) Assume $(\mathbf{P}, \omega \cdot an_1 \cdots an_k, \lambda' \cdot \chi, t, K_a\varphi)$ in d, for $t \in \{\mathbf{!}, \mathbf{?}\neg\}$. In this case, rule $\mathbf{?}U$ is also applicable. Then, the application of this rule generates either d_1' with $(\mathbf{O}, \omega \cdot an_1' \cdots an_{k'}', \lambda', \mathbf{?}U, \neg\chi)$ or d_2' with $(\mathbf{O}, \omega \cdot an_1' \cdots an_{k'}', \lambda' \cdot \chi, \mathbf{?}RK, \neg\varphi)$, for some $\omega \cdot an_1' \cdots an_{k'}'$ in d. (Recall that the opponent can apply it only once, due to the structural rule Opponent's Restriction.) By assumption, we have $(\mathcal{M}, f(\omega \cdot an_1 \cdots an_k)) \not\models \lambda' \cdot \chi$ or $(\mathcal{M}^{\lambda' \cdot \chi}, f(\omega \cdot an_1 \cdots an_k)) \not\models K_a\varphi$, iff $(\mathcal{M}, f(\omega \cdot an_1 \cdots an_k)) \not\models \lambda' \cdot \chi$ or $(\mathcal{M}, f(\omega \cdot an_1' \cdots an_{k'}')) \models \lambda' \cdot \chi$ and $(\mathcal{M}^{\lambda' \cdot \chi}, f(\omega \cdot an_1' \cdots an_{k'}')) \models \neg\varphi$, for some $\omega an_1' \cdots an_{k'}'$ in d. Therefore, d' is satisfiable.

6. Challenge of Shared Knowledge Operator: Similar to the case Challenge of Knowledge Operator above.
7. Challenge of Accessibility Relation for Shared Knowledge Operator: Similar to the case Challenge of Accessibility Relation for Knowledge Operator above.
8. Challenge of Common Knowledge Operator: Similar to the case Challenge of Knowledge Operator above.
9. Challenge of Accessibility Relation for Common Knowledge Operator: Similar to the case Challenge of Accessibility Relation for Knowledge Operator above.
10. Challenge of Public Announcement Operator: There are three cases to be considered:
    (a) Assume $(\mathbf{O},\omega,\lambda,t,\langle\varphi\rangle\psi)$ in d, for $t\in\{\mathbf{!},\mathbf{?}\neg\}$. Then, the application of this rule generates d' with both $(\mathbf{P},\omega,\lambda,\mathbf{?}\langle\,\rangle,\neg\varphi)$ and $(\mathbf{P},\omega,\lambda\cdot\varphi,\mathbf{?}\langle\,\rangle,\neg\psi)$. By assumption, we have $(\mathcal{M},f(\omega))\vDash\lambda$ and $(\mathcal{M}^{\lambda},f(\omega))\vDash\langle\varphi\rangle\psi$, iff $(\mathcal{M},f(\omega))\vDash\lambda$ and $(\mathcal{M}^{\lambda},f(\omega))\nvDash\neg\varphi$ and $(\mathcal{M}^{\lambda\cdot\varphi},f(\omega))\nvDash\neg\psi$. Therefore, d' is satisfiable.
    (b) Assume $(\mathbf{P},\omega,(),t,\langle\varphi\rangle\psi)$ in d, for $t\in\{\mathbf{!},\mathbf{?}\neg\}$. Then, the application of this rule generates either d_1' with $(\mathbf{O},\omega,(),\mathbf{?}\langle\,\rangle,\neg\varphi)$ or d_2' with $(\mathbf{O},\omega,\varphi,\mathbf{?}\langle\,\rangle,\neg\psi)$. By assumption, we have $(\mathcal{M},f(\omega))\nvDash\langle\varphi\rangle\psi$, iff $(\mathcal{M},f(\omega))\vDash\neg\varphi$ or $(\mathcal{M}^{\varphi},f(\omega))\vDash\neg\psi$. Therefore, d' is satisfiable.
    (c) Assume $(\mathbf{P},\omega,\lambda'\cdot\chi,t,\langle\varphi\rangle\psi)$ in d, for $t\in\{\mathbf{!},\mathbf{?}\neg\}$. In this case, rule $\mathbf{?}U$ is applicable. Then, the application of this rule generates either d_1' with $(\mathbf{O},\omega,\lambda',\mathbf{?}U,\neg\chi)$ or d_2' with $(\mathbf{O},\omega,\lambda'\cdot\chi,\mathbf{?}\langle\,\rangle,\neg\varphi)$ or d_3' with $(\mathbf{O},\omega,\lambda'\cdot\chi\cdot\varphi,\mathbf{?}\langle\,\rangle,\neg\psi)$. By assumption, we have $(\mathcal{M},f(\omega))\nvDash\lambda'\cdot\chi$ or $(\mathcal{M}^{\lambda'\cdot\chi},f(\omega))\nvDash\langle\varphi\rangle\psi$, iff $(\mathcal{M},f(\omega))\nvDash\lambda'\cdot\chi$ or $(\mathcal{M}^{\lambda'\cdot\chi},f(\omega))\vDash\neg\varphi$ or $(\mathcal{M}^{\lambda'\cdot\chi\cdot\varphi},f(\omega))\vDash\neg\psi$. Therefore, d' is satisfiable.
11. Challenge of Propositional Permanence: There are two cases to be considered:
    (a) Assume $(\mathbf{O},\omega,\lambda'\cdot\chi,t,p)$ in d, for $t\in\{\mathbf{!},\mathbf{?}\neg\}$. Then, the application of this rule generates d' with $(\mathbf{P},\omega,\lambda',\mathbf{?}P,\neg p)$. By assumption, we have $(\mathcal{M},f(\omega))\vDash\lambda'\cdot\chi$ and $(\mathcal{M}^{\lambda'\cdot\chi},f(\omega))\vDash p$. Then, $(\mathcal{M},f(\omega))\vDash\lambda'$ and $(\mathcal{M}^{\lambda'},f(\omega))\nvDash\neg p$. Therefore d' is satisfiable.
    (b) Assume $(\mathbf{P},\omega,\lambda'\cdot\chi,t,p)$ in d, for $t\in\{\mathbf{!},\mathbf{?}\neg\}$. In this case, rule $\mathbf{?}U$ is also applicable. Then, the application of one of these rules generates d_1' with $(\mathbf{O},\omega,\lambda',\mathbf{?}U,\neg\chi)$ or d_2' with $(\mathbf{O},\omega,\lambda',\mathbf{?}P,\neg p)$. By assumption, we have $(\mathcal{M},f(\omega))\nvDash\lambda'\cdot\chi$ or $(\mathcal{M}^{\lambda'\cdot\chi},f(\omega))\nvDash p$. Then, $(\mathcal{M},f(\omega))\nvDash\lambda'\cdot\chi$ or $(\mathcal{M}^{\lambda'},f(\omega))\vDash\neg p$. Therefore, d' is satisfiable.
12. Challenge of Update: It has been treated in the cases above.

