SOLE AXIOMS FOR PARTIALLY ORDERED SETS

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This paper deals with an aspect of the aesthetics of logic and mathematics which has barely been touched; namely, the construction of sole axioms for mathematical systems. Everyone is aware of sole axioms for certain propositional calculi, but beyond this little has been done. Leśniewski has produced sole axioms for group theory and commutative group theory. He, as well as Sobociński, Lejewski, and Grzegorczyk, has given sole axioms for certain of his logical systems These, plus a few other scattered results, constitute the totality of work done in the field of sole axioms.

Of independent interest is the new short formulation of supremum (see propcsition **S**).

Of course, if a system is axiomatized, one can construct a sole axiom by conjoining the axioms, but this is hardly aesthetical. Leśniewski therefore, set out certain criteria for a "good" sole axiom. Some he terms "essential" and others "desirable". He also gives criteria for stating in a given system, that one sole axiom is "better" than another (1). These are nicely set forth in [3]. We shall state here the two "essential" criteria which we we require for this paper:

I. The axiom shall contain only one term, necessarily primitive, of the theory. (It will, of course, contain terms of the logic and/or mathematics on which the theory is based.)

II. The axiom shall have the form of a definition of the primitive term in the sense that the scope of the main quantifier possesses the following properties:

1) The main connective is equivalence ($\equiv$).

2) Only the primitive term and variables occur to the left of the main connective.

(1) The most basic is "the shorter the better".

3 The variables which occur to the left of the main connective are those which occur in the main quantifier.

4) No variable occurs more than once to the left of the main connective.

A desirable property is organicity, that is, no sub-formula of the axiom, when closed by quantification, is a thesis of the system. The following are examples of sole axioms for the theories of equivalence and partial ordering respectively. They are both organic.

E. $[AB]:. A \sim B. \equiv :[C]:B \sim C. \equiv .A \sim C.$

P. $[AB]:. A \leqslant B. \equiv :A. \supset .A = B:[C]:B \leqslant C. \supset .A \leqslant C.$

As always in dealing with an axiom system, the question of preaxiomatic assumptions arises. In the hope that some mathematicians will also read this paper, I will make such assumptions as are natural to mathematicians.

α) Classical logic.

β) Properties of identity.

We have given a set U so that

γ) $\leqslant$ (our primitive term) is a relation in U.

δ) The quantifiers are restricted to range over U.

η) A fragment of set theory (Enough to make γ and δ meaningful).

Since some logicans may find γ and δ distasteful, I will also state some of the axioms without using them. Such axioms will be designated by a prime, e.g.,

P.′ $[AB]:. A \leqslant B. \equiv :A \in U.B \in U:B \leqslant A. \supset .A = B:[C]:B \leqslant C. \supset .A \leqslant C.$ (2)

Lastly, since I am of the Leśniewski school, I will give some of the axioms based on his ontology, as opposed to α, β, and η. These will be designated by an asterisk, e.g.

P.* $[AB]:. A \leqslant B. \equiv :A \varepsilon A.B \varepsilon B:B \leqslant A. \supset .A \varepsilon B:[C]:B \leqslant C. \supset .A \leqslant C$ (3).

(2) $\in$ is the set-theoretic membership relation.

(3) In axioms designated by an asterisk, ε refers to the primitive term of Leśniewski's ontology.

P is inferentially equivalent to **p**1 through **p**3 ($\{\mathbf{P}\} \Longleftrightarrow \{\mathbf{p}1, \mathbf{p}2, \mathbf{p}3\}$)

p1. $[A].A \leqslant A.$

p2. $[AB]:A \leqslant B.B \leqslant A.\supset.A = B.$

p3. $[ABC]:A \leqslant B.B \leqslant C.\supset.A \leqslant C.$

$\{\mathbf{P}\} \Rightarrow \{\mathbf{p}1, \mathbf{p}2, \mathbf{p}3\}$

P1. $[A].A \leqslant A$

1) $[A]:.A \leqslant A.\equiv:A \leqslant A.\supset.A = A:[C]:A \leqslant C.\supset.A \leqslant C$ $[\mathbf{P},B|A]$

2) $[A]:.A \leqslant A.\equiv:A \leqslant A.\supset.A = A$ $[1,p \supset p]$

$[A].A \leqslant A$ $[2,A = A]$

P2. $[AB]:A \leqslant B.B \leqslant A.\supset.A = B.$

Hyp$\supset$:

3) $B \leqslant A.\supset.A = B$: $[\mathbf{P},1]$

$A = B$ $[3,2]$

P3. $[ABC]:A \leqslant B.B \leqslant C.\supset.A \leqslant C$

Hyp. $\supset$:

