A BEGRIFFSSCHRIFT FOR SENTENTIAL LOGIC

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1. INTRODUCTION

The title of this essay is probably a misnomer, but the aim is to achieve for sentential logic something very similar to Frege's achievement for general logic.

A good starting point is the thought that sentential logic is truth functional in the narrow sense that each of its propositions is sufficiently specified by a mere permutation of truth values. This suggests constructing the logic in a script entirely comprised by truth tables, though it might be supposed that the result would be intolerably cumbersome. Fortunately, however, a type of diagram discussed by Hubbeling is sufficiently compact to partially overcome this difficulty and moreover, many worthwhile insights can be obtained in a discussion confined to two variables, which further simplifies the tables. The resulting script displays features of sentential logic that are concealed by more familiar symbolism.

2. BASIC SYMBOLISM

The following are examples of the diagrams described by Hubbeling.

$$\left|\begin{array}{cc} + & + \\ - & - \end{array}\right| p, \quad \left|\begin{array}{cc} - & - \\ + & + \end{array}\right| \sim p, \quad \left|\begin{array}{cc} + & - \\ + & - \end{array}\right| q, \quad \left|\begin{array}{cc} - & + \\ - & + \end{array}\right| \sim q, \quad \left|\begin{array}{cc} + & - \\ - & - \end{array}\right| p \cdot q, \quad \left|\begin{array}{cc} + & + \\ + & - \end{array}\right| p \vee q.$$

In each case the corresponding Peano-Russel expression has been written after the diagram. It will be seen that the arguments of each table have been entered on the table and combined in a manner very similar to the use of Euler or

Venn type diagrams. The upper half of the square is reserved for p, the lower half for $\sim$p, the left hand half for q and the right hand half for $\sim$q. Consequently a «+» in the left upper half denotes p . q, in the right upper half p . $\sim$q and so on. Finally, when there is more than one plus the table denotes the disjunction of the combination; for example, pluses filling both the upper half and the left hand half denotes p $\vee$ q.

The tables can be extended to three or more arguments, but the present discussion will be confined to two arguments.

As is well known, to each function specified by a truth table there corresponds a connective, though only four connectives are in regular use. When interpreted as a connective the table will be placed in square brackets and the four connectives in regular use are as follows.

$$\begin{bmatrix} + & - \\ - & - \end{bmatrix} . , \quad \begin{bmatrix} + & + \\ + & - \end{bmatrix} \vee , \quad \begin{bmatrix} + & - \\ + & + \end{bmatrix} \supset , \quad \begin{bmatrix} + & - \\ - & + \end{bmatrix} \equiv .$$

To avoid all punctuation, the syntax of Polish notation will be adopted: eg. we shall write

$$\begin{bmatrix} + & - \\ + & + \end{bmatrix} \begin{vmatrix} + & - \\ + & - \end{vmatrix} \begin{vmatrix} - & + \\ + & - \end{vmatrix} \text{ and not } \begin{vmatrix} + & - \\ + & - \end{vmatrix} \begin{bmatrix} + & - \\ + & + \end{bmatrix} \begin{vmatrix} - & + \\ + & - \end{vmatrix} .$$

The connective tables can be interpreted as an operation assignment for the reduction of the expression to simpler form. The following are a key to positions in the connective, an expression and its reduced form.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad \begin{bmatrix} + & - \\ + & + \end{bmatrix} \begin{vmatrix} + & - \\ + & - \end{vmatrix} \begin{vmatrix} - & + \\ + & - \end{vmatrix}, \qquad \begin{vmatrix} - & + \\ + & + \end{vmatrix} .$$

The plus in position 1 of the connective table means «wherever both the first and second argument tables have a plus, place a plus in the resultant table» and this occurs in position 3 of the two argument tables. The minus in position 2 of the connective table means «wherever a plus occurs in the first argument table and a minus in the second argument table, place a minus in the resultant table» and this occurs in position 1 of the two argument tables. The plus in position 3 of the connective means «wherever a minus occurs in the first argument and a plus in the second, place a plus in the resultant» and similarly for the last position.

The operations just described will be called «contraction» and any composite expression can be contracted to a single table by first contracting the sub-formulae and working up to the main connective. The single truth table resulting from these operations will be called the «fully contracted form» of the expression. Expressions that are not fully contracted will be called «composite».

