THREE GENERALIZATIONS OF A THEOREM OF BETH'S

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Exploiting an argument of Henkin's in [4], Beth established in *The Foundations of Mathematics,* Section 89, that a set S of *closed* wffs of QC_1, the first-order quantificational calculus, is consistent in QC_1 if and only if (every member of) S is satisfied by at least one assignment of truth-values to the atomic wffs of QC_1 (2). The result has two interesting corollaries: one to the effect that a wff A of QC_1 (A possibly open) is provable in QC_1 if and only if A is satisfied by every assignment of truth-values to the atomic wffs of QC_1, the other one to the effect that a wff A of QC_1 is derivable in QC_1 from a set S of closed wffs of QC_1 if and only if A is satisfied by every assignment of truth-values to the atomic wffs of QC_1 that satisfies S.

In I I shall obtain a first generalization of Beth's Theorem, and show that a set S of wffs of QC_1 (said wffs possibly *open*) is consistent in QC_1 if and only if at least one set of wffs of QC_1 that is isomorphic to S (in a sense to be explicated below) is satisfied by at least one assignment of truth-values to the atomic wffs of QC_1. It will readily follow from the result that a wff A of QC_1 is derivable in QC_1 from a set of wffs of QC_1 if and only if no set of wffs of QC_1 that is isomorphic to $S \cup \{\sim A\}$ is satisfied by any assignment of truth-values to the atomic wffs of QC_1.Since A is implied by S or, in Tarski's language, A is a semantic consequence of S if and only if A is derivable from S in QC_1, it will further follow from my first result that A is implied by S if and only if no set of wffs of QC_1 that is isomor-

(1) The results in this paper were announced at a meeting of the Association for Symbolic Logic, hold at the University of California, Los Angeles, on March 22, 1968.

(2) As in the literature I take a set of wffs of QC_1 to be consistent in QC_1 if 'p & $\sim$p', say, is not derivable from S in QC_1; and throughout I take a set of wffs to be satisfied by a truth-value assignment (eventually, by a truth-value function) if every member of the set is.

phic to $S \cup \{\sim A\}$ is satisfied by any assignment of truth-values to the atomic wffs of QC_1. As announced in an earlier paper of mine, the notion of first-order implication (and by rebound that of first-order validity) can thus be explicated without recourse to models (3).

Passing on to QC_2, the second-order quantificational calculus, and taking a set of wffs of QC_2 to be consistent* in QC_2 if — roughly — 'p & ~ p' is not derivable from S in Henkin's fragment F* of QC_2 in [6] (4), I shall establish in II that S is consistent* in QC_2 if and only if at least one set of wffs of QC_2 that is isomorphic to S is satisfied by at least one assignment of truth-values to the atomic wffs of QC_2, or — to use a fresh terminology explicated below — if and only if at least one set of wffs of QC_2 that is isomorphic to S is satisfied by at least one truth-value function for QC_2. And, taking a set S of wffs of QC_2 to be consistent** in QC_2 if — roughly again — 'p & ~ p' is not derivable from S in Henkin's version F** of QC_2 (5), I shall establish in III that S is consistent** in QC_2 if and only if at least one set of wffs of QC_2 that is isomorphic to S is satisfied by at least one *general* truth-value function for QC_2. It will follow from these two extra generalizations of Beth's Theorem that notions of second-order implication* and second-order implication** (roughly equivalent, the first to derivability in Henkin's F*, the second to derivability in Henkin's F**) can be explicated without mention of models, and hence — in particular — that Henkin's notion of general validity, which is tantamount to second-order implication** by ø, can be ex-

(3) See [8]. I was considerably helped, when devising this new account of first-order implication by J. Hintikka, R. H. Thomason, and B. van Fraassen; for further details on the matter, see [9].

(4) Roughly, because Henkin's account of provability in F* from an arbitrary set is presumably to run mutatis mutandis Church's account in [2] of provability in F^{2p} from an arbitrary set, and hence is weaker than my account of derivability* in QC_2. Footnote 17 has further details on the matter.

(5) Roughly again, because Henkin's account of provability in F** from an arbitrary set is presumably weaker than my account of derivability** in QC_2.

plicated without mention of general models of the Henkin sort. The last result may be welcome news to those who find the notion of general validity somewhat *ad hoc*.

Professor Robert K. Meyer and myself have an account of order omega validity** (a notion tantamount to provability in the quantificational calculus of order omega), and one of order omega validity, which likewise make no mention of models. Publication of these additional results is in the offing.

