EVERY SENTENTIAL LOGIC HAS A TWO-VALUED WORLDS SEMANTICS

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No one anywhere will design a sentential logic without a quite familiar kind of semantics, and no one can now scorn any such logic just because it lacks a semantics. For just as every sentential logic has a characteristic Lindenbaum algebra, so, and less trivially, every such logic has a bivalent relational (and also an operational) semantics.

Each semantical model is built on a class K — of worlds, scenarios, indices, set-ups, context-situations, or whatever — which contains a (distinguished) element 0 — the actual world, current scenario, etc. As well as K each model contains a function, f say, which assigns to each connective of the logic analysed a relation of one-place more than the connective, e.g. where C^n is an n-place connective $S^n = f(C^n)$ is an $(n + 1)$-place relation. S^n is a relation whose first place is on K and whose remaining places are on ∇, the set of ranges of the logic, i.e. S^n is defined on the $(n + 1)$-place Cartesian product $K \times \nabla \times \ldots \times \nabla$. A range or LA-proposition is a set of worlds, and the range of formula B is the set of worlds where B holds (as in Carnap [6]). Thus far the models are orthodox (and like those reached in [7], § 13). What is different, but not new, as the idea has already been used in the semantical analysis of tense logics and indeed much earlier by Henkin [11] in the semantics for type theory, is to add a set ∇ to each model. This set ∇, which is a subset of the power set of K, is to be construed as the class of LA-propositions of the logic analysed, and represents all that the logic can assert. Finally, each logic contains a standard bivalent valuation function v such that for each initial formula p and each world a of K, v(p, a) has one of the values T or F. The evaluation of compound formulae is accomplished by a simple extension of «neighbourhood» semantical rules (already used in [7]); for example,

$$I(C^n A_1 \ldots A_n, a) = T \text{ iff } S^n a[A_1] \ldots [A_n],$$

where $[A_i] = \{\alpha\varepsilon K: I(A_i, a) = T\}$, i.e. canonically $[A_i]$ is the range of A_i. In sum, a uniform general semantics is not obtained by some sort of cheating, by some new device that trivializes the modellings. There is nothing in the semantics that has not already been used and accepted in semantical analyses of intensional logics; nor for that matter are the proof methods used conspicuously new, but mainly an elaboration of those of [1].

The fact that *every* sentential logic has a semantics does not show that semantics is pointless (despite the vacuous contrast), any more than Gauss's theorem that every polynomial has a root shows that the notion of having roots is trivial. Semantical analyses remain the important foundations of full-blown accounts of meaning; and they provide easy keys to a great many logical doors.

Not surprisingly the uniform semantics provided by this method often do not supply anything like the best or most illuminating semantics for a system. For example, the «second-order» semantics generated for the Lewis system S2 by the uniform method are cumbersome indeed compared with the neat «first-order» semantics Kripke found (though they in fact facilitate the derivation of Kripke's semantics for S2). Furthermore the semantics are, in a sense, skew for such connectives as disjunction and negation. Though the normal bivalent disjunction rule

$$I(AvB, a) = T \text{ iff } I(A, a) = T \text{ or } I(B, a) = T$$

holds for a very large class of systems, the models for these systems furnished by the uniform method do not in general admit the rule; and it can be recovered only after the class of models is heavily pruned (in the manner of [1]). Also because classical, or even normal, negation rules cannot be generally recovered, the having of a model does not ensure consistency and a semantically sound and complete logic may be inconsistent or even trivial. The method leads, in short, to a more general model theory than that currently propounded (as explained in [10]).

But even if the models the method generates are sometimes skew, or even inconsistent, the method promises a big payoff not only in logic, but also for linguistics and philosophy. This payoff will be increased still further when the methods are extended to logics and languages far richer than sentential ones, as they can be (see [8]). Indeed the method presented below already suffices for all zero-order logics under a truth-valued interpretation; but for an objectual semantics further features have to be included in the models (see [10]). We conjecture that every *logic has a two-valued worlds semantics*; but there remain some conspicuous problems in the way of proving such a result, e.g. the problem of characterising logics and logical languages generally.

The logical payoff comes through the theories and results semantical analyses of logics and languages open up, for example, theories of truth, reference, meaning and consequence, and, less generally, results such as compactness, separation and decidability. The philosophical and linguistic payoff derives from this logical payoff: it is that any area of language that can be supplied with an exact logical syntax and set of principles can automatically be furnished with an extensional semantics, and so provided with associated logical theories of truth, meaning, consequence and so on. If, for example, the theory of propositional attitudes or notions such as belief or perception have a logic, or a structure, then they have a worlds semantics.

§ 1. *Sentential languages and logics.*

A *sentential language* SL is represented by a structure <IWff, Imp, Wff>, where IWff is the class of initial wff (well-formed formulae) which we represent by sentential parameters p, q, r, p', q', etc.; Imp is the class of improper symbols consisting of (finitely many, in the main case considered) connectives, and possibly containing as well various sorts of parentheses; and Wff is the set of wff built as usual (in the case to be considered using bracket-free Polish rules) from elements of IWff and connectives in Imp. Where SL is finitary we can suppose, without loss of generality, that Imp contains zero or more distinct connectives of each number of places from 1 to m for some finite m and that it has these connectives:

n_1 connectives of 1-place $C^1_1, C^1_2, \ldots, C^1_{n_1}$;

$\vdots$

n_k connectives of k-places $C^k_1, C^k_2, \ldots, C^k_{n_k}$;

$\vdots$

n_m connectives of m-places $C^m_1, \ldots, C^m_{n_m}$.

The familiar formation rules are stated explicitly since the argument requires inductions over the rules: —

1) Each element of IWff is a wff, i.e. IWFF $\subseteq$ Wff.
2_{n_1}) If A is a wff then each C^1_j A is a wff for $1 \leqslant j \leqslant n_1$.

$\vdots$

2_{n_k}) If $A_1, A_2, \ldots, A_k$ are wff then each C^k_j $A_1 \ldots A_k$ is a wff, for $1 \leqslant j \leqslant n_k$.

$\vdots$

2_{n_m}) If $A_1, A_2, \ldots, A_m$ are wff then each C^m_j $A_1 \ldots A_m$ is a wff, for $1 \leqslant j \leqslant n_m$.
3) These are the only wff.

To avoid heavy duplication of similar cases, already evident in the statement of formation rules, we work throughout with a representative connective C^i_j, the jth connective of i-places. In this way too we can encompass infinitary sentential languages; for neither i nor j need be assumed finite in what follows. That is, we consider also both sentential languages with infinitely many connectives, and sentential languages with infinitely long expressions. Some details will, however, be sketchy.