Consider $\begin{bmatrix} - & - \\ + & + \end{bmatrix} \begin{vmatrix} + & + \\ - & + \end{vmatrix} \begin{vmatrix} + & - \\ - & + \end{vmatrix}$. It is easy to see that the second argument is really only a dummy, it can be anything whatever without affecting the fully contracted form of the expression. For the connective is a degenerate binary connective: it is one of the two forms of negation, and these are unary. Such expressions can be written $\left[\begin{array}{cc|} - & - \\ + & + \end{array}\right] \begin{vmatrix} + & + \\ - & + \end{vmatrix}$, $\left[\begin{array}{|cc} + & - \\ + & - \end{array}\right] \begin{vmatrix} + & + \\ - & + \end{vmatrix}$, etc., ie. in the first example one can add a dummy second argument if one likes and in the second example a dummy first argument, if one likes.

From time to time comparisons will be made with Principia Mathematica. In many cases the point made is valid for most or all other presentations of sentential logic, P.M. merely being taken as the example.

Alternative presentations of a Boolian function in the present script do not correspond with alternative presentations in P.M. Eg. compare:—

$\begin{vmatrix} + & - \\ + & + \end{vmatrix}$, $\begin{bmatrix} + & - \\ + & + \end{bmatrix} \begin{vmatrix} + & + \\ - & - \end{vmatrix} \begin{vmatrix} + & - \\ + & - \end{vmatrix}$ and $\begin{bmatrix} + & + \\ + & - \end{bmatrix} \begin{vmatrix} - & - \\ + & + \end{vmatrix} \begin{vmatrix} + & - \\ + & - \end{vmatrix}$,

with $p \supset q$ and $\sim p \vee q$

3. FORMAL INFERENCE

The following is an example of a proof:—

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}\begin{bmatrix} + & + \\ + & - \end{bmatrix}\begin{bmatrix} + & + \\ + & - \end{bmatrix}\begin{vmatrix} + & - \\ - & - \end{vmatrix}\ \begin{vmatrix} - & - \\ + & - \end{vmatrix}\ \begin{vmatrix} - & - \\ + & + \end{vmatrix}\ \begin{vmatrix} + & - \\ + & + \end{vmatrix} \quad 1.$$

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}\begin{bmatrix} + & + \\ + & - \end{bmatrix}\begin{vmatrix} + & - \\ + & - \end{vmatrix}\ \begin{vmatrix} - & - \\ + & + \end{vmatrix}\ \begin{vmatrix} + & - \\ + & + \end{vmatrix} \quad 2.$$

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}\begin{vmatrix} + & - \\ + & + \end{vmatrix}\ \begin{vmatrix} + & - \\ + & + \end{vmatrix} \quad 3.$$

$$\begin{vmatrix} + & + \\ + & + \end{vmatrix} \quad 4.$$

Clearly, by such proofs all theorems can be proved and the method is a decision procedure. Also, it is easy to see that every line in the proof is equivalent to very other line (even if the function is not a tautology). Now in the sentential part of P.M. every line of every proof is equivalent to every other line of every proof but the point is concealed by the symbolism. Indeed, equivalence is in an important sense the basis of proof in P.M., for all theorems are so many different presentations of tautology. In the present script this point is made evident.

The fact that all lines of the above proof are equivalent leads one to ask whether there is not a valid proof running from step 4 to step 1.

Consider lines 3 and 4. The connective is analogous to say «$\times$» or «$+$» in arithmetic. In seeking an inference from step 4 to step 3, for the given connective ($\begin{bmatrix} + & - \\ - & + \end{bmatrix}$), it is as if one were seeking a pair of factors for line 4 (c.p. «$\times$») or two members whose sum is line 4 (c.p. «$+$»). Or in seeking the second argument for the given connective and the first argument ($\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ and $\begin{vmatrix} + & - \\ + & + \end{vmatrix}$), it is as if one were asking whether a number (the first argument) is a factor of line 4.

Also, unlike the questions usually asked in arithmetic there is the further type of enquiry in which, given both the arguments in line 3, one asks whether there is a connective that will contract them into line 4. On any of these alternatives a problem of inverse operation is posed that should be capable of solution, though it certainly has some unusual features.

In the present paper we shall only investigate the procedure in which an inference is made from a given connective and the procedure will be called «expansion by a given connective», or more briefly, «expansion». The function to be expanded will be called «the contraction» and the two functions forming the members of the expansion will be called its «first and second arguments». In what follows the diagram $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ will be used as a key to positions in the connective.