I

When establishing that *if a set S of wffs of QC_1 is consistent in QC_1, at least one set of wffs of QC_1 that is isomorphic to S is satisfied by at least one assignment of truth-values to the atomic wffs of QC_1,* I consider first the case where infinitely many individual variables of QC_1 do not occur free in any member of S, then turn to the contrary case. To abridge matters, I call a set of the first sort a *wff set of Type I,* and one of the second sort a *wff set of Type II.* I also use the following substitution convention. A being a wff of QC_1, X_1, X_2, ..., and X_n ($n \geqslant 1$) being distinct individual variables of QC_1, and for each i from 1 to n Y_i being an individual variable of QC_1 not necessarily distinct from X_1, I should take $(A)[Y_1/X_1]$ to be the result of replacing every free occurrence of X_1in A by an occurrence of Y_1 if no component of A of the sort $(\forall Y_1)B$ contains a free occurrence of X_1 in A; otherwise, I shall take $(A)[Y_1/X_1]$ to be $(A')[Y_1/X_1]$, where A' is the result of replacing every occurrence of Y_1 in every component of A of the sort $(\forall Y_1)B$ that contains a free occurence of X_1 in A by an occurrence of the alphabetically earliest individual variable of QC_1 that is foreign to that component of A; and I shall take $(A)[Y_1, Y_2, \ldots, Y_n/X_1, X_2, \ldots, X_n]$ to be $((A)[Y_1, Y_2, \ldots, Y_{n-1}/X_1, X_2, \ldots, X_{n-1}])[Y_n/X_n]$.

Proof of Case 1 is as follows. (1) S being a wff set of Type I, let S_0 be S; for each i from 1 on let S_i be $S_{i-1} \cup \{A_i[Y/X_i] \supset (\forall X_i)A_i\}$, where $(\forall X_i)A_i$ is in some predetermined order the i-th wff of QC_1 of the sort $(\forall X)A$ and Y is the alphabetically first individual variable of QC_1 that does not occur free in any member of S_{i-1} and is foreign to $(\forall X_i)A_i$; and let S_∞ be the union of $S_0, S_1, S_2, \ldots$. It is easily verified that S_∞ is consistent in QC_1 if S is, and that for each i from 1 on there is an individual variable Y of QC_1 such that $A_i[Y/X_i] \supset (\forall X_i)A_i$ belongs to S_∞. (2) S^0_∞ being the set S_∞ of 1, let s^i_∞ be for each i from 1 on $S^{i-1}_\infty \cup \{A_i\}$, where A_i is in some predetermined order the i-th wff of QC_1, if $S^{i-1}_\infty \cup \{A_i\}$ is consistent in QC_1, otherwise let S^i_∞ be S^{i-1}_∞; and let S^∞_∞ be the union of $S^0_\infty, S^1_\infty, S^2_\infty$, $\ldots$. It is easily verified that S^∞_∞ is consistent in QC_1 if S_∞ is, and that, if S^∞_∞ is consistent in QC_1, then: (i) $\sim A$ belongs to S^∞_∞ if and only if A does not, (ii) $A \supset B$ belongs to S^∞_∞ if and only if A does not or B does, and (iii) $(\forall X)A$ belongs to S^∞_∞ if and only if $A[Y/X]$ does for every individual variable Y of QC_1 (6). (3) Let a wff A of QC_1 be said to be satisfied by an assignment of truth values to the atomic wffs of QC_1 if: (i) in the case that A is atomic, A is assigned the truth-value T in Asst, (ii) in the case that A is of the sort $\sim B$, B is not satisfied by Asst, (iii) in the case that A is of the sort $B \supset C$, B is not satisfied by Asst or C is, and (iv) in the case A is of the sort $(\forall X)B$, $B[Y/X]$ is satisfied by Asst for every individual variable Y of QC_1. Next, let Asst be the result of assigning the truth-value T to every atomic wff of QC_1 that belongs to the set S^∞_∞ of (2), the truth-value F to every other one. It is easily verified that, if S^∞_∞ is consistent in QC_1, then in view of (i)-(iii) in (2) a wff A of QC_1 belongs to S^∞_∞ if and only if A is satisfied by Asst (7). Hence, if S is consistent in QC_1, there is an assignment of truth-values to the atomic wffs of QC_1 that satisfies S. (4) M being a one-to-one mapping of the set of the individual variables of QC_1 into itself, A being a wff of QC_1, and $X_1, X_2, \ldots$, and X_n being in alphabetical order all the individual variables of QC_1

(6) My construction of the two sets S_∞ and S^∞_∞ is obviously reminiscent of a like construction in Henkin's [4].