A formal *sentential logic* L is represented as a triple <SL, Ax, Rℓ >, where SL is a sentential language, Ax is a set of axioms or axiom schemes, and Rℓ is a set of (primitive) transformation rules. An axiom scheme $Ax_j = sm_j (A_1, \dots, A_m)$ in Ax is a *formula scheme* built up from wff $A_1, \dots, A_m$ according to the formation rules ($2n_1$)-$2n_m$) in the finitary case); and each rule R_j in R is of the form
$A_1, \dots, A_n \rightarrow B$, where $A_1, \dots, A_n$ and B are formulae schemes.

Proof and theoremhood are defined in the standard way (e.g. Church [3], p. 49), except that the restriction to finite sequences of wff is inessential. $\vdash_L A$, often shortened to $\vdash A$ where the logic L in question is obvious, abbreviates: A is a theorem of L. Thus $A_1, \dots, A_m \rightarrow B$ iff, where $\vdash A_1, \dots, \vdash A_m$, then $\vdash B$.

It is assumed that $Ax = \{Ax_i\}$ and $R1 = \{R\ell_j\}$ are both indexed sets but the sets need not be finite or even recursive. Indeed we impose *no* further restrictions on the class of axioms or axiom schemes or rules that a formal sentential logic may have. Although the central cases we have in view in the way we develop the semantics are single-sorted systems formulated with axiom schemes and without rules of substitution on (some) sentential parameters, systems with axioms and (restricted) substitution rules can also be encompassed in the theory. For rules of such forms as $A \rightarrow S^p_B A|$ and $A, C^1_i B \rightarrow S^r_B A|$ may of course be included in Rℓ (the substitution notation is from [3]).

General semantical investigations turn in part on the equivalence relations a logic can furnish. When a sentential logic includes an equivalence its semantical analysis can often be improved upon (and in special cases even finitized) by the familiar method of taking equivalence classes of old elements as new elements. Where no equivalence is definable in a logic, (type-) identity of wff can serve as the requisite, rockbottom, equivalence used for the semantic theory. Those logics which contain definable equivalence relations, *structural logics,* are worth separate investigation (as Polish logicians have emphasized). An important subclass (see [13]) of structural logics consists of those logics where a basic implication C is definable in terms of which the equivalence E of a structural logic can in turn be defined.

A *structural* sentential language SSL is then a sentential language such that Imp includes (or permits the definition of) one or other of the two-place connectives E (abbreviating C^2_1 say) or C (abbreviating C^2_2). A structural logic LS has language SSL. There are two (inclusive) classes of LS systems, LC systems with implication connective C, and LE systems, with equivalence E. It is convenient to define:

$(A \leftrightarrow B) =_{Df} EAB$; $(A \rightarrow B) =_{Df} CAB$. Some brackets are omitted again and dots used in accordance with an obvious modification of Church's conventions in [3].

Logic LC has at least the postulates of *austere implication,* namely: —

A1. $A \rightarrow A$ (Identity)
R1. $A,\ A \rightarrow B \rightarrow B$ (Modus Ponens)
R2. $A \rightarrow B,\ B \rightarrow A \rightarrow C \rightarrow D$, where D results from C by replacing one occurrence of A in C by B. (Leibnitz)
R3. $A \rightarrow B,\ B \rightarrow C \rightarrow A \rightarrow C$ (Rule Syllogism)

The substitutivity rule R2 at once generalises to the replacement of zero or more occurrences of A in C by B.

An LE logic has at least the postulates of *austere* equivalence, namely:

A1¹. $A \leftrightarrow B$, for some wff A and B — or else, what is deductively equivalent:
A1¹¹. $A \leftrightarrow A$ —
R1¹. $A,\ A \leftrightarrow B \rightarrow B$
R2¹. $A \leftrightarrow B \rightarrow C \leftrightarrow D$, where C and D are as in R2.

The analogue of R3 is derivable: for if $A \leftrightarrow B$ and $B \leftrightarrow C$, then

$B \leftrightarrow C \leftrightarrow . A \leftrightarrow C$ by R2[1], and hence $A \leftrightarrow C$ by R1. Austere equivalence is plainly an identity-type connective. Plainly also every LC logic is an LE logic, with $\vdash A \rightarrow B$ iff $\vdash A \rightarrow B$ and $\vdash B \rightarrow A$.

§ 2. *General models.*

A (*basic*) *L model* **M** is a structure $\mathbf{M} = \langle 0, K, \nabla, w \rangle$ where K is a set $0 \varepsilon K$, $\nabla \subseteq P(K)$ (i.e. ∇ is a subset of the power set of K), and w is a function which assigns to each connective C^i_j of SL a relation on the $(i + 1)$-place Cartesian product $K \times \nabla \times \ldots \times \nabla$ and to each initial wff or sentential parameter p an element of II^K, where $II =_{Df} \{T, F\}$. Furthermore ∇ satisfies the conditions:
∇i) $\{a \varepsilon K: w(p)(a) = T\} \varepsilon \nabla$, for each sentential parameter p;
∇ii) if $\alpha_1 \ldots, \alpha_i \varepsilon \nabla$ then $\{a \varepsilon K: w(C^i_j) a \alpha_1 \ldots \alpha_i\} \varepsilon \nabla$, for each i such that $1 \leqslant i \leqslant m$ and corresponding j in $1 \leqslant j \leqslant n_i$. Thus for $a \varepsilon K$, $w(p)(a) = T$ or $= F$. Where $S_1, \ldots, S_n$ are n sets, the *n-place Cartesian product* $S_1 X \ldots X S_n = \{\langle i_1, \ldots i_n \rangle$: $i_1 \varepsilon S_1$ & $i_2 \varepsilon S_2 \ldots$ & $i_n \varepsilon S_n\}$. By counting sentential parameters or initial wff as zero-place connectives, w may be viewed as a function defined on the connectives of L.

It is sometimes illuminating to recast the model in alternative but equivalent ways. In particular, it sometimes pays to separate out a model structure $S = \langle 0, K, \nabla, f \rangle$ and a valuation v in A, where f is a 1 - 1 function which assigns to each connective of SL a relation (just as w does for non-zero place connectives) and v is a bivalent valuation function from initial wff and elements of K to truth-values in II, i.e. $v \varepsilon II^{IWffxK}$. Secondly, 0 can be deleted, and defined as $\xi x(x \varepsilon K)$, i.e. as an arbitrary element of K, where K is a non-null set. (By varying the notion of validity, to truth at all elements of K, 0 can often be dispensed with entirely in the way familiar from modal logic semantics.) Thirdly, the functions f or w can be replaced in models by their (sets of) values. So presented a model structure is a structure $<0, K, \nabla, \mathbf{S}_1, \ldots, \mathbf{S}_m>$, where $\mathbf{S}^i_j =_{Df}$ $w(C^i_j)$ [$= f(C^i_j)$]. For the most part we shall work with basic L models $<0, K, \nabla, \mathbf{S}_1, \ldots \mathbf{S}_m, \mathbf{v}>$.