□

Lemma 2. *The proponent wins the dialog d if and only if d is finite and contains a move* $\mu = (\mathbf{P}, \omega, (), t, p)$*, for some* ω, t *and* $p \in \mathcal{P}$*.*

Proof. For the implication from the left to the right, if the proponent wins the dialog d then, by definition, there are no more allowed moves for the opponent. This means that the opponent cannot make a move, which means that the proponent cannot make a move as well, due to the structural rule Game-playing. Therefore, d is finite. By assumption, and due to the structural rule Winning, the last move in d is a proponent's move. Let the update of such move be λ and its argument be φ. Towards a contradiction, assume that either λ is not empty or φ is not a propositional variable $p \in \mathcal{P}$. In either case, there is at least one particle rule that is applicable. This means that such move is not the last one, which is a contradiction. Therefore, λ is the empty sequence and φ is equal to some $p \in \mathcal{P}$.

For the implication from the right to the left, let $|d(\mathbf{P})|$ denote the number of proponent's move in d up till move μ. Note that $|d(\mathbf{O})| \leq |d(\mathbf{P})|$, due to structural rules Starting and Game-playing. Towards a contradiction, assume that the opponent has some move μ' allowed after μ. Move μ' cannot challenge μ nor it can be a defense against it. Thus, μ' is a reaction to one of the other $|d(\mathbf{P})| - 1$ proponent's moves in d. But, at this point, there is already $|d(\mathbf{O})| - 1$ opponent's moves in d. All of these moves are reactions to previous proponent's moves. Then, due to structural rule Game-playing, which forbids repetitions, μ' is not an allowed move. This means that μ is the last move in d. Therefore, the proponent wins d.

We finally can show soundness of the dialogical approach with respect to the model-theoretical approach to **PAC**.

Theorem 1 (Soundness)**.** *If* $\vDash_{\mathbf{D}} \varphi$ *then* $\vDash \varphi$*.*

Proof. We show its contrapositive. That is, we show that $\nvDash \varphi$ implies $\nvDash_{\mathbf{D}} \varphi$. Towards a contradiction, assume $\nvDash \varphi$ and $\vDash_{\mathbf{D}} \varphi$. Then, there exists a pointed epistemic model $(\mathcal{M}, w)$ satisfying $\neg\varphi$ and the proponent wins all terminal dialogs with thesis φ. Let d be one of such dialogs. By Lemma 2, the last move in d is of the form $(\mathbf{P}, \omega, (), t, p)$, for some ω and t. Due to the structural rule Proponent's Restrictions, the argument p must have been advanced by the opponent in d, i.e., d also contains a move of the form $(\mathbf{O}, \omega, (), t', p)$, for some t'. Now, by the assumption and also by Lemma 1, d is satisfiable. Thus, we have $(\mathcal{M}, f(\omega)) \vDash p$ and $(\mathcal{M}, f(\omega)) \nvDash p$, which is a contradiction. Therefore, we have that $\nvDash_{\mathbf{D}} \varphi$. $\square$