3) $[D]:B \leqslant D.\supset.A \leqslant D$: $[\mathbf{P},1]$

$A \leqslant C$ $[3\ D/C,2]$

$\{\mathbf{p}1, \mathbf{p}2, \mathbf{p}3\} \Rightarrow \{\mathbf{P}\}$.

p4. $[AB]:.A \leqslant B.\supset:B \leqslant A.\supset.A = B$ [**p**2]

p5. $[AB]:.A \leqslant B.\supset:[C]:B \leqslant C.\supset.A \leqslant C$ [**p**3]

p6. $[AB]:.[C]:B \leqslant C.\supset.A \leqslant C:\supset.A \leqslant B$ [C/B,**p**1]

p7. **P** [**p**4, **p**5, **p**6]

$\therefore$ $\{\mathbf{P}\} \Longleftrightarrow \{\mathbf{p}1, \mathbf{p}2, \mathbf{p}3\}$.

Note. $\{[AB]:.A \leqslant B.\equiv:B \leqslant A.\supset.A = B.\} \Longleftrightarrow \{\mathbf{p}1, \mathbf{p}2\}$ and $\{[AB]:.A \leqslant B.\equiv:[C]:B \leqslant C.\supset.A \leqslant C.\} \Longleftrightarrow \{\mathbf{p}1, \mathbf{p}3\}$

P′ is inferentially equivalent to **p**′0 through **p**′3.

p′0. $[AB]:A \leqslant B.\supset.A \in U.B \in U.$

p′1. $[A]:A \in U.\supset.A \leqslant A.$

p′2. $[AB]:A \leqslant B.B \leqslant A.\supset.A = B.$

p′3. $[ABC]:A \leqslant B.B \leqslant C.\supset.A \leqslant C.$

P* is inferentially equivalent to **p***0 through **p***3.

p*0. $[AB]:A \leqslant B. \supset .A\varepsilon A.B\varepsilon B.$

p*1. $[A]:A\varepsilon A. \supset .A \leqslant A.$

p*2. $[AB]:A \leqslant B.B \leqslant A. \supset .A\varepsilon B.$

p*3. $[ABC]:A \leqslant B.B \leqslant C. \supset .A \leqslant C.$

Note. Strict inequality may also be used as primitive in which case **P** may be replaced by

$$[AB]:.A<B. \equiv : \sim (A = B):[C]:B<C. \supset .A<C:A<B.$$

or

$$[AB]:.A<B. \equiv : \sim (B<A):[C]:B<C. \supset .A<C:A<B.$$

We now give a general method for constructing sole axioms for partially ordered sets which have additional properties.

THEOREM: Let φ be a property of a partially ordered set stated in terms of the primitive $\leqslant$, and using variables other than A and B.

Let Φ be given by

Φ. $[AB]:.A \leqslant B. \equiv :B \leqslant A. \supset .A = B.\varphi:[C]:B \leqslant C. \supset .A \leqslant C$

then $\{\mathbf{P},\varphi\} \Longleftrightarrow \{\Phi\}$.

Proof: $\{\Phi\} \Longrightarrow \{\mathbf{P},\varphi\}$.

Φ1. $[A].A \leqslant A$

1) $[A]:.A \leqslant A. \equiv :A \leqslant A. \supset .A = A.\varphi:[C]:A \leqslant C. \supset .A \leqslant C::$ $[\Phi,B/A]$

2) $[A]:.A \leqslant A. \equiv :A \leqslant A. \supset .\varphi::$ $[1,p \supset p;\ A = A]$

$[A].A \leqslant A$ $[2, \text{Contradiction}]$

Φ2 and Φ3. See **P2** and **P3**

Φ4. φ

1) $[AB].:A \leqslant B. \supset :B \leqslant A \supset \varphi$ $[\Phi]$

2) $[A]:.A \leqslant A. \supset :A \leqslant A \supset \varphi$ $[1,B/A]$

φ $[2,\Phi 1]$

$\{\mathbf{P},\varphi\} \Longrightarrow \{\Phi\}$ follows immediately from **P** and Logic.

Note. If φ begins with a general quantifier, a variable may be dropped from this quantifier and A (or B) substituted for the dropped variable at all its free occurrences in the scope. Call the resulting formula $\overline{\varphi}$.

Let Φ be given by:

Φ $[AB]:.A \leqslant B. \equiv :B \leqslant A. \supset .A = B:\Phi:[C]:B \leqslant C. \supset .A \leqslant C.$
then $\{\mathbf{P},\varphi\} \Longleftrightarrow \{\Phi\}$.

Note. $\Phi(\Phi)$ is never organic, since $\varphi(\varphi)$ occurs in it.