L models which are not basic result by adding semantical postulates to the L models. The form these postulates take will be explained after basic LC and LE models have been defined and further semantical notions introduced.

A (*basic*) *LC model* **M** is a (basic) L model, with $\mathbf{R} =_{Df}$ $w(C)$ [or $\mathbf{R} =_{Df} f(C)$] and $\alpha \leqslant \beta =_{Df} \mathbf{R}0\alpha\beta$ for $\alpha, \beta \varepsilon \nabla$, which satisfies the following conditions: —

Ri) $\leqslant$ is a quasi-order on ∇ (i.e. for $\alpha, \beta, \gamma \varepsilon \nabla$, $\alpha \leqslant \alpha$, and if $\alpha \leqslant \beta$ and $\beta \leqslant \gamma$ then $\alpha \leqslant \gamma$) such that if $\alpha \leqslant \beta$ then $\alpha \subseteq \beta$. Hence $\leqslant$ is a partial order on ∇. Further ∇ satisfies as well as conditions ∇i) and ∇ii) also

∇iii) If $\alpha, \beta \varepsilon \nabla$ then $\{a \varepsilon K: \mathbf{R}a\alpha\beta\} \varepsilon \nabla$

A (*basic*) *LE model* **M** is a (basic) L model which contains a relation **T**, where $\mathbf{T} =_{Df} w(E)$, which satisfies these conditions: — Ti) For $\alpha, \beta \varepsilon \nabla$, $\mathbf{T}0\alpha\beta$ iff $\alpha = \beta$. In short **T** is *an* identity at 0 on ∇. ∇i) and ∇ii) are as before but ∇iii) is supplanted by

iii^1) If $\alpha, \beta \varepsilon \nabla$ then $\{a \varepsilon K: \mathbf{T}a\alpha\beta\} \varepsilon \nabla$.

An *interpretation* I associated with valuation v [function w] is a function in $II^{Wff X K}$ [$(II^K)^{Wff}$] satisfying the following conditions, where applicable, for every parameter p and every $a \varepsilon K$;

Ii) $I(p, a) = v(p, a)$

Iii) $I(C^i_j A_1 \ldots A_i, a) = T$ iff $\mathbf{S}^i_j a[A_1] \ldots [A_i]$, where $[A_j] =_{Df}$ $\{a \varepsilon K: I(A_j, a) = T\}$.

Since E is taken as C^2_1, **T**, is $\mathbf{S}^2_1$; hence

Iiii) $I(EAB, a) = T$ iff $\mathbf{T}a[A]\,[B]$

follows from Iii). Similarly it follows from Iii),

Iiii) $I(CAB, a) = T$ iff $\mathbf{R}a[A]\,[B]$

That I is well-defined will follow using lemma 1 below.

A wff A *holds on* a valuation v, or on the associated interpretation I, *in* model structure **S** *at* an element a of K, just in case $I(A, a) = T$; otherwise A *fails on* v, or I, *in* **S** *at* a. Wff is *true on* v, or on associated I, in S just in case $I(A, 0) = T$; and otherwise A is *false on* v, or on I, *in* **S**, and v in **S** *falsifies* A. Notions with respect to models are similarly defined. A wff A is *true in* model **M** if A is true on the valuation v of **M** in **M**; **M** is a *countermodel to* A iff A is falsified in **M**; etc. Wff A is *valid in* model structure **S** just in case A is verified in all valuations therein, and otherwise is *invalid in* **S**. Finally A is *L-valid* just in case A is true in all L models (or valid in all L model structures); otherwise A is *L-invalid.*

For sentential logics which do not admit the rule of uniform substitution (separated out as *nonuniform logics*) the definition of validity and ensuing details should be slightly modified, in the way explained at the end of the section.

In order to accommodate any and every axiom scheme we give a recipe for writing down for each axiom scheme (and hence for each axiom) of L a corresponding semantical postulate. Let $Ax_j = sm_j(A_1, \ldots, A_i)$ be the j^{th} axiom scheme in the given indexing of the schemes in Ax. Ax_j, which will serve as paradigmatic, is built up from elements $A_1, \ldots, A_i$ according to the formation rules of L. The semantical postulate $Sp_j = sp_j0(\alpha_1, \ldots, \alpha_i)$ corresponding to Ax_j is specified by recursion on the construction of Axj, thus:

1. Replace each A_k in Ax_j by α_k, for $1 \leqslant k \leqslant i$.
2. Given that replacements $\beta_1, \ldots, \beta_j$ have been made to the operands of connective C^j_h of Ax_j, replace

$C^j_h\, \beta_1 \ldots \beta_j$ by $\{\alpha \varepsilon K\colon \mathbf{S}^j_h\, a\, \beta_1 \ldots \beta_j\}$ (where of course $\mathbf{S}^j_h = f(C^j_h)$).

3. When no further replacements can be made, i.e. all connectives have been replaced, with the schema γ resulting, the semantical postulate Sp_j is $0\, \varepsilon\, \gamma$.

Equivalently we can restrict step 2 to replacements for connectives other than the main (i.e. initial) connective of Ax_j, and specify for the remaining case.

3^1. Where C^j_h is the initial connective (and letter) of Ax_j and replacements $\beta_1, \ldots, \beta_j$ have been made for the operands of C^j_h replace $C^j_h\, \beta_1 \ldots \beta_j$ by $S^j_h\ 0\ \beta_1 \ldots \beta_j$;

and this resulting scheme is the semantical postulate Sp_j corresponding to Ax_j. These recipes for generating semantical postulates are equivalent since $0 \varepsilon \{a: \mathbf{S}^i_j\, a\beta_1 \ldots \beta_i\}$ iff $\mathbf{S}^i_j\, 0\beta_1 \ldots \beta_i$.

In systems with uniform substitution there remains only the degenerate case where a sentential parameter standing alone is added as an axiom. The case is degenerate since the axiom contains no connectives, and is of limited interest since any such system is absolutely inconsistent, every wff being a theorem. The corresponding semantical postulate is of course $0 \varepsilon \alpha$, for any $\alpha \varepsilon \nabla$.