(7) The proof is by mathematical induction on the number of occurrences of '$\sim$', '$\supset$', and '$\forall$' in A. '$\vee$', '&', '$\equiv$', and '$\exists$' are presumed to be defined in the customary manner.

that occur free in A, let the M-image of A be A itself when $n = 0$, otherwise let it be $A[M(X_1), M(X_2), \ldots, M(X_n)/X_1, X_2, \ldots, X_n]$. And, S and S′ being not necessarily distinct sets of wffs of QC_1, let S′ be said to be isomorphic to S if: (i) in the case that S is empty, S′ is empty as well, and (ii) in the contrary case, S′ consists — for some one-to-one mapping of the set of the individual variables of QC_1 into itself — of the M-images of the various members of S. It immediately follows from (3) that, if S is consistent in QC_1, there is an assignment of truth-values to the atomic wffs of QC_1 that satisfies at least one set of wffs of QC_1 isomorphic to S, the set in question being S itself.

Proof of Case 2 is as follows. S being a wff set of Type II, and M the one-to-one mapping of the set of the individual variables of QC_1 into itself such that, where X is the alphabetically i-th individual variable of QC_1, M(X) is the alphabetically $(2 \cdot i)$-th individual variable of QC_1, let S′ be ø if S is ø, otherwise let S′ consist of the M-images of the various members of S. It is easily verified that S′ is a wff set of Type I, is isomorphic to S, and is consistent in QC_1 if S is. Hence, in view of Case 1, if S is consistent in QC_1, there is an assignment of truth-values to the atomic wffs of QC_1 that satisfies at least one set of wffs of QC_1 isomorphic to S, said set being S′ (8).

S being a wff set of Type I or of Type II, suppose next that S is not consistent in QC_1. Then, as the reader may verify on his own, no set of wffs of QC_1 that is isomorphic to S is consistent in QC_1 either. But, as the reader may again verify on his own, a set of wffs of QC_1 is not satisfied by an assignment of truth-values to the atomic wffs of QC_1 unless it is consistent in QC_1. Hence no set of wffs of QC_1 that is isomorphic to S is satisfied by any assignment of truth-values to the atomic wffs of QC_1.

Hence, my first generalization of Beth's Theorem: *A set S of wffs of QC_1 is consistent in QC_1 if and only if at least one set of wffs of QC_1 that is isomorphic to S is satisfied by at least one assignment of truth-values to the atomic wffs of QC_1.* Hence, as a corollary: *A set S of wffs of QC_1 implies a wff A of QC_1*

(8) For a more detailed proof of both cases of the result, see [9].

(this in the standard sense of the word 'implies') *if and only if no set of wffs of* QC_I *that is isomorphic to* $S \cup \{\sim A\}$ *is satisfied by any assignment of truth-values to the atomic wffs of* QC_I. Hence, as a further corollary: *A wff of* QC_I *is valid* (this in the standard sense of the word 'valid') *if and only if A is satisfied by every assignment of truth-values to the atomic wffs of* QC_I.

A word may be in order, before I move to QC_2, on the notion of derivability in QC_1 (and, hence, consistency in QC_1, a set S of wffs of QC_1 being said here to be consistent in QC_1 if '$p \,\&\, \sim p$' is not derivable from S in QC_1). Following in this the example of Fitch in [3], I count a wff A of QC_1 as an axiom of QC_1 if: (i) it is of one of the six sorts

A1. $A \supset (B \supset A)$,

A2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$,

A3. $(\sim A \supset \sim B) \supset (B \supset A)$,

A4. $(\forall X)(A \supset B) \supset ((\forall X)A \supset (\forall X)B)$

A5. $A \supset (\forall X)A$, where X does not occur free in A,

A6. $(\forall X)A \supset A[Y/X]$ (9), or (ii) the wff is of the sort $(\forall X)A$, where A is an axiom of QC_1. I count a finite column of wffs of QC_1 as a derivation in QC_1 of a wff A of QC_1 from a set S of wffs of QC_1 if: (i) the column closes with A and (ii) every entry in the column belongs to S, is an axiom of QC_1, or follows from two previous entries in the column by application of *Modus Ponens*. And I take a wff A of QC_1 to be derivable in QC_1 from a set of wffs of QC_1 if there is a derivation in QC_1 of A from S, to be provable in QC_1 if A is derivable from ø in QC_1. It immediately follows from this account of things that, if a wff of QC_1 is derivable in QC_1 from a set S of wffs of QC_1, then A is derivable in QC_1 from any set of wffs of QC_1 that has S as a subset. As Montague and Henkin have shown in [10], the result — in the absence of which QC_1 fails to be strongly complete

(9) A6 may of course be weakened to read: $(\forall X)A \supset A[Y/X]$, where as free occurrence of X in A is in a component of A of the sort $(\forall Y)B$.

(10) QC_1 is said to be strongly complete if any wff of QC_1 that is implied by a set of wffs of QC_1 is derivable in QC_1 from that set.