It is easy to see that the interpretation of the connective is as follows. For a sign at position 1, «wherever this sign occurs in the contraction, the sign «+» may be placed in the corresponding position in both arguments of the expansion». For a sign at 2, «wherever this sign occurs in the contraction the sign «+» may be placed in the corresponding position in the first argument of the expansion and the sign «—» in the corresponding position of the second». Similarly for signs in the other two positions. In general, this yields several expansions of a given contraction.

Notice that the position of a given pair of signs in the two arguments corresponds to a given position in the contraction but not a given position in the connective. The positions in the connective correspond to pairs of values in the arguments, e.g. position 1 to a «+» in both arguments in any position.

It is interesting to study the expansion of various functions by various connectives, but in the present essay discussion will be confined to the expansion of tautology by the 16 possible connectives. This is, of course, the domain of two-argument theorems and is the basis of theorems having a greater number of arguments.

4. THE EXPANSION OF TAUTOLOGY

The expansion of tautology by 12 of the 16 possible connectives yields merely trivial theorems. Expansion by the remaining four connectives is more interesting. Here, the trivial cases will be considered first.

Expansion by $\begin{bmatrix} + & + \\ + & + \end{bmatrix}$. Using the rule described in the last section it can easily be seen that the first and second arguments can be anything, ie. the «tautology connective» forms a tautology from any two arguments.

Expansion by $\begin{bmatrix} - & - \\ - & - \end{bmatrix}$. It can easily be seen that the «contradictory connective» will not expand tautology.

Expansion by $\begin{bmatrix} + & - \\ - & - \end{bmatrix}$, $\begin{bmatrix} - & + \\ - & - \end{bmatrix}$, $\begin{bmatrix} - & - \\ + & - \end{bmatrix}$ or $\begin{bmatrix} - & - \\ - & + \end{bmatrix}$. Because each of these truth tables contains one and only one plus, each connective expands tautology into one and only one pair of arguments. These pairs are, taut./taut., taut./contr., contr./taut., contr./contr.

Expansion by $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ or $\begin{bmatrix} - & + \\ + & - \end{bmatrix}$. The first of these expands tautology into two identical arguments, that may be anything, ie. the «equivalence connective» forms the equivalence of any function to itself. The second of these connectives expands tautology into two arguments, one of which can be anything and the other of which is the negation of the first.