4. Completeness

In this section, we show that the dialogical approach is complete with respect to the model-theoretical approach. In other words, we show that $\vDash \varphi$ implies $\vDash_{\mathbf{D}} \varphi$. In fact, we use its contrapositive, i.e., we show that $\nvDash_{\mathbf{D}} \varphi$

implies $\not\vDash \varphi$. In words, this means that, if the proponent does not win all dialogs with thesis φ then there exists a pointed epistemic model satisfying $\neg\varphi$ (i.e., a counter-model for φ). First, however, we need two definitions and an auxiliary lemma.

Definition 17 (Length of Formulas)**.** The length of a formula $\varphi \in \mathcal{L}_{\mathbf{PAC}}(Ag, \mathcal{P})$, given by the expression $len(\varphi)$, is inductively defined as follows:

$$
\begin{aligned}
&len(p) = 1\\
&len(\neg\varphi) = 1 + len(\varphi)\\
&len(\varphi \wedge \psi) = 1 + len(\varphi) + len(\psi)\\
&len(K_a\varphi) = 2 + len(\varphi)\\
&len(E_G\varphi) = 1 + |G| + len(\varphi)\\
&len(C_G\varphi) = 1 + |G| + len(\varphi)\\
&len(\langle\varphi\rangle\psi) = 2 + len(\varphi) + len(\psi)
\end{aligned}
$$

Definition 18 (Length of Sequences of Formulas)**.** The length of a sequence of formulas λ, given by the expression $len(\lambda)$, is the number of formulas in λ, i.e., it is inductively defined by:

$$
\begin{aligned}
&len() = 0\\
&len(\lambda \cdot \varphi) = 1 + len(\lambda)
\end{aligned}
$$

Lemma 3. *If there is a terminal dialog with thesis φ that is not won by the proponent then there is a pointed epistemic model satisfying $\neg\varphi$.*

Proof. Let d be a terminal dialog with thesis φ that is not won by the proponent. We show the following claim: There exists an epistemic model $\mathcal{M}$ such that we have both:

- $(\mathcal{M}, \omega) \not\vDash \lambda$ or $(\mathcal{M}^\lambda, \omega) \not\vDash \psi$, for all moves $(\mathbf{P}, \omega, \lambda, t, \psi)$ in d, for any t;
- $(\mathcal{M}, \omega) \vDash \lambda$ and $(\mathcal{M}^\lambda, \omega) \vDash \psi$, for all moves $(\mathbf{O}, \omega, \lambda, t, \psi)$ is in d, for any t.

Since d starts with the move $(\mathbf{P}, (), (), \mathbf{!}, \varphi)$, the above claim permits us to conclude that the pointed epistemic model $(\mathcal{M}, f())$ satisfies $\neg\varphi$.

It remains to prove the claim. To that end, we construct the epistemic model $\mathcal{M}$ using the moves in the dialog d, as follows. Let $\mathcal{M}$ = $\mathcal{M} = \langle \mathcal{W}, \{\mathcal{R}_a\}_{a \in Ag}, \mathcal{V}\rangle$, where:

- $\mathcal{W} = \{\omega : (\mathbf{O}, \omega, \lambda, t, \psi) \text{ is in } d, \text{for some } \lambda, t, \psi\}$;
- $\mathcal{R}_a$ = the reflexive, transitive and symmetric closure of the set: $\{(\omega, \omega') : \omega' = \omega \cdot a \cdot \omega'' \text{ and } (\mathbf{O}, \omega', \lambda, t, \psi) \text{ is in } d, \text{ for some } \lambda, t, \psi\}$;
- $\mathcal{V}_p = \{\omega : (\mathbf{O}, \omega, \lambda, t, p) \text{ is in } d, \text{ for some } \lambda, t\}$.

Assume that move $(\mathbf{X}, \omega, \lambda, t, \psi)$ is in d. The remainder of the proof is an induction on $len(\lambda) + len(\psi)$.

In the induction base, λ is the empty sequence and $\psi = p$, for some $p \in \mathcal{P}$. Let $\mathbf{X} = \mathbf{P}$. Then, by Lemma 2, the proponent wins the dialog, which contradicts the assumption. Therefore, $\mathbf{X} = \mathbf{O}$. Then, $f(\omega) \in \mathcal{V}_p$ (by assumption), iff $(\mathcal{M}^{\lambda}, \omega) \vDash \psi$ (by the definition of $\mathcal{M}$).

Now, assume that the claim is true for all λ and ψ such that $len(\lambda) + len(\psi) \leq x$. We show that it is true for all λ and ψ such that $len(\lambda) + len(\psi) = x + 1$.