Clauses 1 and 2 of the recipe for constructing a semantical postulate corresponding to each axiom scheme in fact apply more generally to deliver a *semantical scheme* corresponding to each *formula scheme*. (The general recipe can be viewed, as in [10], as providing a translation procedure.) This we use in specifying semantical postulates corresponding to rules. Let the j$^{\text{th}}$ rule in the given indexing of rules of L, rule Rj, be of the form: $A_1, \ldots, A_m \rightarrow B$; and let $\alpha_1, \ldots, \alpha_m, \beta$ be respectively the semantical schemes corresponding to formulae schemes $A_1, \ldots, A_m, B$ (after exhaustive application of rules 1 and 2). Then the semantical postulate RSj corresponding to Rj is as follows: If $0 \varepsilon \alpha_1, \ldots, 0 \varepsilon \alpha_m$ then $0 \varepsilon \beta$. Except in degenerate cases such semantic postulates can be reformulated by taking account of the structure of each wff A_i. As each A_i is of the form $C^j_n\, B_1 \ldots B_j$, $0 \varepsilon \alpha_i$ can be supplanted by $\mathbf{S}^j_h\, 0\, \beta_1 \ldots \beta_j$ for suitable $\beta_1 \ldots, \beta_j$.

As a working example of the method consider a very weak system S0 formulated and studied by Halldén [2]. S0 derives from Lewis's system S1 by simply omitting Lewis's definition of strict implication in terms of logical possibility, and using instead a primitive entailment relation. Thus for S0 Imp is the set $\{C, K, N\}$, with Halldén's connectives defined: $(A \rightarrow B) =_{\mathrm{Df}} CAB$; $(AB) =_{\mathrm{Df}} KAB$; $\sim A =_{\mathrm{Df}} NA$. We re-express Halldén's axioms as schemata so as to eliminate the rule of uniform substitution in the usual way (already exploited).

Ax1. $AB \rightarrow BA$, i.e. CKABKBA	Sp1. $\mathbf{R}0\{a:\mathbf{S}a\alpha\beta\}\{a:\mathbf{S}a\beta\alpha\}$
Ax2. $AB \rightarrow A$, i.e. CKABA	Sp2. $\{a:\mathbf{S}a\alpha\beta\} \leqslant \alpha$
Ax3. $A \rightarrow AA$, i.e. CAKAA	Sp3. $\mathrm{R}0\{a:\mathbf{S}a\alpha\alpha\}$

Ax4. (AB)D → A(BD), i.e. CKKABDKAKBD

Sp4. $\{a{:}\mathbf{S}a\{a{:}\mathbf{S}a\alpha\beta\}\delta\} \leqslant \{a{:}\mathbf{S}a\alpha\{a{:}\mathbf{S}a\beta\delta\}\}$

Ax5. A → ∼ ∼A, i.e. CANNA

Sp5. $\mathbf{R}0\alpha\{a{:}\mathbf{T}a\{a{:}\mathbf{T}a\alpha\}\}$

Ax6. (A → B) (B → D) → (A → D), i.e. CKCABCBDCAD

Sp6. $\{a{:}\mathbf{S}a\{a{:}\mathbf{R}a\alpha\beta\}\{a{:}\mathbf{R}a\beta\delta\}\} \leqslant \{a{:}\mathbf{R}a\alpha\delta\}$

Ax7. (A(A → B)) → B, i.e. CKACABB

Sp7. $\{a{:}\mathbf{S}a\alpha\{a{:}\mathbf{R}a\alpha\beta\}\} \leqslant \beta$

S0 has as rules, as well as R1 and R2,

R4. A, B → AB

RS4. $0 \varepsilon \alpha$ and $0 \varepsilon \beta$ then $\mathbf{S}0\alpha\beta$

Since R3 is a derived rule of S0 (applying R4, Ax6 and R1), S0 includes an austere implication C, and accordingly the general theory for LC systems applies. To specify an S0 model it suffices to specify f (or w) and the semantical postulates. To specify f it is enough to introduce two 3-place relations **R** and **S** on K×∇×∇ and one 2-place relation **T** on K×∇, and set f(C) = **R**, f(K) = **S** and f(N) = **T**. The special semantical postulates for S0 are those, Sp1-Sp8, RS4 displayed above. We illustrate the procedure for obtaining the semantical postulate corresponding to an axiom scheme in the quite typical case of Ax7 and Sp7:

$CKACABB$; $CK\alpha C\alpha\beta\beta$: $CK\alpha\{a{:}\mathbf{R}a\alpha\beta\}\beta$;
$\mathbf{C}\{a{:}\mathbf{S}a\alpha\{a{:}\mathbf{R}a\alpha\beta\}\}\beta$; $0\ \varepsilon\{a{:}\mathbf{R}a\{a{:}\mathbf{S}a\alpha\{a{:}\mathbf{R}a\alpha\beta\}\}\beta\}$, i.e.
$\mathbf{R}0a{:}\mathrm{S}a\alpha\{a{:}\mathrm{R}a\alpha\beta\}\}\beta$, i.e. $\{a{:}\mathbf{S}a\alpha\{a{:}\mathbf{R}a\alpha\beta\}\} \leqslant \beta$.

To accommodate neatly non-uniform logics and logics with sentential constants, the modellings are varied (inessentially), by assigning to each sentential parameter a property on worlds, i.e. parameters are treated like zero-place connectives. Then function w of L-models assigns to each sentential parameter or initial wff p not an element of II^{K} but a property $\mathbf{S}^{p} = w(p)$ on K, and I(p, a) = T iff $\mathbf{S}^{p}$ a. The construction of semantical schemes corresponding to axioms is varied as follows:

0. Replace a sentential parameter p in axiom Ax_j by $\{a\varepsilon K{:}\ \mathbf{S}^{p}a\}$.

For example, corresponding to the axiom (p → p) is the semantical scheme $\mathbf{R}0\{a{:}\mathbf{S}^{p}a\}\{a{:}\mathbf{S}^{p}a\}$. It is easy to see how this semantical postulate guarantees $(p \rightarrow p)$ without guaranteeing $(q \rightarrow q)$ or $(A \rightarrow A)$.

§ 3. *The soundness of sentential logics.*

Lemma 1, $[A] \varepsilon \nabla$, for every wff A.

Proof is by induction on the length of wff A. The induction basis is furnished by ∇i), and the induction steps for each connective by ∇ii), ∇ii') and ∇iii). Suppose, to illustrate, A is of the form $C^i_j\ B_1 \dots B_i$. Now $[A] = \{a{:}I(C^i_j\ B_1 \dots B_i,\ a) = T\} = \{a{:}S^i_j\, a[B_1] \dots [B_i]\}$.
Since by induction hypothesis, $[B_1]\ \varepsilon\ \nabla, \dots, [B_i]\ \varepsilon\ \nabla$, $\{a{:}S^i_j\, a[B_1] \dots [B_i]\}\ \varepsilon\ \nabla$, by ∇iii).