Using the above theorem, the task of finding sole axioms reduces to stating the various properties in terms of $\leqslant$ and (possibly) condensing them. For some the statements are obvious and we omit these. We do not pretend to be all-inclusive in what follows.

THE EXISTENCE OF THE JOIN (SUPREMUM) OF TWO ELEMENTS

Formally this is stated as:

s. $[DE][\exists F]:.D \leqslant F.E \leqslant F:[G]:D \leqslant G.E. \leqslant G. \supset .F \leqslant G$
which we shall prove is inferentially equivalent in a partially ordered set to

S. $[DE][\exists F][G]:D \leqslant G.E \leqslant G. \equiv .F \leqslant G$

Proof of $\{\mathbf{P},\mathbf{s}\} \Longrightarrow \{\mathbf{S}\}$

$[DE][\exists F]:.$

1) $D \leqslant F.$	
2) $E \leqslant F:$	[**S**]
3) $[G]:D \leqslant G.E \leqslant G. \supset .F \leqslant G:$	
4) $[G]:F \leqslant G. \supset .D \leqslant G:$	[**P**3,1]
5) $[G]:F \leqslant G. \supset .E \leqslant G:$	[**P**3,2]
$[G]:D \leqslant G.E \leqslant G. \equiv .F \leqslant G$	[3;4,5]

Proof of $\{\mathbf{P},\mathbf{S}\} \Longrightarrow \{\mathbf{s}\}$

$[DE][\exists F]:.$

1) $[G]:D \leqslant G.E \leqslant G. \equiv .F \leqslant G:$	[**S**]
2) $D \leqslant F.E \leqslant F. \equiv .F \leqslant F:$	[1,G/F]
3) $D \leqslant F.E \leqslant F.$	[**P**1,2]
$D \leqslant F.E \leqslant F:[G]:D \leqslant G.E \leqslant G. \supset .F \leqslant G$	[3,1]

THE EXISTENCE OF THE MEET (INFIMUM) OF TWO ELEMENTS

I. $[DE][\exists F]:[G]:G \leqslant D.G \leqslant E.\equiv.G \leqslant F.$ (Proof analogous to that for **S**.)

For a lattice, we conjoin **S** and **I** to get

L. $[DE]:[\exists F]:[G]:D \leqslant G.E \leqslant G.\equiv.F \leqslant G:.[\exists H]:[G]:G \leqslant D.G \leqslant E.\equiv:G \leqslant H$

THE DISTRIBUTIVE LAW

From [1], we know that in $\{\mathbf{P},\mathbf{L}\}$, a lattice, the distributive law $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ is equivalent to

d. $[ABCDE]:A \leqslant B.B \leqslant C.B \cap D = A.B \cap E = A.B \cup D = C.B \cup E = C.\supset.D = E$

which we shall prove is equivalent to

D. $[ABC]:\because:[G]: :G \leqslant A.\supset:G \leqslant B.\equiv.G \leqslant C:.A \leqslant G.\supset:B \leqslant G.\equiv.C \leqslant G: :\supset.B = C.$

We give some preliminary lemmas.

P4. $[BC]:.[A]:B \leqslant A.\equiv.C \leqslant A:\supset.B = C$

Hyp.$\supset$.

2) $C \leqslant B.$ [1,A/B,**P**1]

3) $B \leqslant C.$ [1,A/C,**P**1]

$B = C$ [**P**2,3,2]

P5. $[ABDE]: :[G]:D \leqslant G.E \leqslant G.\equiv.A \leqslant G:[G]:D \leqslant G.E \leqslant G.\equiv.B \leqslant G:.\supset.A = B.$

Hyp $\supset$:

3) $[G]:A \leqslant G.\equiv.B \leqslant G:$ [1,2]

$A = B$ [**P**4,3]

P5 assures us that in a partially ordered set we may use the following definitional form to define join.

DP1. $[ABC]:.A = B \cup C.\equiv:[G]:B \leqslant G.C \leqslant G.\equiv.A \leqslant G.$

Note. The pre-axiomatic assumption δ assures that $A \in \cup$.

We now work in $\{\mathbf{P},\mathbf{L}\}$ (4)

L1. $[ABG]{:}A\leqslant G.B\leqslant G.\equiv.A\cup B\leqslant G$ [**DP1** $A/B\cup C$,**L**]

Note. L is required because pre-axiomatic assumption δ requires that only objects in U be substituted for variables. Since A and B occur in the quantifier, they are in U, L then assures us that $A\cup B\varepsilon U$.