First, assume that $len(\lambda) > 0$, i.e., $\lambda = \lambda' \cdot \chi$, for some χ:

- If $\mathbf{X} = \mathbf{P}$ then $(\mathbf{O}, \omega, \lambda', \mathbf{?}U, \neg\chi)$ is in d (because d is terminal). Then, $(\mathcal{M}, \omega) \vDash \lambda'$ and $(\mathcal{M}^{\lambda'}, \omega) \vDash \neg\chi$ (by the induction hypothesis). Then, $(\mathcal{M}, \omega) \nvDash \lambda$.
- If $\mathbf{X} = \mathbf{O}$ then both $(\mathbf{P}, \omega, \lambda', \mathbf{?}U, \neg\chi)$ and $(\mathbf{O}, \omega, \lambda', \mathbf{!}, \chi)$ are in d (because d is terminal). Then, $(\mathcal{M}, \omega) \vDash \lambda'$ and $(\mathcal{M}^{\lambda'}, \omega) \nvDash \neg\chi$ (by the induction hypothesis). Then, $(\mathcal{M}, \omega) \vDash \lambda$.

Now, assume that $(\mathcal{M}, \omega) \vDash \lambda$. There are seven cases to be considered:

1. Let $\psi = p$. Note that we have $\lambda = \lambda' \cdot \chi$, for some χ (otherwise this would be the induction base):
   - If $\mathbf{X} = \mathbf{P}$ then $(\mathbf{O}, \omega, \lambda', \mathbf{?}P, \neg p)$, (because d is terminal). Then, $(\mathcal{M}^{\lambda'}, \omega) \vDash \neg p$ (by the induction hypothesis). Then, $(\mathcal{M}^{\lambda'}, \omega) \nvDash p$, iff $(\mathcal{M}^{\lambda}, \omega) \nvDash p$ (by definition).
   - If $\mathbf{X} = \mathbf{O}$ then $(\mathbf{P}, \omega, \lambda', \mathbf{?}P, \neg p)$, (because d is terminal). Then, $(\mathcal{M}^{\lambda'}, \omega) \nvDash \neg p$ (by the induction hypothesis). Then, $(\mathcal{M}^{\lambda'}, \omega) \vDash p$, iff $(\mathcal{M}^{\lambda}, \omega) \vDash p$ (by definition).

2. Let $\psi = \neg\psi'$:
   - If $\mathbf{X} = \mathbf{P}$ then $(\mathbf{O}, \omega, \lambda, t, \psi')$ is in d, for some $t \in \{\mathbf{!}, \mathbf{?}\neg\}$, (because d is terminal). Then, $(\mathcal{M}^{\lambda}, \omega) \vDash \psi$ (by the induction hypothesis). Then, $(\mathcal{M}^{\lambda}, \omega) \nvDash \neg\psi$.
   - If $\mathbf{X} = \mathbf{O}$:
     - If $\psi' = p$ and λ is the empty sequence then $(\mathbf{O}, \omega, \lambda, t', p)$ is not in d, because, if it were the case, we would have $(\mathbf{P}, \omega, \lambda, t'', p)$ is in d (because the dialog is terminal), which means that the proponent wins the dialog (by Lemma 2) and thus, contradicting the assumption. Therefore, $f(\omega) \not\subseteq \mathcal{V}_p$ (by definition). Then, $(\mathcal{M}^{\lambda}, \omega) \vDash \neg p$.
     - Else, $(\mathbf{P}, \omega, \lambda, t, \psi')$ is in d, for some $t \in \{\mathbf{!}, \mathbf{?}\neg\}$, (because d is terminal). Then, $(\mathcal{M}^{\lambda}, \omega) \nvDash \psi$ (by the induction hypothesis). Then, $(\mathcal{M}^{\lambda}, \omega) \vDash \neg\psi$.