— (10) is blocked when derivability in QC_1 is accounted for as in [2] (11).

II

My second generalization of Beth's Theorem calls for the following syntactical and semantical preliminaries.

First, where A is a wff of QC_2, and F and G are (not necessarily distinct) predicate variables of QC_2 of the same degree, I shall take (A)[G/F] to be the result of replacing every free occurrence of F in A by an occurrence of G if no component of A of the sort ($\forall$G)B contains a free occurrence of F in A; otherwise, I shall take (A)[G/F] to be (A')[G/F], where A' is the result of replacing every occurrence of G in every component of A of the sort ($\forall$G)A that contains a free occurrence of F in A by an occurrence of the alphabetically earliest predicate variable of QC_2 that is of the same degree as G and is foreign to that component of A.

(11) In [2] Church adopts along with *Modus Ponens* a rule, called Generalization, that reads: "From A to infer ($\forall$X) A". An individual variable X of QC_1 is then said to be generalized upon in a finite column of wffs of QC_1 if at least one entry in the column is of the sort ($\forall$X) A and follows from a previous entry in the column by application of Generalization. A column of wffs of QC_1 then counts as a derivation in QC_1 of a wff A of QC_1 (in Church's words, as a proof in F^{1p} of A from S) if (in effect): (i) the column closes with A, (ii) every entry in the column belongs to S, is of one of the six sorts A1-A6, or follows from previous entries in the column by application of *Modus Ponens* or Generalization, and (iii) no individual variable of QC_1 that is generalized upon in the column occurs free in any member of S. Finally, a wff A of QC_1 is taken to be derivable in QC_1 from a set S of wffs of QC_1 (in Church's words, to be provable in F^{1p} from S) if: (i) in the case that S is finite, there is a derivation in QC_1 of A from S, (ii) in the contrary case, there is a derivation in QC_1 of A from a finite susbet of S. Montague and Henkin have shown that, under Church's understanding of things, '$(\forall y)(g(y) \supset g(y))$', though derivable in QC_1 from $\emptyset$, is not derivable in QC_1 from $\{g(y)\}$. Montague and Henkin suggest two ways of correcting this anomaly, and I suggest another in [7]; Fitch's, however, is by far the simplest.

Next, where A is a wff of QC_2, $V_1, V_2, \ldots,$ and V_n $(n \geqslant 1)$ are distinct variables of QC_2, and for each i from 1 to n V'_1 is an individual variable of QC_2 when V_i is one, otherwise a predicate variable of QC_2 of the same degree as V_i, I shall take $(A)[V'_1, V'_2, \ldots, V'_n/V_1, V_2, \ldots, V_n]$ to be $((A)[V'_1, V'_2, \ldots, V'_{n-1}/V_1, V_2, \ldots, V_{n-1}])[V'_n/V_n]$.

Next, where (a) M is a one-to-one mapping of the set of the variables of QC_2 into itself such that, for every individual variable X of QC_2, M(X) is an individual variable of QC_2, and, for every m from 1 on and every predicate variable F of QC_2 of degree m, M(F) is a predicate variable of QC_2 of degree m, (b) A is a wff of QC_2, and (c) $V_1, V_2, \ldots,$ and V_n are in alphabetical order all the variables of QC_2 that occur free in A, I shall take the M-image of A to be A itself when $n = 0$, otherwise to be $A[M(V_1), M(V_2), \ldots, M(V_n)/V_1, V_2, \ldots, V_n]$. And, S and S′ being (not necessarily distinct) sets of wffs of QC_2, I shall take S′ to be isomorphic to S if: (i) in the case that S is empty, S′ is empty as well, and (ii) in the contrary case, S′ consists — for some one-to-one mapping M of the aforedescribed sort — of the M-images of the various members of S.

Then, I shall count a wff of QC_2 an axiom* of QC_2 if it is of one of: (i) the aforementioned sorts A1-A6, (ii) the three extra sorts

A7. $(\forall F)(A \supset B) \supset ((\forall F)A \supset (\forall F)B)$,

A8. $A \supset (\forall F)A$, when F does not occur free in A,

A9. $(\forall F)A \supset A[G/F]$ ([12]),

or (iii) the two sorts $(\forall X)A$ and $(\forall F)A$, where A is an axiom * of QC_2. I shall count a finite column of wffs of QC_2 as a derivation* of a wff A of QC_2 from a set S of wffs of QC_2 if: (i) the column closes with A and (ii) every entry in the column belongs to S, is an axiom* of QC_2, or follows from two previous entries in the column by application of *Modus Ponens*. I shall take a wff A of QC_2 to be derivable* in QC_2 from a set S of wffs of

([12]) A9 can of course be weakened to read: $(\forall F)A \supset A[G/F]$, where G is a predicate variable of QC_2 of the same degree as F, and no free occurrence of F in A is in a component of A of the sort $(\forall G)B$.