L2. $[ABC]{::}A\cup C = B\cup C.\equiv{:}.[G]{:}.C\leqslant G.\supset{:}A\leqslant G.\equiv.B\leqslant G.$

1) $A\cup C = B\cup C.\equiv{:}[G]{:}B\leqslant G.C\leqslant G.\equiv.A\cup C\leqslant G{:}$ [**DP1**]

2) $\equiv{:}[G]{:}B\leqslant G.C\leqslant G.\equiv.A\leqslant G.C\leqslant G{:}.$ (1,**L1** B/C]

$\equiv{:}.[G]{:}.C\leqslant G.\supset{:}B\leqslant G.\equiv.A\leqslant G$ [2]

Similarly we may define meet

DP2. $[ABC]{:}.A = B\cap C.\equiv{:}[G]{:}G\leqslant B.G\leqslant C.\equiv.G\leqslant A.$

and arrive at

L3. $[ABC]{:}\ {:}A\cap C = B\cap C.\equiv{:}.[G]\leqslant C.\supset{:}G\leqslant A.\equiv.G\leqslant B.$

L4. $\mathbf{d}\equiv\mathbf{D}$

1) $[ABCDE]{:}A\leqslant B.B\leqslant C.B\cap D = A.B\cap E = A.B\cup D = C.B\cup E = C.\supset.D = E{:}.\equiv{:}.$

2) $[BDE]{:}[\exists AC].A\leqslant B.B\leqslant C.B\cap D = A.B\cap E = A.B\cup D = C.B\cup E = C.\supset.D = E{:}.\equiv{:}.$ [1]

3) $[BDE]{:}B\cap D = B\cap E.B\cup D = B\cup E.\supset.D = E{:}.\equiv{:}\because$ [2,**L**,$A/B\cap D$,$C/B\cup D$;2]

4) $[BDE]{:}\because[G]{:}.G\leqslant B.\supset{:}G\leqslant D.\equiv.G\leqslant E{:}.[G]{:}.B\leqslant G.\supset{:}D\leqslant G.\equiv.E\leqslant G{::}\supset.D = E{::}\equiv{:}\because$ [3,**L2**,**L3**]

$[BDE]{:}\because[G]{::}G\leqslant B.\supset{:}G\leqslant D.\equiv.G\leqslant E{:}.B\leqslant G.\supset{:}D\leqslant G.\equiv.E\leqslant G{::}\supset.D = E.$ [4]

Thus in a lattice, **D** is equivalent to the distributive law· However, **D** is meaningful in any partially ordered set so that it enables us to speak of distributive partially ordered sets (5) if we so desire.

(4) We shall use basic lattice properties without proof and refer to any of them by L.

(5) This is more general than the concept of distributive latticoid in the sense that "oid" is used in [2].

THE MODULAR LAW

From [1], we know that in a lattice the modular law,

$a \leqslant c . \supset . a \cup (b \cap c) = (a \cup b) \cap c$, is equivalent to

m. $[ABC]: A \cap B = A \cap C . A \cup B = A \cup C . \supset . \sim B < C$ which, by means of **L2** and **L3**, is easily proved to equivalent to,

M. $[ABC] \because [G] :: G \leqslant A . \supset : G \leqslant B . \equiv . G \leqslant C :. A \leqslant G . \supset : B \leqslant G . \equiv . C \leqslant G :: B \leqslant C :: \supset . B = C$.

As in the case of **D**, **M** enables us to speak of modular partially ordered sets.

THE SEMI-MODULAR LAW

The semi-modular law in [1],

sm. [xya]: x covers a.y covers a. $\sim (x = y) . \supset . x \cup y$ covers $x . x \cup y$ covers y, is trivially equivalent to,
[xyza]:. $z = x \vee z = y . \supset . z$ covers $a : \sim (x = y) : \supset . [\exists b] . b$ covers x. b covers y, which translates directly into,

SM. $([ABC] :: [G] : G = B \vee G = C . \supset A \leqslant G . \sim (G = A) . [Z] . \sim (Z \geqslant G . \sim (G = Z) . A \leqslant Z . \sim (Z = A)) :. \sim (B = C) :. \supset : [\exists D] : [G] : G = B \vee G = C . \supset . G \leqslant D . \sim (G = D) . [Z] \sim (Z \leqslant D . \sim (D = Z)) . G \leqslant Z \sim (Z = G))$.

COMPLEMENT

The existence of complement can be given by.

EXC. $[A][\exists E] :: [C] : C \leqslant A . C \leqslant E . \supset . [D] . C \leqslant D :. [C] : A \leqslant C . E \leqslant C . \supset . [D] . D \leqslant C$.

Note. This is meaningful even if there is no zero or one.

The existence and uniqueness of complement can be given by

C. $[AB] \because A = B . \equiv :: [\exists E] :: [C] :. C \leqslant E : C \leqslant A \vee C \leqslant B : \supset . [D] . C \leqslant D :. [C] :. E \leqslant C : A \leqslant C \vee B \leqslant C : \supset . [D] . D \leqslant C$.