Theorem 1. (i) For any structural sentential logic L, i.e. LC or LE, and any wff A, if A is a theorem of L then A is L-valid.

(ii) For any sentential logic L and any wff A, if $\vdash_L A$ then A is L-valid.

Proof: (ii) is established in the course of proving the more specific (i); for the restriction to structural logics is inessential. Proof of (i) is by induction over the length of the proof of A. As usual it is shown that the axiom schemes are L-valid and that the rules preserve L-validity. Consider first the postulates of the basic systems.

ad A1. Since $[A]\ \varepsilon\ \nabla$ and $\leqslant$ is reflexive on ∇, $[A] \leqslant [A]$, i.e. $\mathbf{R}0[A]\,[A]$. Thus $I(A \rightarrow A, 0) = T$, for an arbitrary model $\mathbf{M}$.

ad R1. Suppose in arbitrary model $\mathbf{M}$, $I(A, 0) = T = I(A \rightarrow B, 0)$. Then $\mathbf{R}0[A]\,[B]$, whence by Ri), $[A] \subseteq [B]$. Hence as $I(A, 0) = T$, $I(B, 0) = T$.

ad R2. Suppose, again for arbitrary $\mathbf{M}$, $I(A \rightarrow B, 0) = T = I(B \rightarrow A, 0)$ Then, by Ri), $[A] = [B]$. Then the semantic evaluation of C and D as specified in R2 must coincide since they differ only with respect to one replacement of A by B, i.e. $[C] = [D]$. (The detailed verification of this point is by induction on the length of C.) Hence, as in the case of A1, $I(C \rightarrow D, 0) = T$

ad R3. Suppose for arbitrary **M**, $I(A \rightarrow B, 0) = T = I(B \rightarrow C, 0)$. Then $[A] \leqslant [B] \leqslant [C]$, so by transitivity of $\leqslant$ on ∇ (from Ri), $[A] \leqslant [C]$, whence $I(A \rightarrow C, 0) = T$.
ad A1″. For $[A] = [A]$ implies $\mathbf{T}0[A]\,[A]$, by Ti).
ad R1′. Suppose for any **M**, $I(A, 0) = T = I(A \leftrightarrow B, 0)$. Then, by Ti) and the rule for $\leftrightarrow$, $[A] = [B]$, whence since $I(A, 0) = T$, $I(B, 0) = T$, which is sufficient for preservation of L-validity.
ad R2′. Similar to R2.

In the case of the further axiom schemes we can establish the general case if we can show, for arbitrary j, that Ax_j is L-valid given Sp_j holds in all L models. Let Ax_j be of the form $sm_j(A_1, \ldots, A_i)$ with elements $A_1, \ldots, A_i$, and suppose Ax_j is the formula $C^h_f\, D_1, \ldots, D_h$ for some connective C^h_f and some operands $D_1, \ldots, D_h$. Let Sp_j be the corresponding derived semantic scheme with form $sp_j0(\alpha_1, \ldots, \alpha_n)$; and let $f(C^h_f) = \mathbf{S}^h_f$. Since $[A_k] \varepsilon \nabla$ for each k with $1 \leqslant k \leqslant i$, $sp_j0([A_1], \ldots, [A_j])$.

We show by induction on the number $n \geqslant 0$ of connectives in the semantic scheme, $sp([A_1], \ldots, [A_i])$ say, corresponding by recursion rules 1 and 2 to sm $([A_1], \ldots, [A_i])$, that where C^i_j is the connective occurring at the n^{th} stage:

$$[C^i_j\, B_1, \ldots, B_i] = \{a{:}S^i_j a\, [B_1] \ldots [B_i]\}. \qquad (\delta).$$

The induction basis is provided by the equations $[A_k] = [A_k]$ for each k such that $1 \leqslant k \leqslant i$. We suppose the construction has proceeded to the connective C^i_j with $C^i_j\, \beta_1 \ldots \beta_i$ replaced by $\{a{:}S^i_j\, a\, \beta_1 \ldots \beta_i\}$. Suppose further that the connective C^i_j has the arguments $B_1 \ldots, B_i$ in Ax_j. Then to complete the induction it suffices to establish (δ). But this is immediate from the rule for evaluating $I(C^i_j\, B_1 \ldots B_i, a) = T$

It follows from the induction that $\{a{:}\mathbf{S}^h_f a\, [D_1] \ldots [D_h]\} = [C^h_f\, D_1 \ldots D_h] = [Ax_j]$. Hence $\mathbf{S}^h_f 0[D_1] \ldots [D_h]$ iff $0 \varepsilon [Ax_j]$, i.e. Sp_j iff $I(Ax_j, 0) = T$; whence $I(Ax_j, 0) = T$.

[An illustration more quickly clarifies the method. Suppose

then Sp2 of Halldén's system holds, and it is required to verify Ax2. Since $[A], [B] \varepsilon \nabla$, consider (what proves to suffice) $R0\{a:\mathbf{S}a[A]\,[B]\}\,[A]$. By a first induction step since $\{a:\mathbf{S}a[A]\,[B]\} = [KAB]$ it follows:
$\mathbf{R}0[KAB]\,[A]$. Since similarly $\{\mathbf{a}:Ra[KAB]\,[A]\} = [CKABA]$, $\mathbf{R}0[KAB]\,[A]$ iff $I(Ax, 0) = T$. Hence, since $\mathbf{R}0[KAB]\,[A]$ by Sp2 and the induction, $I(Ax, 0) = T$.]

Finally, to show that the further rules preserve L-validity, consider the j^{th} rule R_j, and suppose that all L models conform to the semantical requirement RS_j. Let R_j be of the form $A_1, \ldots, A_m \rightarrow B$. Let $\mathbf{M}$ be an arbitrary model for which $I(A_1, 0) = T, \ldots, I(A_m, 0) = T$. To verify R_j it suffices to show that $I(B, 0) = T$. But applying semantic postulate RS_j and the inductive argument employed in vindicating axiom schemes, if $0 \varepsilon [A_1], \ldots, 0 \varepsilon [B]$. Hence on the given assumptions $I(B, 0) = T$.

§ 4. *The completeness of sentential logics.*

There are three cases, LC logics, LE logics and general L logics, and we proceed from special to more general.