3. Let $\psi = \psi_1 \wedge \psi_2$:
   - If $\mathbf{X} = \mathbf{P}$ then either $(\mathbf{O}, \omega, \lambda, \mathbf{?}\wedge, \neg\psi_1)$ or $(\mathbf{O}, \omega, \lambda, \mathbf{?}\wedge, \neg\psi_2)$ is in d. (because d is terminal and due to Opponent's Restriction). Then, either $(\mathcal{M}^\lambda, \omega) \vDash \neg\psi_1$ or $(\mathcal{M}^\lambda, \omega) \vDash \neg\psi_2$ (by the induction hypothesis). Then, $(\mathcal{M}^\lambda, \omega) \nvDash \psi$.
   - If $\mathbf{X} = \mathbf{O}$ then both $(\mathbf{P}, \omega, \lambda, \mathbf{?}\wedge, \neg\psi_1)$ and $(\mathbf{P}, \omega, \lambda, \mathbf{?}\wedge, \neg\psi_2)$ are in d (because d is terminal). Then, $(\mathcal{M}^\lambda, \omega) \nvDash \neg\psi_1$ and $(\mathcal{M}^\lambda, \omega) \nvDash \neg\psi_2$ (by the induction hypothesis). Then, $(\mathcal{M}^\lambda, \omega) \vDash \psi$.
4. Let $\psi = K_a\psi'$ and let $\omega = \omega' \cdot an_1 \cdots an_k$:
   - If $\mathbf{X} = \mathbf{P}$ then $(\mathbf{O}, \omega' \cdot an'_1 \cdots an'_{k'}, \lambda, t, \neg\psi')$ is in d, for some $t \in \{\mathbf{?}K, \mathbf{?}R\}$ and some sequence $an'_1 \cdots an'_{k'}$ (because d is terminal and due to Opponent's Restriction). Then, $(\mathcal{M}^\lambda, \omega' \cdot an'_1 \cdots an'_{k'}) \vDash \neg\psi'$ for some sequence $an'_1 \cdots an'_{k'}$ (by the induction hypothesis). Then, $(\mathcal{M}^\lambda, \omega) \nvDash K_a\psi$, because, by definition, $(\omega, \omega' \cdot an'_1 \cdots an'_{k'}) \in \mathcal{R}_a$.
   - If $\mathbf{X} = \mathbf{O}$ then $(\mathbf{P}, \omega' \cdot an'_1 \cdots an'_{k'}, \lambda, t, \neg\psi')$ is in d, for some $t \in \{\mathbf{?}K, \mathbf{?}R\}$ and all sequences $an'_1 \cdots an'_{k'}$ in d (because d is terminal), iff $(\mathcal{M}^\lambda, \omega' \cdot an'_1 \cdots an'_{k'}) \nvDash \neg\psi'$ for all sequences $an'_1 \cdots an'_{k'}$ in d (by the induction hypothesis). Then, $(\mathcal{M}^\lambda, \omega) \vDash K_a\psi$, because, by definition, $(\omega, \omega' \cdot an'_1 \cdots an'_{k'}) \in \mathcal{R}_a$, for all sequences $an'_1 \cdots an'_{k'}$ in d.
5. Let $\psi = E_G\psi'$:
   - If $\mathbf{X} = \mathbf{P}$ then $(\mathbf{O}, \omega' \cdot an'_1 \cdots an'_{k'}, \lambda, t, \neg\psi')$ is in d, for some $t \in \{\mathbf{?}E, \mathbf{?}RE\}$ and some sequence $an'_1 \cdots an'_{k'}$ (because d is terminal and due to Opponent's Restriction). Then, $(\mathcal{M}^\lambda, \omega' \cdot an'_1 \cdots an'_{k'}) \vDash \neg\psi'$ for some sequence $an'_1 \cdots an'_{k'}$ (by the induction hypothesis). Then, $(\mathcal{M}^\lambda, \omega) \nvDash E_G\psi$, because, by definition, $(\omega, \omega' \cdot an'_1 \cdots an'_{k'}) \in \mathcal{R}_a$.
   - If $\mathbf{X} = \mathbf{O}$ then $(\mathbf{P}, \omega' \cdot an'_1 \cdots an'_{k'}, \lambda, t, \neg\psi')$ is in d, for some $t \in \{\mathbf{?}E, \mathbf{?}RE\}$ and all sequences $an'_1 \cdots an'_{k'}$ in d (because d is terminal), iff $(\mathcal{M}^\lambda, \omega' \cdot an'_1 \cdots an'_{k'}) \nvDash \neg\psi'$ for all sequences $an'_1 \cdots an'_{k'}$ in d (by the induction hypothesis). Then, $(\mathcal{M}^\lambda, \omega) \vDash E_G\psi$, because, by definition, $(\omega, \omega' \cdot an'_1 \cdots an'_{k'}) \in \bigcup_{a \in G}\mathcal{R}_a$, for all sequences $an'_1 \cdots an'_{k'}$ in d.
6. Let $\psi = C_G\psi'$:
   - If $\mathbf{X} = \mathbf{P}$ then $(\mathbf{O}, \omega' \cdot a_1n'_1 \cdots a_kn'_{k'}, \lambda, t, \neg\psi')$ is in d, for some $t \in \{\mathbf{?}C, \mathbf{?}RC\}$ and some sequence $a_1n'_1 \cdots a_{k'}n'_{k'}$ (because d is terminal and due to Opponent's Restriction). Then, $(\mathcal{M}^\lambda, \omega' \cdot an'_1 \cdots an'_{k'}) \vDash \neg\psi'$ for some sequence $a_1n'_1 \cdots a_{k'}n'_{k'}$ (by the induction hypothesis). Then, $(\mathcal{M}^\lambda, \omega) \nvDash C_G\psi$, because, by definition, $(\omega, \omega' \cdot a_1n'_1 \cdots a_{k'}n'_{k'}) \in \left(\bigcup_{a \in G}\mathcal{R}_a\right)^*$.
   - If $\mathbf{X} = \mathbf{O}$ then $(\mathbf{P}, \omega' \cdot a_1n'_1 \cdots a_{k'}n'_{k'}, \lambda, t, \neg\psi')$ is in d, for some $t \in \{\mathbf{?}C, \mathbf{?}RC\}$ and all sequences $a_1n'_1 \cdots a_{k'}n'_{k'}$ in d (because d is terminal), iff $(\mathcal{M}^\lambda, \omega' \cdot a_1n'_1 \cdots a_{k'}n'_{k'}) \nvDash \neg\psi'$ for all sequences $a_1n'_1 \cdots a_{k'}n'_{k'}$