QC_2 if there is a derivation* of A from S in QC_2, to be provable* in QC_2 if A is derivable* from ø in QC_2 (13). And I shall take a set S of wffs of QC_2 to be consistent* in QC_2 if 'p& ~ p' is not derivable* from S in QC_2.

Finally, where A is a wff of QC_2 and Asst an assignment of truth-values to the atomic wffs of QC_2. I shall say that A is satisfied by Asst if: (i) in the case that A is atomic, A is assigned the truth-value T in Asst, (ii) in the case that A is of the sort $\sim B$, B is not satisfied by Asst, (iii) in the case that A is of the sort $B \supset C$, B is not satisfied by Asst or C is, (iv) in the case that A is of the sort $(\forall X)B$, $B[Y/X]$ is satisfied by Asst for every individual variable Y of QC_2, and (v) is the case that A is of the sort $(\forall F)B$, $B[G/F]$ is satisfied by Asst for every predicate variable G of QC_2 that is of the same degree as F.

Proof that *if a set S of wffs of QC_2 is consistent* in QC_2, at least one set of wffs of QC_2 that is isomorphic to S is satisfied by at least one assignment of truth-values to the atomic wffs of QC_2*, can be had by essentially the same argument as in I. First, count a set S of wffs of QC_2 as a wff set of Type I if infinitely many individual variables of QC_2 and — for each m from 1 on — infinitely many predicate variables of QC_2 of degree m do not occur free in any member of S, otherwise count S as a wff set of Type II. Second, where S is a wff set of Type I, let S_0 be S; for each i from 1 on let S_i be $S_{i-1} \cup \{A_i(Y/X_i) \supset (\forall X_i)A_i\}$ where $(\forall X_i)A_i$ is in some predetermined order the i-th of QC_2 of the sort $(\forall X)A$ and Y is the alphabetically earliest individual variable of QC_2 that does not occur free in any member of S_{i-1} and is foreign to $(\forall X_i)A_i$; let S_{∞_0} be the union of S_0, S_1, S_2, ...; for each i from 1 on let S_{∞_i} be $S_{\infty_{i-1}} \cup \{A_i[G/F_i] \supset (\forall F_i)A_i\}$, where $(\forall F_i)A_i$ is in some predetermined order the i-th wff of QC_2 of the sort $(\forall F)A$ and G is the alphabetically earliest predicate variable of QC_2 that is of the same degree as F_i, does not occur free in any member of $S_{\infty_{i-1}}$ and is foreign to $(\forall F_i)A_i$; let $S^{o}_{\infty_\infty}$ be the

(13) My account of derivability* in QC_2 is patterned after Henkin's account in [6] of provability in F*, and my later account of derivability** in QC_2 patterned after his account of provability in F**.

union of $S_{\infty_0}, S_{\infty_1}, S_{\infty_2}, \ldots$; for each i from 1 on let $S^i_{\infty_\infty}$ be $S^{i-1}_{\infty_\infty} \cup \{A_i\}$, where A_i is in some predetermined order the i-th wff of QC_2, if $S^{i-1}_{\infty_\infty} \cup \{A_i\}$ is consistent* in QC_2; otherwise let $S^i_{\infty_\infty}$ be $S^{i-1}_{\infty_\infty}$; and let $S^\infty_{\infty_\infty}$ be the union of $S^0_{\infty_\infty}, S^1_{\infty_\infty}, S^2_{\infty_\infty}, \ldots$ It is easily verified that $S^\infty_{\infty_\infty}$ is consistent* in QC_2 if S is, and — in particular — that if $S^\infty_{\infty_\infty}$ is consistent* in QC_2, then an arbitrary wff of QC_2 of the sort $(\forall F)A$ belongs to $S^\infty_{\infty_\infty}$ if and only if $A[G/F]$ does for every predicate variable G of QC_2 of the same degree as F. Third, let Asst be the result of assigning T to every atomic wff of QC_2 that belongs to $S^\infty_{\infty_\infty}$, F to every other one. It is easily verified that, if $S^\infty_{\infty_\infty}$ is consistent* in QC_2, then a wff A of QC_2 belongs to $S^\infty_{\infty_\infty}$ if and only if A is satisfied by Asst [14]. Hence, if S is consistent* in QC_2, there is an assignment of truth-values to the atomic wffs of QC_2 that satisfies S, and hence that satisfies at least one set of wffs of QC_2 isomorphic to S.