If we wish to speak of complete lattices we may alter our pre-axiomatic assumptions as follows:

δ′) Upper case letters in quantifiers are restricted to range over U. Lower case letters in quantifiers are restricted to range over subsets of U.

η′) Enough set theory to make γ and δ′ meaningful.

Then analogous to the supremum and infimum for two elements we have:

SA. $[a][\exists F][G]:.[D]:D \in a. \supset .D \leqslant G: \equiv .F \leqslant G$ and

A. $[a][\exists F][G]:.[D]:D \in a. \supset .G \leqslant D: \equiv .G \leqslant F.$

This concludes the section dealing with construction of statements for use in the THEOREM.

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We now give a sole axiom for complete boolean algebra, **B**. In the study of the relation of Leśniewski's mereology to boolean algebra (6) one needs the notion of a "complete boolean algebra with zero deleted", **BD**. To this end we introduce a "complete boolean algebra with or without zero deleted", **W**. To be specific $\{\mathbf{B}\} \Longleftrightarrow \{\mathbf{W}, [\exists A][B].A \leqslant B\}$. If one deletes the zero from a boolean algebra the resulting system is without a minimum element except in the case of a boolean algebra with exactly two elements. To allow for this exceptional case we must use the non-existence of zero under the hypothesis of the existence of at least two elements in the deleted system, i.e., $\{\mathbf{DB}\} \Longleftrightarrow \{\mathbf{W},[\exists AB].\sim A = B: \supset .[C][\exists D].\sim C \leqslant D\}$.

According to Tarski, [4] p. 330, Theorem 2, the following is an axiom system for complete boolean algebra (7).

T1. $[ABC]:A \leqslant B.B \leqslant C. \supset .A \leqslant C.$

T2. $[a][\exists A]::[D]:D \in a. \supset .D \leqslant A::[D]::D \leqslant A:[EF]:E \in a.F \leqslant D.F \leqslant E. \supset .[G].F \leqslant G:. \supset .[H].D \leqslant H.$

(6) A relation is noted in [4] p. 333, footnote. The statement there is inexact in the sense that it ignores the logical base which Leśniewski specifically constructed for his mereology.

(7) This includes the boolean algebra consisting of a single element.

T3. $[ABa]:::[D]:D \in a. \supset .D \leqslant A::[D]::D \leqslant A:[EF]:E \in a.F \leqslant D.F \leqslant G. \supset .[G].F \leqslant G:. \supset .[H].D \leqslant H:\cdot:[D]:D \in a. \supset .D \leqslant B::[D]:: D \leqslant B:[EF]:E \in a.F \leqslant D.F \leqslant E. \supset .[G].F \leqslant G:. \supset .[H].D \leqslant H:\cdot: \supset .A = B$

We shall prove this system is equivalent to

B. $[AB::\cdot:A \leqslant B. \equiv :::[K]:B \leqslant K. \supset .A \leqslant K:::B \leqslant B. \supset :\cdot:[ab]:\cdot: C \in b. \equiv :[D]:D \in a \supset .D \leqslant C:[DH]:D \leqslant C. \sim D \leqslant H. \supset . [\exists EFG].E \in a.F \leqslant D.F \leqslant E. \sim F \leqslant :: \supset .[\exists L].b = \{L\}$.

First we give a formula depending only on the pre-axiomatic assumptions:

F. $[D]::D \leqslant A:[EF]:E \in a.F \leqslant D.F \leqslant E. \supset .[G].F \leqslant G:. \supset .[H]. D \leqslant H:\cdot: \equiv :.[DH]:D \leqslant A. \sim D \leqslant H. \supset .[\exists EFG].E \in a.F \leqslant D.F \leqslant E. \sim F \leqslant G.$

B1. $[A].A \leqslant A$ [**B**,B/A]

B2. $[ABC]:A \leqslant B.B \leqslant C. \supset .A \leqslant C.$ (**T**1) [**B**,K|C]

DB1. $[Aa]:.A\varepsilon \text{ sup}(a). \equiv :[D]:D \in a. \supset .D \leqslant A:[DH]:D \leqslant A. \sim D \leqslant H. \supset .[\exists EFG].E \in a.F \leqslant D.F \leqslant E. \sim F \leqslant G$

B3. $[a][\exists A]. \text{ sup } (a) = \{A\}$ [**B**,B/A,b/sup(a),**B**1,**B**1,**DB**1]

B4. $[a][\exists A].A \in \text{ sup } (a)$ [**B**3]

B5. **T**2 [**B**4,**F**,**DB**1]