in d (by the induction hypothesis). Then, $(\mathcal{M}^{\lambda}, \omega) \vDash C_G\psi$, because, by definition, $(\omega, \omega' \cdot a_1 n'_1 \cdots a_{k'} n'_{k'}) \in \left(\bigcup_{a \in G} \mathcal{R}_a\right)^*$, for all sequences $a_1 n'_1 \cdots a_{k'} n'_{k'}$ in d.

7. Let $\psi = \langle \psi_1 \rangle \psi_2$:
   - If $\mathbf{X} = \mathbf{P}$ then either $(\mathbf{O}, \omega, \lambda, \mathbf{?}\langle\,\rangle, \neg\psi_1)$ or $(\mathbf{O}, \omega, \lambda \cdot \psi_1, \mathbf{?}\langle\,\rangle, \neg\psi_2)$ is in d (because d is terminal and due to Opponent's Restriction). Then, either $(\mathcal{M}^{\lambda}, \omega) \vDash \neg\psi_1$ or $(\mathcal{M}^{\lambda}, \omega) \vDash \psi_1$ and $(\mathcal{M}^{\lambda \cdot \psi_1}, \omega) \vDash \neg\psi_2$ (by the induction hypothesis). Then, $(\mathcal{M}^{\lambda}, \omega) \nvDash \langle \psi_1 \rangle \psi_2$.
   - If $\mathbf{X} = \mathbf{O}$ then both $(\mathbf{P}, \omega, \lambda, \mathbf{?}\langle\,\rangle, \neg\psi_1)$ and $(\mathbf{P}, \omega, \lambda \cdot \psi_1, \mathbf{?}\langle\,\rangle, \neg\psi_2)$ are in d (because d is terminal). Then, $(\mathcal{M}^{\lambda}, \omega) \nvDash \neg\psi_1$, $(\mathcal{M}^{\lambda}, \omega) \vDash \psi_1$ and $(\mathcal{M}^{\lambda \cdot \psi_1}, \omega) \nvDash \neg\psi_2$ (by the induction hypothesis). Then, $(\mathcal{M}^{\lambda}, \omega) \vDash \langle \psi_1 \rangle \psi_2$.

□

Theorem 2 (Completeness). *If* $\vDash \varphi$ *then* $\vDash_{\mathbf{D}} \varphi$.

Proof. We show its contrapositive. That is, we show that if $\nvDash_{\mathbf{D}} \varphi$ then $\nvDash \varphi$. Assume that $\nvDash_{\mathbf{D}} \varphi$. Then, there is a terminal dialog with thesis φ that is not won by the proponent. By Lemma 3, there is a pointed epistemic model $(\mathcal{M}, w)$ satisfying $\neg\varphi$. By definition, $(\mathcal{M}, w)$ does not satisfy φ. Therefore, φ is not valid. □

5. Discussion and Conclusion

In this paper we presented a novel approach to **PAL** and **PAC**, namely, one based on the dialogical framework. The main result is the proof of soundness and completeness with respect to the model-theoretical approach. In this section, we discuss some similar approaches and alternatives as well as possibilities for future work.

The Standard Particle Rule for Conjunction. The reader familiar with dialogical logic may find our particle rule Challenge of Conjunction awkward. The difference from the standard approach is that our rule, when challenging a conjunction $\varphi \wedge \psi$, advances an argument $\neg\varphi$ or $\neg\psi$, instead of simply indicate which of the conjuncts is challenged. We find that this change, apart from being harmless to the correction of the approach, is more elegant. First, because we end up with a dialogical system in which all arguments are formulas. That is, we do not need to add to the language a new special symbol to indicate which conjunct is being challenged. Second, our dialogs respect a so-called "sub-formula and negation of sub-formula property". That is, all arguments are either a sub-formula of the thesis or a negation of a sub-formula of the thesis.

Other Knowledge Operators. We would like to note that simple changes on the rules **?***RK*, **?***RE* and **?***RC* can be made to make this approach work for other kinds of knowledge operators as well. We have already presented rule **?***R* (Definition 11), which can deal with knowledge operators that do not respect truth, positive and negative introspection. Another example is the following version of **?***RK*′, which deals with a knowledge operator respecting truth and positive introspection but not respecting negative introspection.