Proof of the same result for the case where S is a wff set of Type II, and proof of the converse result when S is a wff set either of Type I or of Type II, proceeds as in I.

Hence, my second generalization of Beth's Theorem: *A set S of wffs of QC_2 is consistent* in QC_2 if and only if at least one set of wffs of QC_2 that is isomorphic to S is satisfied by at least one assignment of truth-values to the atomic wffs of QC_2*. Hence, as a corollary: *A wff A of QC_2 is derivable* in QC_2 from a set S of wffs of QC_2 if and only if no set of wffs of QC_2 that is isomorphic to $S \cup \{\sim A\}$ is satisfied by any assignment of truth-values to the atomic wffs of QC_2*. Hence, as

(14) The proof is by mathematical induction on the number of occurrences of '$\sim$', '$\supset$', and '$\forall$' in A.

a further corollary: *A wff A of QC_2 is provable* in QC_2 if and only if A is satisfied by every assignment of truth-values to the atomic wffs of QC_2*. Because of the parallelism between these corollaries and the ones on p. 5-6, I suggest that a set S of wffs of QC_2 be said to imply* a wff A of QC_2 if no set of wffs of QC_2 that is isomorphic to $S \cup \{\sim A\}$ is satisfied by any assignment of truth-values to the atomic wffs of QC_2, and that a wff A of QC_2 be said to be valid* if A is satisfied by every assignment of truth-values to the atomic wffs of QC_2. The reader will note that under this understanding of things a wff A of QC_2 is valid* if and only if A is completely valid with respect to every model of the sort described by Henkin on p. 206 of [6].

The present generalization of Beth's Theorem can be phrased in another — but, of course, equivalent — manner. TV being a function from the set of the set of the wffs of QC_2 to $\{T, F\}$, count TV as a truth-value function for QC_2 if: (i) $TV(\sim A) = T$ if and only if $TV(A) = F$, (ii) $TV(A \supset B) = T$ if and only if $TV(A) = F$ or $TV(B) = T$, (iii) $TV((\forall X)A) = T$ if and only if $TV(A[Y/X]) = T$ for every individual variable Y of QC_2, and (iv) $TV((\forall F)A) = T$ if and only if $TV(A[G/F]) = T$ for every predicate variable G of QC_2 of the same degree as F. And, A being a wff of QC_2 and TV a truth-value function for QC_2, take A to be satisfied by TV if $TV(A) = T$.

Return now to the extension S^{∞}_{∞} of S that we constructed four

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paragraphs back, and let TV be the function from the set of the wffs of QC_2 to $\{T, F\}$ such that, for every wff A of QC_2, $TV(A) = T$ if and only if A belongs to S^{∞}_{∞}. TV readily proves to be a

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truth-value function for QC_2, if S^{∞}_{∞} is consistent* in QC_2, and

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one that satisfies every member of S. Hence, if S is consistent* in QC_2, then at least one set of wffs of QC_2 that is isomorphic to S is satisfied by at least one truth-value function for QC_2. But, as the reader may verify on his own, no set wffs of QC_2 is satisfied by a truth-value function for QC_2 unless consistent* in QC_2. Hence, *a set S of wffs of QC_2 is consistent* in QC_2 if and only if at least one set of wffs of QC_2 that is isomorphic to S is* *satisfied by at least one truth-value function for QC_2.* But, if so, then a set S of wffs of QC_2 may be said to imply* a wff A of QC_2 if and only if no set of wffs of QC_2 that is isomorphic to $S \cup \{\sim A\}$ is satisfied by any truth-value function for QC_2, and a wff A of QC_2 may be termed valid* if A is satisfied by every truth-value function for QC_2.

III

My last generalization of Beth's Theorem calls for an extra substitution convention.

Let A and B be (not necessarily distinct) wffs of QC_2; let F be for some m from 1 on a predicate variable of QC_2 of degree m; and let X_1, X_2, ..., and X_m be distinct individual variables of QC_2.

Case 1: None of the free occurrences of F in A is in a component of A of the sort $(\forall V)C$, where V is an individual variable of QC_2 distinct from each one of X_1, X_2, ..., and X_m that occurs free in B, or a predicate variable of QC_2 that occurs free in B. Then $A[B/F(X_1, X_2, \ldots, X_m)]$ is to be the result of substituting $B[Y_1, Y_2, \ldots, Y_m/X_1, X_2, \ldots, X_m]$ for every component of A of the sort $F(Y_1, Y_2, \ldots, Y_m)$ that contains one of the free occurrences of F in A.