B6. $[ABa]:A \in \text{ sup } (a).B \in \text{ sup } (a). \supset .A = B$ [**B**3]

B7. **T**3 [**B**6, **DB**1, **F**]

$\therefore$ $\{\mathbf{B}\} \Longrightarrow \{\mathbf{T}1,\mathbf{T}2,\mathbf{T}3\}$

T4. $[K]:B \leqslant K. \supset .A \leqslant K: \supset .A = B$ [K/B,**P**1]

DT1. **DB**1

T5. $[a][\exists A].A \in \text{ sup } (a)$ [**T**2,**F**,**DT**1]

T6. $[ABa]:A \in \text{ sup } (a).B\varepsilon \text{ sup } (a). \supset .A = B$ [**T**3,**F**,**DT**1]

T7. $[a][\exists A]. \text{ sup } (a) = \{A\}$ [**T**5,**T**6]

T8. $[ab:\cdot:[C]:.C \in b. \equiv :[D]:D \in a. \supset .D \leqslant C:[DH]:D \leqslant C. \sim D \leqslant H. \supset .[\exists EFG].E \in a.F \leqslant D.F \leqslant E. \sim G:: \supset .[\exists L].b = \{L\}$

Hyp.$\supset$.

2) $b = \text{sup}(a)$. [**DT**1,1]

$[\exists L].b = \{L\}$ [**T**7,2]

T9. **B** [**T**1,**T**8,**T**4]

:. $\{\mathbf{T}1,\mathbf{T}2,\mathbf{T}3\} \Longrightarrow \{\mathbf{B}\}$:. $\{\mathbf{B}\} \Longleftrightarrow \{\mathbf{T}1,\mathbf{T}2,\mathbf{T}3\}$.

Next we consider the following two statements

W. $[AB]::\cdot:A\leqslant B.\equiv:::B\leqslant B.\supset:\cdot:[ab]:\cdot:B\in a:[C]:.C\in b.\equiv:[D]:D\in a.\supset.D\leqslant C:[DH]:D\leqslant C.\sim D\leqslant H.\supset.[\exists EFG].E\in a.F\leqslant D.F\leqslant E.\sim F\leqslant G::\supset.[\exists L].b=\{L\}.A\leqslant L.$ (8)

Z. $[\exists A]:[B].A\leqslant B.$

We shall show that $\{\mathbf{W},\mathbf{Z}\} \Longleftrightarrow \{\mathbf{B}\}$.

B10. $[ABab]:\cdot:A\leqslant B.B\in a:.[C]:.C\in b.\equiv:[D]:D\in a.\supset.D\leqslant C:[DH]:D\leqslant C.\sim D\leqslant H.\supset.[\exists EFG].E\in a.F\leqslant D.F\leqslant E.\sim F\leqslant G::\supset.[\exists L].b=\{L\}.A\leqslant L.$

Hyp. $\supset::[\exists L]::$

4)	$b=\{L\}:.$	[**B**,1,3]
5)	$[C]:.C\in\{L\}.\supset:[D]:D\in a.\supset.D\leqslant C:.$	[3,4]
6)	$[D]:D\in a.\supset.D\leqslant L:$	[5,C/L]
7)	$B\leqslant L$	[6,2]
8)	$A\leqslant L.$	[**B**2,1,7]
	$[\exists L].b=\{L\}.A\leqslant L.$	[4,8]

B11. $[AB]::\cdot:[ab]:\cdot:B\varepsilon a:.[C]:.C\varepsilon b.\equiv:[D]:D\varepsilon a.\supset.D\leqslant C:[DH]:D\leqslant C.\sim D\leqslant H.\supset.[\exists EFG].E\varepsilon a.F\leqslant D.F\leqslant E.\sim F\leqslant G::\supset.[\exists L].b=\{L\}.A\leqslant L:\cdot:\supset.A\leqslant B.$

Hyp. $\supset:\cdot:$

2) $[C]:.C=B.\equiv:[D]:D=B.\supset.D\leqslant C:[DH]:D\leqslant C.\sim D\leqslant H.\supset.[\exists FG].F\leqslant D.F\leqslant B.\sim F\leqslant G::\supset.A\leqslant B::$ [1 a|{B},b|{B},E|B]

$A\leqslant B$ [2 C|B,F|D,G|H,**B**1]

B12. **W** [**B**10,**B**11,**B**1]

Since **B** *is* an axiom for complete boolean algebra, **Z** must follow from it. Therefore $\{\mathbf{B}\} \Longrightarrow \{\mathbf{W},\mathbf{Z}\}$.

W1. $[A],A\leqslant A.$ [**W**,B/A]

(8) This formula is based on Sobociński's single axiom for Leśniewski's mereology.