In proving completeness of LC logics we appeal to sets of LC-theories. An *LC-theory* a is a class of wff of L closed under C, i.e. if $A \varepsilon a$ and $\vdash_{LC} A \rightarrow B$, then $B \varepsilon a$. Then a canonical LC model $\mathbf{M} = <L, K, \nabla, f, v>$ is defined as follows: L is the set of theorems of LC, K is the set of LC-theories; ∇ is the set of elements α of the power set of K such that, for some wff B, $B \varepsilon c$ iff $c \varepsilon \alpha$ for every $c \varepsilon K$,

i.e. $\nabla = \{\alpha \varepsilon P(K): (PB)\,(c \varepsilon K)\,(c \varepsilon \alpha \equiv B \varepsilon c)\}$;

for $a \varepsilon K$ and $\alpha, \beta \varepsilon \nabla$, $\mathbf{R}a\alpha\beta$ iff for some wff B, C, $B \rightarrow C \varepsilon a$ and $\{c \varepsilon K: B \varepsilon c\} = \alpha$ and $\{c \varepsilon K: C \varepsilon c\} = \beta$, i.e. $Ra\alpha\beta$ iff $(PB, C)\,(B \rightarrow C \varepsilon a \;\&\; |B| = \alpha \;\&\; |C| = \beta$, where $|B| =_{Df} \{c \varepsilon K: B \varepsilon c\}$; for $a \varepsilon K$ and $\alpha_1, \ldots, \alpha_i \varepsilon \nabla$, $S^i_j a\; \alpha_1 \ldots \alpha_i$ iff $(PB_1, \ldots, B_i)\,(C^i_j B_1 \ldots B_i \varepsilon a \;\&\; |B_1| = \alpha_1 \;\&\ldots\&\; |B_i| = a_i)$; where $\mathbf{R}$ and $\mathbf{S}^i_j$ are as defined as before, $f(C^i_j) = \mathbf{S}^i_j$; and for every sentential parameter p and each $a \varepsilon K$, $v(p, a) = T$ iff $p \varepsilon a$, or, for non-uniform logics, $\mathbf{S}^p a$ iff $w(p)a$ iff $p \varepsilon a$.

For LE logics we define an *LE-theory* a as a class of wff closed under E, i.e. if $A \varepsilon a$ and $\vdash_{LE} A \leftrightarrow B$ then $B \varepsilon a$. A *canonical* LE *model* $\mathbf{M} = \langle L, K, \nabla, f, v \rangle$ differs from a canonical LC model only in taking L as the set of theorems of LE, K as the class of LE-theories, and in defining $\mathbf{T}a\alpha\beta$ as (PB) (PC) $(B \leftrightarrow C \, \varepsilon \, a \;\&\; |B| = \alpha \;\&\; |C| = \beta)$. ∇ and v are defined as before, though relative to the new characterisation of K.

A *canonical* L *model* $\mathbf{M} = \langle L, K, \nabla, f, v \rangle$ for a general sentential logic L differs in definition from canonical LC and LE models only in taking L as the set of theorems of L and K as the class of all sets of wff of L, i.e. an L-theory is a set of wff of L.

Lemma 2. (1) If $A \rightarrow B$ is not a theorem of LC (an LC logic) then for some $a \varepsilon K$, $A \varepsilon a$ and $B \varepsilon a$.

(2) If $A \leftrightarrow B$ is not a theorem of LE then for some $a \varepsilon K$, $A \varepsilon a$ and $B \varepsilon a$.

Proof. (1) Suppose $\sim \vdash A \rightarrow B$, and define $d = \{D: \vdash A \rightarrow D\}$. Since $\sim \vdash A \rightarrow B$, $B \not\varepsilon d$, and since $\vdash A \rightarrow A$, $A \varepsilon d$. Further d is an LC-theory; for suppose $B \varepsilon d$ and $\vdash B \rightarrow C$. Then $\vdash A \rightarrow B$, so by R3, $\vdash A \rightarrow C$, i.e. $C \varepsilon d$ as required.

(2) Vary (1) redefining $d = \{D: \vdash A \leftrightarrow D\}$. The proof that d is an LE-theory applies $R2^1$.

Lemma 3. Where I is the interpretation associated with valuation v of the canonical LC or LE or L model **M**,

$I(A, a) = T$ iff $A \varepsilon a$,

for every $a \varepsilon K$ and every wff A.

Proof. The induction, on the length of A, is based on the definition of v in **M**. There are induction steps for each connective.

ad $\rightarrow$ (for LC). A is of the form $B \rightarrow C$. If $B \rightarrow C \varepsilon a$ then $B \rightarrow C \varepsilon a \;\&\; [B] = |B| \;\&\; [C] = |C|$. For the induction hypothesis ensures that $I(B, a) = T$ iff $B \varepsilon a$, and hence that $[B] = |B|$. Hence, by particularization, (PB, C) $(B \rightarrow C \, \varepsilon a \;\&\; [B] = |B| \;\&\; [C] = |C|)$, whence $\mathbf{R}a[B]\,[C]$, by definition of $\mathbf{R}$; and so $I(B \rightarrow C, a) = T$. Conversely, suppose $I(B \rightarrow C, a) = T$, that is $\mathbf{R}a[B]\,[C]$. Then for some wff D and E, $D \rightarrow E \varepsilon a$, $|D| = [B]$ and $|E| = [C]$. Hence for every $c \varepsilon K$, $D \varepsilon c$ iff $I(B, c) = T$, i.e.

by induction hypothesis $B \varepsilon c$; and $E \varepsilon c$ iff $C \varepsilon c$. Therefore, by contraposing the previous lemma, $\vdash D \leftrightarrow C$ and $\vdash E \leftrightarrow C$. Hence, by R2, $\vdash D \rightarrow E \rightarrow . B \rightarrow C$; and so as $D \rightarrow E \varepsilon a$, $B \rightarrow C \varepsilon a$.

ad C^i_j (for LC and LE). A is of the form $C^i_j B_1 \dots B_i$. If $A \varepsilon a$ then $C^i_j B_1 \dots B_i \varepsilon a \;\&\; [B_1] = |B| \;\&\dots\&\; [B_i] = |B_i|$; so $(PB_1 \dots B_i)(C^i_j B_1 \dots B_i \varepsilon a \;\&\; [B_1] = |B_1| \;\&\dots [B_i] = |B_i|)$, whence $\mathbf{S}^i_j a[B_1] \dots [B_i]$, and $I(C^i_j B_1 \dots B_i, a) = T$. Conversely, suppose $I(C^i_j B_1 \dots B_i, a) = T$. Then for some $D_1, \dots, D_i$, $C^i_j D_1 \dots D_i \varepsilon a$, $|D_1| = [B_1], \dots,$ and $|D_i| = [B_i]$. Hence, by the previous lemma $\vdash D_1 \leftrightarrow B_1, \dots, \vdash D_i \leftrightarrow B_i$. Thus, by R2 [or R2^1], $\vdash C^i_j D_1 \dots D_i \rightarrow C^i_j B_1 \dots B_i$ [or $\vdash C^i_j D_1 \dots D_j \leftrightarrow C^i_j B_1 \dots B_i$], and so as a is a theory, $C^i_j B_1 \dots B_i \varepsilon a$.

ad C^i_j (for L). Only the converse argument differs from the previous case. Suppose $I(C^i_j B_1 \dots B_i, a) = T$. Then, as before, for some wff $D_1, \dots, D_i$, $C^i_j D_1 \dots D_i \varepsilon a$, $|D_1| = [B_1], \dots,$ and $|D_i| = [B_i]$. Consider $|D_k| = [B_k]$ for arbitrary k, $1 \leqslant k \leqslant i$. By induction hypothesis, for every class a of wff of L, $D_k \varepsilon a$ iff $B_k \varepsilon a$. Hence D_k is the same wff as B_k, othewise some class of wff would distinguish them. Therefore $C^i_j B_1 \dots B_i \varepsilon a$, as required.