Case 2: At least one of the free occurrences of F in A is in a component of A of the sort $(\forall V)C$, where V is an individual variable of QC_2 distinct from each one of X_1, X_2, ..., and X_m that occurs free in B, or a predicate variable of QC_2 that occurs free in B. Let $(\forall V_1)C_1$, $(\forall V_2)C_2$, ..., and $(\forall V_n)C_n$ be in order of decreasing length all the components of A of the sort $(\forall V)C$ (15) where V is as just described; let A_0 be A; and, V'_i being the alphabetically earliest individual variable of QC_2 that is foreign to C_i when V_i is an individual variable of QC_2, other-

(15) Where A and B are two components of a wff C of QC_2, let A precede B in order of decreasing length if A has fewer symbols than B or, in the case that A and B have the same number of symbols, if A alphabetically precedes B.

wise the alphabetically earliest predicate variable of QC_2 of the same degree as V_i that is foreign to C_i, let A_i be for each i from 1 to n the result of substituting $(\forall V'_i)C_i[V'_i/V_i]$ for every occurrence of $(\forall V_i)C_i$ in A that contains one of the free occurrences of F in A_{i-1}. Then $A[B/F(X_1, X_2, \ldots, X_m)]$ is to be $A_n[B/F(X_1, X_2, \ldots, X_m)]$ (16).

Next, count as an axiom** of QC_2 that is of one of the aforedescribed sorts A1-A8, or is of the sort

A9. $(\exists F)(\forall X_1)(\forall X_2)\ldots(\forall X_m)(F(X_1, X_2, \ldots, X_m) \equiv A)$, where for some m from 1 on F is a predicate variable of QC_2 of degree m that does not occur free in A, and $X_1, X_2, \ldots$, and X_m are distinct individual variables of QC_2, or is of one of the two sorts $(\forall X)A$ and $(\forall F)A$, where A is an axiom** of QC_2. Count as a derivation** in QC_2 of a wff A of QC_2 from a set S of wffs of QC_2 any column of wffs of QC_2 that closes with A and everyone of whose entries belongs to S, is an axiom** of QC_2, or follows from two previous entries in the column by application of *Modus Ponens*. Take a wff A of QC_2 to be derivable** in QC_2 from a set S of wffs of QC_2 if there is a derivation** of A from S in QC_2, to be provable** in QC_2 if A is derivable** from $\emptyset$ in QC_2 (17). And take a set S of wffs of QC_2 to be consistent** in QC_2 if '$p \,\&\, \sim p$' is not derivable** from S in QC_2.

Finally, TV being a function from the set of the wffs of QC_2 to $\{T, F\}$, count TV as a general truth-value function for QC_2 if: (1) $TV(\sim A) = T$ if and only if $TV(A) = F$, (ii) $TV(A \supset B) =$

(16) $A[B/F(X^1, X_2, \ldots, X_m)]$ is a generalization of what Church understands by $\breve{S}^F_B(X_1, X_2, \ldots, X_m)A\,|$ in [2], and Henkin by $\bar{S}^F_B(X_1, X_2, \ldots, X_m)A\,|$ in [6].

(17) In [2] Church adopts a second rule of Generalization, that reads: "From A to infer $(\forall F)A$". A predicate variable F of QC_2 is then said to be generalized upon in a finite column of wffs of QC_2 if at least one entry in the column is of the sort $(\forall F)A$ and follows from a previous entry in the column by application of Generalization. A column of wffs of QC_2 then counts as a derivation** in QC_2 of a wff A of QC_2 from a set S of wffs of QC_2 if (in effect): (i) the column closes with A, (ii) every entry in the column belongs to S, is of one of the nine sorts A1-A8 and $(\forall F)A \supset A[B/F(X_1, X_2, \ldots, X_m)]$, where $X_1, X_2, \ldots$, and X_m

T if and only if TV(A) = F or TV(B) = B, (iii) TV((∀X)A) = T if and only if TV(A[Y/X]) = T for every individual variable Y of QC_2, and (iv) TV((∀F)A) = T, where for some m from 1 on F is a predicate variable of QC_2 of degree m, if and only if TV(A[B/F(X_1, X_2, ..., X_m)]) = T for every wff B of QC_2 and every m individual variables X_1, X_2, ..., and X_m of QC_2 that are distinct from one another. And, A being a wff of QC_2 and TV a general truth-value function for QC_2, take A to be satisfied by TV if TV(A) = T.