W2. $[AB] \therefore [K]:B \leqslant K. \supset .A \leqslant K: \supset .A \leqslant B$ [**W1**]

DW1. DB1

W3. $[ABa]:A \leqslant B \cdot B \in a. \supset .[\exists L]:\sup(a) = \{L\}.A \leqslant L.$
[**W**,b|sup(a),**W1**, **DW1**]

W4. $[Aa]:A \in a..[\exists L].\sup(a)\ \{L\}$ [**W**,B/A;**W1**]

DW2. $[AB]:A \in 1b(B). \equiv .A \equiv .A \leqslant B$

W5. $[A].A \in \sup(1b(A))$ [**DW1**,a|1b(a),C/A,E|D,F|D,
G|H,**DW2**,**W1**]

W6. $[A].\{A\} = \sup(1b(A))$

1) $A\varepsilon 1b(A).$ [**DW2**,**W1**]
$[\exists L].$
2) $\sup(1b(A)) = \{L\}.$ [**W4**,1]
3) $A \in \{L\}.$ [**W5**,2]
$\sup(1b(A)) = \{A\}$ [2,3]

W7. $[ABC]:A \leqslant B.B \leqslant C. \supset .A \leqslant C.$

Hyp.$\supset$.

3) $B \in 1b(C).$ [**DW1**,2]
$[\exists L].$
4) $\sup(1b(C)) = \{L\}$ [**W3**,1,3]
5) $A \leqslant L$
6) $L = C$ [**W6**,4]
$A \leqslant C$ [5,6]

W8. $[AB]:A \leqslant B.B \leqslant A. \supset .A = B$

Hyp $\supset$:

3) $[D]:D \leqslant A. \equiv .D \leqslant B:$ [**W7**,1;**W7**,2]
4) $1b(A) = 1b(B).$ [**DW2**,3]
5) $\{A\} = \{B\}$ [**W6**,4]
$A = B$ [5]

W9. $[AB]:.[C].A \leqslant C.[D].B \leqslant D: \supset .A = B$ [C/B,D/A,**W8**]

Therefore the following definitional form is valid:

BW3. $[A]:A = \mathbf{O}. \equiv .[B].A \leqslant B.$

W10. $\{\mathbf{O}\} = \sup(\varnothing)$

1) $[C]C \in \sup(\varnothing). \equiv .[DH].\sim(D \leqslant C. \sim D \leqslant H):$ [**DW1**]

2) [C]:.C$\in$ sup($\varnothing$).$\equiv$:[D]:D$\leqslant$C.$\supset$.[H].D$\leqslant$H:. [1]
3) [C]:.C$\in$ sup($\varnothing$).$\equiv$:[D]:D$\leqslant$C.$\supset$.D = **O**: [**DW**3,2]
4) [C]:.C$\in$ sup($\varnothing$).$\equiv$.C = **O**: [3,**W**1;**W**8,**DW**3]
{**O**} = sup($\varnothing$) [5,**Z**]

Note. **Z** is required for the last step, since due to δ', we may not substitute **O** for C until we know **O**$\in$U.

W11. [a][$\exists$L].sup(a) = {L} [**BW**4,**BW**10]

W12. **B** [**W**7;**W**1,**DW**1,**W**11,W2]

Therefore {**B**} $\Longleftrightarrow$ {**BW**,**Z**}.

Next we wish to prove that {**BW**,**ZD**} $\Longleftrightarrow$ {**BD**} for

DB. [AB]::·:A$\leqslant$B.$\equiv$:::B$\leqslant$B.$\supset$:·:[ab]:·:B$\in$a:[C]:.C$\in$b.$\equiv$: [D]:D$\in$a.$\supset$.D$\leqslant$C:[D]:D$\leqslant$C.$\supset$[$\exists$F].E$\in$a.F$\leqslant$D.F$\leqslant$ E::$\supset$.[$\exists$L].b = {L}.A$\leqslant$L and

ZD. [$\exists$AB].$\sim$A = B:$\supset$:[C].[$\exists$D].$\sim$C$\leqslant$D
Using only the pre-axiomatic assumptions we have

X. [L]:L$\leqslant$L.$\supset$.Lε{L}.L$\leqslant$L.L$\leqslant$L

P6. [AB].A = B:$\supset$:[CDE]:C$\leqslant$C.$\equiv$.D$\leqslant$E. [**P**1]

P7. [AB].A = B:$\supset$:[Ca]:[D]:D$\leqslant$C.$\supset$.[$\exists$EF]·E$\in$a·F$\leqslant$ D.F$\leqslant$E. [**D**6,**X**]