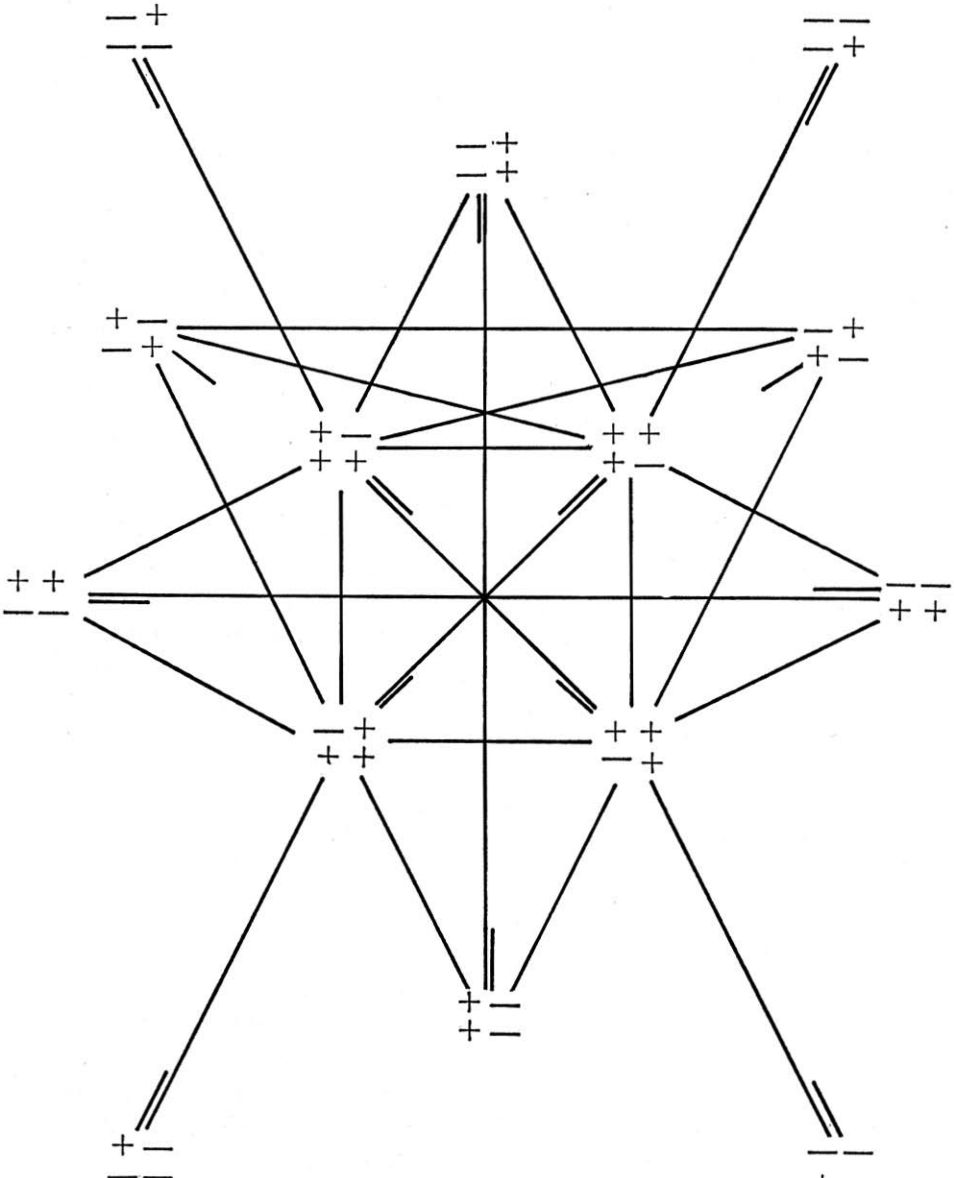
Expansion by $\begin{bmatrix} + & + \\ - & - \end{bmatrix}$, $\begin{bmatrix} + & - \\ + & - \end{bmatrix}$, $\begin{bmatrix} - & - \\ + & + \end{bmatrix}$ or $\begin{bmatrix} - & + \\ - & + \end{bmatrix}$. These are all unary connectives, one argument can be anything, the expansions are $\left[\begin{array}{cc|} + & + \\ - & - \end{array}\right]\left|\begin{array}{cc} + & + \\ + & + \end{array}\right|$, $\left[\begin{array}{|cc} + & - \\ + & - \end{array}\right]\left|\begin{array}{cc} + & + \\ + & + \end{array}\right|$, $\left[\begin{array}{cc|} - & - \\ + & + \end{array}\right]\left|\begin{array}{cc} - & - \\ - & - \end{array}\right|$ and $\left[\begin{array}{|cc} - & + \\ - & + \end{array}\right]\left|\begin{array}{cc} - & - \\ - & - \end{array}\right|$.

The above expansions have points of interest but scarcely enumerate an exciting or rich domain of theorems. The four remaining connectives are more interesting.

Expansion by $\begin{bmatrix} + & - \\ + & + \end{bmatrix}$ or $\begin{bmatrix} - & + \\ + & + \end{bmatrix}$. The first of these connectives expands tautology into all combinations of arguments except those having a plus in some position in the first argument with a minus in the corresponding position in the second. The second of these connectives expands tautology into all combinations except those having a plus in some position in the second argument with a minus in the corresponding position in the first. Therefore the two connectives form a pair whose expansions differ only in the order of the arguments. Only the first connective (implication) will be studied here.

Clearly, the three pluses in the connective give three possible pairs of values for each position in the arguments of the expansion and as there are four positions, the total number of expansions is $3^4 = 81$. These can, in fact, be presented as the familiar Boolian lattice, by taking advantage of the fact that the connective is reflexive and transitive (as can easily be shown). The nodes of the lattice are the 16 fully contracted functions. Then 16 pairs of arguments are the reflexive connections of the nodes with themselves, a further 32 pairs are formed by the lines of the lattice and the remaining 33 pairs are formed by the transitive connections of the nodes joined by two or more intervening lines.

Expansion by $\begin{bmatrix} + & + \\ + & - \end{bmatrix}$ or $\begin{bmatrix} - & + \\ + & + \end{bmatrix}$. These form a pair in that the pairs of arguments of the expansions of the one are the negations of the pairs of the other. The first connective (disjunction) will be studied here. The total number of expansions is once again 81 and one might expect that they could be grouped into a lattice, but this is not so. It is, indeed, easy to show that the connective is only reflexive for tautology and that it is not transitive. On the other hand, the connective is symmetric, so that if the expansions can be presented as a node-and-line diagram, the lines can be read either way. A



suggested diagram is presented above and bears no obvious relation to a lattice. On the diagram tautology has been omitted as it disjuncts with every function, but its immagined presence, as it were, is indicated as if it were in the centre, by the short, inward directed line from each function. Contradiction is also omitted; it only disjuncts with tautology.