Corollaries. 1. For $a \varepsilon K$, $B \rightarrow C \varepsilon a$ iff $Ra[B][C]$, in the case of LC. 2. For $a \varepsilon K$, $B \leftrightarrow C \varepsilon a$ iff $\mathbf{T}a[B][C]$, in the case of LE logics. 3. For $a \varepsilon K$, $C^i_j B_1 \dots B_i \varepsilon a$ iff $\mathbf{S}^i_j a[B_1] \dots [B_i]$.

Lemma 4.
(1) The canonical LC model is a LC model.
(2) The canonical LE model is a LE model.
(3) The canonical L model is an L model.

Proof. In virtue of R1, or R1^1, $L \varepsilon K$. It is immediate that $\nabla \subseteq P(K)$, that $\mathbf{R}$ is a relation on $K X \nabla X \nabla$, that each $\mathbf{S}^i_j$ is a relation on $K X \nabla X \dots X \nabla$, and that f and v are well-defined.
ad ∇i). By definition of ∇, $\alpha \varepsilon \nabla$ iff $(PB)(\alpha = |B|)$. Hence

$|p| \varepsilon \nabla$, and so $\{a \varepsilon K\colon v(p, a) = T\} \varepsilon \nabla$, by definition of v. More generally, $|B| \varepsilon \nabla$ for every wff B.

ad ∇ii). Suppose $\alpha \varepsilon \nabla$ and $\beta \varepsilon \nabla$. Let A and B be wff such that $\alpha = |A|$ and $\beta = |B|$. Since for arbitrary $c \varepsilon K$, $\mathbf{R}c\alpha\beta$ iff iff $A \rightarrow B \varepsilon c$ by the previous corollary,
$(c \varepsilon K)\ (c \varepsilon \{a \varepsilon K\colon \mathbf{R}a\alpha\beta\} \equiv .\ A \rightarrow B \varepsilon c)$, whence
$\{a \varepsilon K\colon \mathbf{R}a\alpha\beta\} \varepsilon \nabla$ follows.

ad ∇iii). Suppose $\alpha_1, \ldots, \alpha_i \varepsilon \nabla$. Let $A_1, \ldots, A_i$ be wff such that $|A_1| = \alpha_1, \ldots, |A_i| = \alpha_i$. Since for every $c \varepsilon K$, $C^i_j A_1 \ldots A_i \varepsilon c$ iff $S^i_j c[A_1] \ldots [A_i]$, $C^i_j A_1 \ldots A_i$ iff $S^i_j c\alpha_1 \ldots \alpha_i$. Hence $c \varepsilon \{a \varepsilon K\colon S^i_j\ a\ \alpha_1 \ldots \alpha_i\}$ ifff $C^i_j\ A_1 \ldots A_i \varepsilon c$ for every $c \varepsilon K$. Therefore, by definition of ∇, $\{a \varepsilon K\colon S^i_j\ a\ \alpha_1 \ldots \alpha_i\} \varepsilon \nabla$, in every case.

ad Ri (for (1)). Suppose $\alpha \leqslant \beta$. Then for some wff B, C say, $B \rightarrow C \varepsilon L$, $|B| = \alpha$ and $|C| = \beta$. Suppose for arbitrary $b \varepsilon K$, $b \varepsilon \alpha$. To prove $\alpha \subseteq \beta$, it suffices to show that $b \varepsilon \beta$. Given $b \varepsilon \alpha$, $B \varepsilon b$. Hence, as $|\!\!-_{LC} B \rightarrow C$, $C \varepsilon b$. Accordingly, since $|C| = \beta$, $b \varepsilon \beta$. It follows at once that $\leqslant$ is anti-symmetric.

For each $\alpha \varepsilon \nabla$ there is some wff A such that $\alpha = |A|$. Further by A1, $A \rightarrow A \varepsilon L$; hence $\mathbf{R}0\alpha\alpha$ by definition of R. Thus $\leqslant$ is reflexive. For transitivity suppose that $\alpha \leqslant \beta$ and $\beta \leqslant \gamma$. Then for some wff B_1, C_1, B_2, C_2, $B_1 \rightarrow C_1 \varepsilon L$, $|B_1| = \alpha$, $|C_2| = \gamma$ and $|C_1| = \beta = |B_2|$. Since for every $d \varepsilon K$, $C_1 \varepsilon d$ iff $B_2 \varepsilon d$, $|\!\!- C_1 \leftrightarrow B_2$. Hence by R2, $\vdash B_1 \rightarrow C_1 \rightarrow . B_1 \rightarrow B_2$, so $B_1 \rightarrow B_2 \varepsilon L$; and thus as $B_2 \rightarrow C_2 \varepsilon L$ by R3 $B_1 \rightarrow C_2 \varepsilon L$. Hence by the definition of $\mathbf{R}$, $\alpha \leqslant \gamma$.

ad Ti) (for (2)). Suppose for $\alpha, \beta \varepsilon \nabla$, $\mathbf{T}0\alpha\beta$. Then for some wff B and C, $|\!\!- B \leftrightarrow C$, $|B| = \alpha$ and $|C| = \beta$. Suppose further $b \varepsilon \alpha$; then $B \varepsilon b$, hence $C \varepsilon b$ by $\leftrightarrow$ -closure, so $b \varepsilon \beta$. Similarly supposing $b \varepsilon \beta$, $b \varepsilon \alpha$, for arbitrary $b \varepsilon K$. Hence $\alpha = \beta$. Conversely suppose $\alpha = \beta$. Let C be a wff such that $|C| = \alpha = \beta$. Then since $|\!\!- C \leftrightarrow C$, $\mathbf{T}0\alpha\beta$.

Lemma 5. For each of the axiom schemes or rule schemes which extend LC or LE or L beyond the basic system the corresponding semantical postulates holds.