P8. [AB].A = B:$\supset$.[DH].D$\leqslant$H. [**P**1]

P9. [AB].A = B:$\supset$:[Ca]:[DH]:D$\leqslant$C.$\sim$D$\leqslant$H.$\supset$.[$\exists$EFG]. E$\in$a.F$\leqslant$D.F$\leqslant$E.$\sim$F$\leqslant$G [**P**8]

P10. [AB].A = B:$\supset$:W.$\equiv$.**BD**. [**P**7,**P**9]

P11. [AB].A = B:$\supset$.**ZD**

P12. [AB].A = B:$\supset$:**W**.**ZD**.$\equiv$.**BD** [**P**10,**P**11]

BD1. [A].A $\leqslant$A [**BD**,B/A]

DBD1. [Aa]:.A$\in$ sup(a).$\equiv$:[D]:D$\in$a.$\supset$.D$\leqslant$A:[D];D$\leqslant$A.$\supset$. [$\exists$EF].E$\in$a.F$\leqslant$D.F$\leqslant$E

BD2. [Aa]:A$\in$a.$\supset$·[$\exists$L].sup(a) = {L}. [**BD**,B/A,b|sup(a), **BD**1,**DBD**1]

BD3. [A]:A$\in$ sup({A}) **DBD**,a|{A},E|A,F|**D**,**BD**1]

BD4. $[AC]:.[G].C \leqslant G : \supset . A \in \sup(\{C\})$ [**DBD**1,a|{C},E|C,F|C]

BD5. $[ABC]:.[G].C \leqslant G : \supset . A = B$

Hyp $\supset$.

2) $A \in \sup(\{C\})$. [**BD**4,1]
3) $B \varepsilon \sup(\{C\})$. [**BD**4,1]
4) $[\exists L].\sup(\{C\}) = \{L\}$. [**BD**2,a|{C}]
$A = B$ [4,2,3]

BD6. $[ABD]. \sim A = B. \supset .[\exists C]. \sim D \leqslant C$ [**BD**5]

BD7. **ZD** [**BD**6]

Note now that in both {**W**,**ZD**} and {**BD**}, if we are under the hypothesis $[\exists AB]. \sim A = B$, the statements $[\exists H] \sim D \leqslant H$ and $[\exists G]. \sim F \leqslant G$ are true, so that as conjuncts their insertion or removal yields equivalent statements. Thus in both systems we have **DBD**1 $\equiv$ **DW**1 and **W** $\equiv$ **BD**. Therefore

$[\exists AB]. \sim A = B : \supset :$ **W**.**ZD**.$\equiv$.**BD**.

From this and P12 we have {**W**,**ZD**} $\Longleftrightarrow$ {**BD**}. That is, **BD** is a sole axiom for a complete boolean algebra with zero deleted.

BD′. $[AB]:::A \leqslant B. \equiv :::A \in U.B \in U:::B \leqslant B. \supset :\cdot:[ab]:\cdot:a \subset U.b \subset U.B \in a:[C]:.C \in b. \equiv :[D]:D \in a. \supset .D \leqslant C:[D]:D \leqslant C. \supset .[\exists EF].E \in a.F \leqslant D.F \leqslant E:: \supset .[\exists L].b = \{L\}.A \leqslant L.$

BD*. $[AB]:::A \leqslant B. \equiv :::A \varepsilon A.B \varepsilon B:::B \leqslant B. \supset :\cdot:[ab]:\cdot:B \in a:[C]:.C \varepsilon b. \equiv :[D]:D \in a \supset .D \leqslant C:[D]:D \leqslant C. \supset .[\exists EF].E \in a.F \leqslant D.F \leqslant E:: \supset .A \leqslant b.$

It is of interest to note that organicity is a function of the pre-axiomatic assumptions, since **BD** is not organic, but **BD**′ and **BD*** are.

We close with a result due to V. F. Rickey, namely, that the following is a sole axiom for a totally ordered set.

$$[AB]::A \leqslant B. \equiv :.B \leqslant A. \supset .A = B:\cdot A \leqslant B. \supset :[C]:B \leqslant C. \supset .A \leqslant C.$$

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REFERENCES

[1] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Coll. Pub. XXV, New York, 1948 (Revised Edition).

[2] R. E. Clay and S. K. Sehgal, Boolean algebroids, *Notre Dame Jour. of Formal Logic*, Vol. V (1964), pp. 154-157.

[3] B. Sobociński, On well-constructed axiom systems, *Polskie Towarzystwo Naukowe Na Obczyznie* (London 1956), pp. 1-12.

[4] A. Tarski, On the foundations of boolean algebra, *Logic, Semantics, Metamathematics*, Clarendon Press, Oxford (1956), pp. 320-341.