Including the lines to tautology there are 39 lines on the diagram and these with the line from tautology to contradiction make 40 lines, each of which can be taken in either direction, making 80 pairs of arguments. The reflexive disjunction of tautology with itself brings the total to 81.

In conclusion, the richest realm of theorem are those based on disjunction and on implication and their pairs. Also, the system of theorems based on disjunction differs more radically from the system based on implication than one would expect from the similarity of their truth tables or from the disjunctive equivalent of implication in the system of P.M.

5. MATERIAL TRUTH AND FALSITY

Consider a fully contracted function, eg. $\begin{vmatrix} + & - \\ - & - \end{vmatrix}$ and its equivalent in P.M., vis. p.q.

Whitehead and Russell endeavoured to distinguish between the positing of the function and the assertion of its truth, writing «p . q .» for the former and «⊢ . p . q .» for the latter, but as, in any case, every line of the relevant part of P.M. is a theorem, the distinction was null. My impression is that in any case it is bound to remain obscure in that symbolism. We take up the point they were trying to make and endeavour to clarify it, using fresh terminology as it is debateable how close is the correspondence between P.M. and the present system.

A function such as $\begin{vmatrix} + & - \\ - & + \end{vmatrix}$ is a mere permutation of truth and falsity possibilities and as such it is neither true nor false. Moreover, because it contains both plus and minus signs it does not specify either a true or a false sentence: for this reason it will be said not to be «actualised» (c.p. P.M. «merely posited»).

A function analogous to this may, however, specify a sentence that is true as a matter of contingent fact and in that event the function will be said to be «actualised» (c.p. P.M. «materially true» and «asserted»). Also, the positions in the truth table that would have been occupied by minus signs, had the function not been actualised, are now replaced by dots, thus $\left[\begin{vmatrix} + & \cdot \\ \cdot & + \end{vmatrix}\right]$, because the function specifies a true sentence by virtue of one of its truth possibilities but has no falsity possibilities.

The tautology, $\begin{vmatrix} + & + \\ + & + \end{vmatrix}$, has in any case no falsity possibilities and as such may be regarded as vacuously actualised (c.p. all theorems being asserted in P.M.). The contradiction has, correspondingly, no truth possibilities and so cannot be actualised.

When a composite function is actualised the minus sign need only be replaced by dots on the main connective, for a little thought will show that it is a category mistake to suppose that one part of a composite expression may be actualised and not another. If such a function is contracted, or further expanded, the dots can then be preserved on the main connective.

Three modes of inference from premisses, that have some unusual features, can be listed in the present symbolism. T hese are 1. Inference by equivalence. Any premiss can be expanded or contracted. As remarked earlier, this simply presents different forms of the one function. 2. Inference by implication. Let one or more dots in a premiss be replaced by pluses, thus forming a function having a wider range of truth possibilities. The narrower range of truth possibilities was actualised, therefore the wider range must be actualised, ie. the inference is valid. 3. Inference by adjunction of premisses. Fully contract the premisses, form a table by placing a plus in every position where every premiss has a plus and placing a dot in every other position. Then because all the premisses are actualised, their common truth possibilities must be, ie. the inference is valid.

Notice that after a set of premisses have been adjuncted, the original premisses can be restored by implication. Indeed, if one wishes to develop all the consequences of a set of premisses, one can first adjunct them all into a single fully contracted premiss and then systematically apply inference by implication and by equivalence. Then any conclusion that can validly be infered from the original premisses can validly be inferred from the single fully contracted premiss.

6. SOME META-THEOREMS

The present section refers to the system of non-actualised (and vacuously actualised) functions. By means of relatively informal meta-proofs of well known theorems the relation between implication, equivalence, proof and theoremhood will be explored. Besides its intrinsic interest this will implicitly display something of the relationship beween the present system and more familiar systems.

The turnstile, «$\vdash$» will be used for «is a theorem», ie. «contracts to tautology» and positions in the tabled will be identified by the key $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$.

6.1 If Φ and Ψ are steps in the same proof, then $\vdash . \begin{bmatrix} + & - \\ - & + \end{bmatrix} \Phi\ \Psi$. For ex hypothesis Ψ is a contraction of Φ or vice versa. Suppose the former and that the fully contracted form of Ψ is ξ, then the following proof can be constructed. $A. \begin{bmatrix} + & - \\ - & + \end{bmatrix} \Phi\ \Psi$ ...... $B. \begin{bmatrix} + & - \\ - & + \end{bmatrix} \Psi\ \Psi$ ...... $C. \begin{bmatrix} + & - \\ - & + \end{bmatrix} \xi\xi$ $D. \begin{vmatrix} + & + \\ + & + \end{vmatrix}$ Hence all steps of a proof are equivalent.