Proof. Consider first the j^{th} axiom scheme $sm_j(A_1, \ldots, A_i) \equiv Ax_j$. To show $sp_j0(\alpha_1, \ldots, \alpha_i)$, we shall appeal back to the inductive definition of the semantical postulate. Suppose that Ax_j is of the form $C^k_h C_1 \dots C_k$, and so Sp_j is of the corresponding form $S^k_h \delta_1 \dots \delta_k$ in virtue of its definition, with the form of δ_j determined by C_j. (In the degenerate case with uniform substitution where a sentential parameter alone is a thesis, let p be the wff such that $|p| = \alpha$. Then, since $p \varepsilon L$, $L \varepsilon \alpha$.) To establish Sp_j it is enough to show, since $Ax_j \varepsilon L$ by the assumption that Ax_j extends the basic system, that $|C_1| = \delta_1 \& \dots \& |C_k| = \delta_k$; for then the result follows by the definition of S^k_h. That $|C_j| = \delta_j$ for $1 \leqslant j \leqslant k$ we establish by induction on the construction of Sp_j from Ax_j. The basis is provided by ∇i) according to which for some $A_1, \dots, A_i$, $|A_1| = \alpha_1, \dots, |A_i| = \alpha_i$.

Suppose the construction of Sp_j has reached the stage where $C^i_j \beta_1 \dots \beta_i$ is replaced by $\{a{:}S^i_j a\beta_1 \dots \beta_i\}$. By induction hypothesis there are wff $D_1, \dots, D_i$ such that $|D_1| = \beta_1, \dots, |D_i| = \beta_i$. It suffices to show that

$$|C^i_j D_1 \dots D_i| = \{a{:}S^i_j a\beta_1 \dots \beta_i\} \qquad (\beta)$$

Suppose first $C^i_j D_1 \dots D_i \varepsilon \alpha$. Then (strictly) $C^i_j D_1 \dots D_i \varepsilon a \& |D_1| = \beta_1 \& \dots \& |D_i| = \beta_i$, whence $S^i_j a\beta_1 \dots \beta_i$ by definition of S^i_j. For the converse, suppose that for some wff $B_1, \dots, B_i$, $C^i_j D_1 \dots D_i \varepsilon a \& |B_1| = \beta_1 \& \dots \& |B_i| = \beta_i$. Since then $|D_1| = = |B_1|, \dots, |D_i| = |B_i|$, for LC and LE systems, $\vdash D_1 \leftrightarrow B_1, \dots, \vdash D_i \leftrightarrow B_i$. Hence $\vdash C^i_j D_1 \dots D_i \rightarrow C^i_j B_1 \dots B_i$ by R2 (or $\vdash C^i_j D_i \dots D_i \leftrightarrow C^i_j B_1 \dots B_i$ by R2[1]). Thus $C^i_j B_1 \dots B_i \varepsilon a$. For L logics it follows that B_k is the same wff as D_k for each k, $1 \leqslant k \leqslant i$. Hence again $C^i_j B_1 \dots B_i \varepsilon a$. Thus ($\beta$) is established.

Consider next the j^{th} rule scheme, R_j: $A_1, \dots, A_m \rightarrow B$. Then RS_j is: if $0 \varepsilon \alpha_1, \dots, 0 \varepsilon \alpha_m$ then $0 \varepsilon \beta$ where $\alpha_1, \dots, \alpha_m, \beta$ correspond to $A_1, \dots, A_m, B$. Then the proof of (β) above establishes that $\alpha_1 = |A_1|, \dots, \alpha_m = |A_m|$, $\beta = |B|$; and generally that for each formula scheme $E(A_1, \dots, A_k)$ with components $A_1, \dots, A_k$, if $|A_i| = \alpha_i$ for $1 \leqslant i \leqslant k$ and semantical scheme $sp(\alpha_1, \ldots, \alpha_k)$ corresponds to E, then $sp = |E|$. Now suppose $0 \varepsilon \alpha_1, \ldots, 0 \varepsilon \alpha_m$. Then $0 \varepsilon |A_1|, \ldots 0 \varepsilon |A_m|$, so $I(A, 0) = T, \ldots, I(A_m, 0) = T$, i.e. $\vdash A_1, \ldots, \vdash A_m$. Hence by $R_j \vdash B$, whence $I(B, 0) = T$; so $0 \varepsilon |B|$. Hence $0 \varepsilon \beta$ as required.

Theorem 2. (i) For any structural logic L, i.e. LC or LE, and any wff A of L, if A is L-valid then A is a theorem of L.

(ii) For any sentential logic L and any wff A, if A is L-valid then $\vdash_L A$.

Proof. Suppose A is not a theorem of L. Then A ε L. Hence in the canonical L model **M**, $I(A, L) \neq T$. Hence A is not L-valid.

Corollaries. 1. A is L-valid iff A is a theorem of L. Thus every sentential logic is semantically complete; but not every such logic will be consistent.

2. There is an effective procedure for writing down a semantics for any effectively given sentential logic.

Conversely, for any semantical system which can be transformed to an effectively given canonical form (that of an L model), there is an effective method of specifying an axiomatisation of L.

3. Every many-valued sentential logic can be reexpressed as a two-valued intensional logic. (This proves a conjecture of Scott [12].)

4. The thesis of extensionality holds, at least at the sentential level, in the following form: every intensional logic has a strong translation (in the sense explicated, e.g., by Bressan [9]) into an extensional logic.

Completeness can be proved alternatively by algebraic means, along the following lines: Suppose A is not a theorem of sentential logic L. Then A is falsified in a Lindenbaum algebra $\mathbf{K} = \langle Wff, 0, h \rangle$, with wff as values, with the class of theorems 0 of L as designated elements, and with h a function which maps each connective of L onto the corresponding operation on the elements of the algebra. Define a model structure $\mathbf{K}^+$ on **K**, e.g. with 0 as 0, $K \subseteq P(Wff)$, and ∇ that subset of P(K) determined by L; prove a general representation theorem; and finally define a valuation v in K^+ which falsifies A.

The details of this construction, which we reserve for another occasion, are a fairly direct generalisation of the standard methods, applied, e.g. by Lemmon [4] for a class of modal logics, and by Meyer and Routley [5] for a band of entailment logics.

The main result reveals, very clearly, that there are various general theories awaiting investigation, one of the more important being the necessary and/or sufficient (syntactical and semantical) conditions for the transformation of the uniform semantics for a sentential logic into some simpler or familiar canonical form, e.g. for the elimination of ∇ from the model structure, for the recovery of classical connectives, for the replacement of a relation **R** on $K \times \nabla \times \nabla$ by a relation R defined on $K \times K \times K$ as in Kripke-style semantics for modal logics, and so on. But what little we do know about this general theory of semantics we also reserve for another occasion.

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