- **Challenge of Accessibility Relation for Knowledge (?***RK*′**):**
  If $(\mathbf{X}, \omega, \lambda, t, K_a\varphi)$, $(\mathbf{X}, \omega \cdot an_1 \cdot an_k, \lambda', t', \psi)$ is in d,
  for any $t \in \{\mathbf{!}, \mathbf{?}\neg\}$, any t', λ', ψ and any (possibly empty) sequence $an_1, \ldots, an_k$,
  then $(\mathbf{Y}, \omega \cdot an_1 \cdots an_k, \lambda, \mathbf{?}RK', \neg\varphi)$ is allowed.

Analogous changes are done to rules **?***RE* and **?***RC* to make these operators reflect the corresponding operators for shared and common knowledge.

Dialogs without Defensive Moves. We would like to note that it is possible to propose an alternative dialogical system without particle rules for defense. This alternative system would have only challenges and counter-challenges. First of all, it is easy to see that all rules for defense are the same: they simply advance argument φ in reaction to a challenge that advances argument $\neg\varphi$. Second, note that it is exactly the same behavior as the one of particle rule Challenge of Negation. Therefore, a dialogical system without defensive moves is very simple to achieve: Just drop all the rules for defense and allow the current particle rule Challenge of Negation to challenge all the other challenges. It is easy to see that the resulting system is sound and complete.

According to our view, a dialogical system without defensive rules is more elegant. It also seems to give a better account to the dynamics of a debate: The proponent utters the thesis, then the opponent disagrees and attacks it, then the proponent counter-attacks the opponent's argument, and so on. Note that it also permits to better explain the particularity of an atomic argument (i.e., one consisting of propositional variable $p \in \mathcal{P}$). It it is the only kind of argument that does not admit attacks. This is so because, in such games, we only focus on the structure of arguments and not on their contents.

Logical Consequences. With a small modification, the dialogical approach to **PAC** presented here can also handle logical consequence, which is defined as follows.

Definition 19. A formula φ *i*s a logical consequence of a set of formulas Γ, which is noted $\Gamma \vDash \varphi$, if and only if all pointed epistemic models $(\mathcal{M}, w)$ that satisfy all formulas in Γ also satisfy φ.

1.	$\langle \varepsilon, 0, \langle p \rangle \neg K_a p \rangle$		
2.	$\langle p, 0, \neg K_a p \rangle$		$(R\langle\,\rangle : 1)$
3.	$\langle \varepsilon, 1, p \rangle$	$\langle a, 0, 1 \rangle \in S$	$(R\hat{K} : 2)$
4.	$\langle p, 1, \neg p \rangle$		$(R\hat{K} : 2)$
5.	$\langle \varepsilon, 1, \neg p \rangle$		$(RSB : 4)$
	closed		

Table 11: Tableau for the formula $\neg\langle p \rangle \neg K_a p$

The only modification necessary is on the structural rule Starting. In this alternative system, we permit the opponent to start by uttering the set of formulas Γ and then the proponent utters the thesis. In this case, each formula in Γ are also available during the dialog, i.e., they can also be challenged and/or used by the players during the dialog.

Comparison with Tableaux. The dialogical approach to **PAL** may also be thought as a reasoning method for it. Since it is sound and complete with respect to the model-theoretical approach, it can be used to check weather a formula is valid, satisfiable, etc. Indeed, the dialogical approach is somewhat similar to the tableau method proposed in Balbiani *et al.* [1]. For example, consider the development of the tableau for the formula $\neg\langle p \rangle \neg K_a p$ in Table 11[6]. It displays exactly the same formulas as the ones advanced as arguments by the opponent in Table 9.

In fact, we strongly believe the dialogical approach could present the same computational complexity as the tableau method, if it were equipped with some kind of "inclusion test" to guarantee termination, as in the tableaux method. We do not develop such technique here though, leaving it for future work. The advantage being that the dialogical approach offers the possibility to study announcement logics through an argumentative process which had lead us to bridge link between this type of logic and conditional precedent in law.

Future Work. We believe that different dialogical approaches to **PAL** and **PAC** might be interesting to explore. For instance, we intend to study the consequences of decreasing the strategic power of the player in order to get closer to real argumentative contexts. In such dialogs, the players could, for example, "forget" to use an argument and then permit the adversary to win the dialog even if the point of view defended is inconsistent.

[6] Note that we always use the negation of the input formula in the tableau.

Other possibilities are the proposition of different rules in order to study different kinds of announcement operators.