It is easily verified that, for every wff A of QC_2, every predicate variable F of QC_2 of, say, degree m, every predicate variable G of QC_2 of degree m that is foreign to A, and every m individual variables X_1, X_2, ..., and X_m of QC_2 that are distinct from another, A[G/F] and A[G(X_1, X_2, ..., X_m)/F(X_1, X_2, ..., X_m)] are the same. Consider then the set S^0_∞ on p. 10. Since for each i from 1 on there is a predicate variable G of QC_2 of the same degree, say, m, as F_i such that A_i[G/F_i] ⊃)(∀F_i)A_i belongs to S^0_∞, then for each i from 1 on there are distinct individual variables X_1 X_2, ..., and X_m of QC_2 (the alphabetically first m individual variables of QC_2 will do), and a wff B of QC_2 (to wit: G(X_1, G_2, ..., X_m)) such that A_i[B/F_i(X_1, X_2, ..., X_m)] ⊃ (∀F_i)A_i belongs to S^0_∞. Now for each i from 1 on let S^i_∞ be

are m (the degree of F) distinct individual variables of QC_2, or follows from previous entries in the column by application of Modus Ponens or Generalization, and (iii) no individual or predicate variable of QC_2 that is generalized upon in the column occurs free in any member of S. Finally, a wff A of QC_2 is taken to be derivable** in QC_2 from a set S of wffs of QC_2 if: (i) in the case that S is finite, there is a derivation** in QC_2 of A from S, (ii) in the contrary case, there is a derivation** in QC_2 of A from a finite subset of S. Henkin in [6] exclusively deals with provability in F**. As the paper precedes [10] by three years, he would presumably have handled derivability** along the above lines, but with A9 doing duty for (∀F) A ⊃ A[B/F(X_1, X_2, ..., X_m)]. Montague and Henkin's argument in [10] is easily extended to show that, under Church's and Henkin's understanding of things, '(∀g) (g(y) ⊃ g(y))', though derivable** and derivable* in QC_2 from ø, is neither derivable** nor derivable* in QC_2 from {g(y)}.

$S^{i-1}_{\infty\infty} \cup \{A_i\}$, where A_i is in some predetermined order the i-th wff of QC_2, if $S^{i-1}_{\infty\infty} \cup \{A_i\}$ is consistent** in QC_2; otherwise let $S^i_{\infty\infty}$ be $S^{i-1}_{\infty\infty}$; and let $S^{\infty}_{\infty\infty}$ be the union of $S^0_{\infty\infty}$, $S^1_{\infty\infty}$, $S^2_{\infty\infty}$, ... It is easily verified that $S^{\infty}_{\infty\infty}$ is consistent** in QC_2 if S is, and — since $(\forall F)A \supset A[B/F(X_1, X_2, \ldots, X_m)]$ is provable** in QC_2 for any predicate variable F of QC_2, any two wffs A and B of QC_2, and any m (m the degree of F) individual variables $X_1, X_2, \ldots, X_m$ of QC_2 that are distinct from one another — ([18]) that, if $S^{\infty}_{\infty\infty}$ is consistent** in QC_2, then an arbitrary wff of QC_2 of the sort $(\forall F)A$ belongs to $S^{\infty}_{\infty\infty}$ if and only if $A[B/F(X_1, X_2, \ldots, X_m)]$ does for every wff B of QC_2 and every m individual variables $X_1, X_2, \ldots$, and X_m of QC_2 that are distinct from one another. Let then TV be the function from the set of the wffs of QC_2 to $\{T, F\}$ such that, for every wff A of QC_2, $TV(A) = T$ if and only ifA belongs to $S^{\infty}_{\infty\infty}$. TV readily proves to be a general truth-value function for QC_2, if $S^{\infty}_{\infty\infty}$ is consistent** in QC_2, and one that satisfies every member of S.

Hence, my third generalization of Beth's Theorem: *A set S of wffs of QC_2 is consistent** in QC_2 if and only if at least one set of wffs of QC_2 that is isomorphic to S is satisfied by at least one general truth-function for QC_2*. Hence, as a corollary: *A wff A of QC_2 is derivable** from a set S of wffs of QC_2 if and only if no set of wffs of QC_2 that is isomorphic to $S \cup \{\sim A\}$ is satisfied by any general truth-value function for QC_2*. Hence, as a further corollary, *A wff A of QC_2 is provable** in QC_2 if and only if A is satisfied by every general truth-value function for QC_2*. Because of the parallelism between these corollaries and the ones on p. 5-6, I suggest that a set S of wffs of QC_2 be said to imply** a wff A of QC_2 if no set of wffs of QC_2 that is iso-

([18]) Proof of the result can be found (in effect) in [6].

morphic to $S \cup \{\sim A\}$ is satisfied by any general truth-value function for QC_2, and that a wff A of QC_2 be said to be valid** if A is satisfied by every general truth-value function for QC_2. The reader will note that under this understanding of things a wff A of QC_2 is valid** if and only if A is valid with respect to every general model of the sort described by Henkin on p. 84 of [5] (19).

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