Corollary. All steps in the proof of a theorem are theorems.

6.2. If Φ and Ψ are any functions such that $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ Φ Ψ $\begin{bmatrix} + & - \\ + & + \end{bmatrix}$ then Φ Ψ and $\begin{bmatrix} + & - \\ + & + \end{bmatrix}$ Ψ Φ. For there must be a proof:—

$A.$ $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ Φ Ψ ...... $B.$ $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ Φ' Ψ' $C.$ $\begin{vmatrix} + & + \\ + & + \end{vmatrix}$, where Φ' and Ψ' are the fully contracted forms of Φ and Ψ. Now the connective $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ is not used in the proof untill the last step and inspection shows that only positions 1 and 4 are then used. Consequently, the connective $\begin{bmatrix} + & - \\ + & + \end{bmatrix}$ is equally valid. Moreover, inspection of this last step shows that $\Phi' = \Psi'$, therefore $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ Ψ Φ and $\begin{bmatrix} + & - \\ + & + \end{bmatrix}$ Ψ Φ. Hence equivalent functions mutually entail each other.

6.3. If Φ and Ψ are any functions such that $\begin{bmatrix} + & - \\ + & + \end{bmatrix}$ Φ Ψ and $\begin{bmatrix} + & - \\ + & + \end{bmatrix}$ Ψ Φ, then $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ Φ Ψ (and $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ Ψ Φ). For there must be proofs ending ...... $A.$ $\begin{bmatrix} + & - \\ + & + \end{bmatrix}$ Φ' Ψ' $B.$ $\begin{vmatrix} + & + \\ + & + \end{vmatrix}$ and ...... $A.$ $\begin{bmatrix} + & - \\ + & + \end{bmatrix}$ Ψ' Φ' $B.$ $\begin{vmatrix} + & + \\ + & + \end{vmatrix}$. From the first of these no pair of signs in Φ' Ψ' can be +, —, and from the second no pair in Ψ' Φ' can be +, —, and therefore no pair in Φ' Ψ' can be —, +, Therefore only positions 1 and 4 of the connective can be used, etc. Hence functions that mutually entail each other are equivalent.

6.4. If Φ is any function $\vdash.$ $\begin{bmatrix} + & - \\ + & + \end{bmatrix}$ Φ $\begin{vmatrix} + & + \\ + & + \end{vmatrix}$. For there must be a proof $A.$ $\begin{bmatrix} + & - \\ + & + \end{bmatrix}$ Φ $\begin{vmatrix} + & + \\ + & + \end{vmatrix}$ ...... $B.$ $\begin{bmatrix} + & - \\ + & + \end{bmatrix}$ Φ' $\begin{vmatrix} + & + \\ + & + \end{vmatrix}$

and this last line cannot contain a + with a —, ie. position 2 is not used in further contraction and therefore one may add

C. $\begin{vmatrix} + & + \\ + & + \end{vmatrix}$. Hence all functions entail tautology.

6.5. If Φ is any function such that $\vdash . \begin{bmatrix} + & - \\ + & + \end{bmatrix} \begin{vmatrix} + & + \\ + & + \end{vmatrix} \Phi$ then $\vdash . \Phi$. Proof by 6.3 and 6.4.

6.6. If $\vdash . \begin{bmatrix} + & - \\ - & + \end{bmatrix} \Phi \ \Psi$ then Φ and Ψ can be presented as steps in the same proof. For there must be a proof ...... *A.* $\begin{bmatrix} + & - \\ - & + \end{bmatrix} \Phi' \ \Psi'$ *B.* $\begin{vmatrix} + & + \\ + & + \end{vmatrix}$ and from this last step $\Phi' = \Psi'$. Therefore there must be a proof *A.* Φ ...... *B.* Φ' $(= \Psi')$ ...... *C.* Ψ.

These various theorems can only be presented as meta-theorems in the present script and they are, indeed, meta-logical, a point that is obscured in P.M., where they are carried down into the object system. Relating this to earlier sections of this paper it will be seen that the Begriffsschrift displays and distinguishes three aspects of sentential logic, the logic of the manipulation of functions (and notably, tautology), the logic of inference from premisses and meta-logic. In P.M. all three aspects are confused.

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