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A. Summary of Particle Rules

X Utterance	Y Challenge	X Defense	Conditions
Boolean Operators			
$\omega, \lambda, t, \neg\varphi$	$\omega, \lambda, \mathbf{?}\neg, \varphi$	—	$t \in \{\mathbf{!}, \mathbf{?}\neg\}$
$\omega, \lambda, t, \varphi \wedge \psi$	$\omega, \lambda, \mathbf{?}\wedge, \neg\varphi$ $\omega, \lambda, \mathbf{?}\wedge, \neg\psi$	$\omega, \lambda, \mathbf{!}, \varphi$ $\omega, \lambda, \mathbf{!}, \psi$	$t \in \{\mathbf{!}, \mathbf{?}\neg\}$
Knowledge Operators			
$\omega, \lambda, t, K_a\varphi$	$\omega \cdot an, \lambda, \mathbf{?}K, \neg\varphi$	$\omega \cdot an, \lambda, \mathbf{!}, \varphi$	$t \in \{\mathbf{!}, \mathbf{?}\neg\}$ and fresh n
$\omega \cdot an_1 \cdots an_k, \lambda, t, K_a\varphi$ $\omega \cdot an'_1 \cdots an'_k, \lambda', t', \psi$	$\omega \cdot an'_1 \cdots an'_k, \lambda, \mathbf{?}RK, \neg\varphi$	$\omega \cdot an'_1 \cdots an'_k, \lambda, \mathbf{!}, \varphi$	$t \in \{\mathbf{!}, \mathbf{?}\neg\}$ and any t', λ', ψ and any finite (possibly empty) sequence $n_1 \cdots n_k$ and $n'_1 \cdots n'_k$
Shared and Common Knowledge Operators			
$\omega, \lambda, t, E_G\varphi$	$\omega, \lambda, t, \mathbf{?}E, \neg\varphi$	$\omega \cdot an, \lambda, \mathbf{!}, \varphi$	$t \in \{\mathbf{!}, \mathbf{?}\neg\}$ for a fresh s and any $a \in G$
$\omega \cdot an_1 \cdots an_k, \lambda, t, E_G\varphi$ $\omega \cdot an'_1 \cdots an'_k, \lambda', t', \psi$	$\omega \cdot an'_1 \cdots an'_k, \lambda, \mathbf{?}RE, \neg\varphi$	$\omega \cdot an'_1 \cdots an'_k, \lambda, \mathbf{!}, \varphi$	$t \in \{\mathbf{!}, \mathbf{?}\neg\}$, for any λ', t', ψ, any $a \in G$, and any finite (possibly empty) sequence $n_1 \cdots n_k$ and $n'_1 \cdots n'_k$
$\omega, \lambda, t, C_G\varphi$	$\omega \cdot a_1n_1 \cdots a_kn_k, \lambda, \mathbf{?}C, \neg\varphi$	$\omega \cdot n_1, \cdots, n_k, \lambda, \mathbf{!}, \varphi$	$t \in \{\mathbf{!}, \mathbf{?}\neg\}$, for a fresh $n_1, \cdots, n_k \in \mathbb{N}$, and $a_1, \cdots, a_k \in G$
$\omega \cdot a_1n_1 \cdots a_kn_k, \lambda, t, C_G\varphi$ $\omega \cdot a'_1n'_1 \cdots a'_{k'}n'_{k'}, \lambda't', \psi$	$\omega \cdot a'_1n'_1 \cdots a'_{k'}n'_{k'}, \lambda, \mathbf{?}RC, \neg\varphi$	$\omega \cdot a_1n'_1 \cdots a_kn'_k, \lambda, \mathbf{!}, \varphi$	$t \in \{\mathbf{!}, \mathbf{?}\neg\}$, any t', λ', ψ and any finite (possibly empty) sequence $n_1 \cdots n_k$ and $n'_1 \cdots n'_k$ and $a_1, \cdots a_1n'_{k'}$
Announcement Operators			
$\omega, \lambda, t, \langle\varphi\rangle\psi$	$\omega, \lambda, \mathbf{?}\langle\,\rangle, \neg\varphi$ $\omega, \lambda \cdot \varphi, \mathbf{?}\langle\,\rangle, \neg\psi$	$\omega, \lambda, \mathbf{!}, \varphi$ $\omega, \lambda \cdot \varphi, \mathbf{!}, \psi$	$t \in \{\mathbf{!}, \mathbf{?}\neg\}$
$\omega, \lambda \cdot \varphi, t, p$	$\omega, \lambda, \mathbf{?}P, \neg p$	$\omega, \lambda, \mathbf{!}, p$	$t \in \{\mathbf{!}, \mathbf{?}\neg\}$
$\omega \cdot an, \lambda \cdot \varphi, t, \psi$	$\omega \cdot an, \lambda, \mathbf{?}U, \neg\varphi$	$\omega \cdot an, \lambda, \mathbf{!}, \varphi$	$t \in \{\mathbf{!}, \mathbf{?}\neg, \mathbf{?}\wedge, \mathbf{?}K$ $\mathbf{?}RK, \mathbf{?}E, \mathbf{?}RE, \mathbf{?}C,$ $\mathbf{?}RC, \mathbf{?}\langle\,\rangle, \mathbf{?}P\}$ and any ψ