ON FINITELY MANY-VALUED LOGICS

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Our language contains the following symbols:

(1) the logical constants II ('is 2-valued'), $\vdash$ ('assertion'), $\sim$ ('not'), $\rightarrow$ ('only if'), $\wedge$ ('and'), $\vee$ ('or'), $\leftrightarrow$ ('if and only if'), $\bigwedge$ ('for all'), $\bigvee$ ('for some'), ι ('the'), and I ('is identical with'); we call the first seven of these sentential connectives and all of the rest except I variable binders;

(2) a denumerable infinity of distinct
   (a) individual variables,
   (b) individual constants, and
   (c) predicates of any positive number of places among which I is the first two-place predicate.

In the metalanguage, we use '$\langle$', '$\rangle$' and '{', '}' to mark the boundaries of non-empty finite sequences and sets respectively and '$^\frown$' and '$_{-1}$' as standing for the operations of concatenating two finite sequences and of removing the first term of a non-empty finite sequence respectively. Also, we use 'n' as a metalinguistic variable ranging over all positive integers greater than 1. TF, terms, and formulas will be understood as follows:

(1) TF = the intersection of all sets k such that
   (a) for any variable or individual constant t, the pair t, $\langle tIt\rangle$ is in k;
   (b) for any positive integer m, m-place predicate p, and m-term sequence of members of the domain of k t, the pair t(1), $(\langle t(1)\rangle {}^\frown \langle p\rangle)^\frown t_{-1}$ is in k;
   (c) for any variable v and for f and g in the range of k,
      (i) the pair $\langle \iota vf\rangle$, $\langle vIv\rangle$ is in k and
      (ii) for any h in $\{\langle IIf\rangle\ \langle\vdash f\rangle\ \langle\sim f\rangle\ \langle f\rightarrow g\rangle\ \langle f\wedge g\rangle\ \langle f\vee g\rangle\ \langle f\leftrightarrow g\rangle\ \langle\bigwedge vf\rangle\ \langle\bigvee vf\rangle\}$,
      the pair v, h is in k;

(2) t is a term just in case t is in the domain of TF; and

(3) f is a formula just in case f is in the range of TF.

An atomic formula is, of course, an object occurring on the right side of one of the pairs under (b) above.

In what follows, we omit sequence marks according to the usual conventions for the omission of parentheses.

1. n-VALUED SEMANTICS IN EMPTY AND NON-EMPTY UNIVERSES (1)

If x is a set, a is an assigner in x just in case a is a function such that
(1) the domain of a = the set of variables and
(2) for any v in the domain of a,
   (a) if x is empty, then a(v) = the empty set, and
   (b) if x is not empty, then, for some m in x, a(v) = {m}.

If a is an assigner in x, v is a variable, and y is an object of any kind, then $a(^v{}_y)$ = a with the pair v, a(v) removed and the pair v, y added in its place.

By an n-interpreter, we mean a function i such that
(1) the domain of i = the set of all individual constants and predicates and
(2) there is a set u such that
   (a) for any individual constant c, either i(c) = the empty set or, for some m in u, i(c) = {m};
   (b) for any positive integer m and m-place predicate p, i(p) is an (n—1)-term sequence of sets of m-term sequences of members of u, and
   (c) i(I) = the (n—1)-term sequence s such that, for any k in the domain of s, s(k) = the set of all t such that, for some m in u, t = ⟨mm⟩.

The interpetations given here to predicates, and particularly to I, seem somewhat artificial, but the author has not been able to find any more natural ones.

Given an n-interpreter i, Ui (the universe of i) is the u under (2) above.

Given an n-interpreter i and an assigner in Ui a, we understand Int ia (the interpretation with respect to i and a of...) as follows:
(1) for any variable v, Int ia (v) = a(v);
(2) for any individual constant c, Int ia(c) = i(c);
(3) for any positive integer m, m-place predicate p, and m-term sequence of terms t, Int ia $((\langle t(1)\rangle \frown \langle p\rangle) \frown t_{-1})$ = the z such that either there is an m-term sequence u such that u(k) is in Int ia (t(k)) for any k in the domain of u and z = the number of members of the set of all k in the domain of i(p) such that u is in (i(p))(k) divided by n-1 or not and z = O;
(4) for any variable v and formulas f and g,

(1) The adaptation of the definitions of this and the following sections to non-empty universe semantics and logics is, of course, not difficult.

(a) (Int ia (IIf) = the z such that either Int ia (f) is in {O1} and z = 1 or not and z = O;
(b) Int ia ($\vdash$ f) = the z such that either Int ia (f) = 1 and z = 1 or not and z = O;
(c) Int ia ($\sim$f) = 1—Int ia (f);
(d) Int ia (f$\rightarrow$g) = the smallest member of {1, (1—Int ia (f)) + Int ia (g)};
(e) Int ia (f $\wedge$ g) = the smallest member of {Int ia (f) Int ia (g)};
(f) Int ia (f $\vee$ g) = the greatest member of {Int ia (f) Int ia (g)};
(g) Int ia (f$\leftrightarrow$g) = (1—the greatest member of {Int ia (f) Int ia (g)}) + the smallest member of {Int ia (f) Int ia (g)};
(h) Int ia (ɿvf) = the z such that either Int ia ($^{v}\{_{m}\}$) (f) is in {O1} for any m in Ui, there is a k in Ui such that, for any m in Ui, Int ia ($^{v}\{_{m}\}$) (f) = 1 just in case m = k, and z = {k}, or not and z = the empty set;
(i) Int ia ($\bigwedge$vf) = the z such that either Ui is empty and z = 1 or not and z = the smallest member of the set of all r such that, for some m in Ui, Int ia ($^{v}\{_{m}\}$) (f) = r; and
(j) Int ia ($\bigvee$vf) = the z such that either Ui is empty and z = O or not and z = the greatest member of the set of all r such that, for some m in Ui, Int ia ($^{v}\{_{m}\}$) (f) = r.

If i is an n-interpreter, then Ti (the truth values of i) = the set of all r such that, for some formula f and assigner in Ui a, Int ia (f) = r. By Tn (the truth values of n-valued semantics), we mean the set of all r such that, for some n-interpreter i, r is in Ti. It can be shown that

Theorem 1. Tn = the set of all r such that, for some natural number smaller than n k, r = k divided by (n—1) [2].

Given a formula f, f is i-true just in case Int ia (f) = 1 for any assigner in Ui a, f is n-valid in case f is i-true for any n-interpreter i, and f is valid just in case f is n-valid for any n. It follows that

Theorem 2. If 2 is smaller than n, then the set of all n-valid formulas is a proper subset of the set of all 2-valid formulas.

For assume the antecedent. If f is n-valid, then, for any n-interpreter i such that Ti = {O1}, f is i-true; also, for any 2-interpreter j, there is

[2] Thus, our n truth values are just those given in J. Łukasiewicz's and A. Tarski's 'Investigations into the sentential calculus' (in TARSKI's book *Logic, Semantics, Metamathematics*, Oxford, 1956).

an n-interpreter i such that Ui = Uj, Ti = $\{O1\}$, and g is i-true just in case g is j-true for any formula g. Hence, f is also 2-valid. On the other hand, if f is an atomic formula in which neither I nor ɿ occurs, then IIf is 2-valid, but not n-valid.

We say that a formula f is nonzero just in case there are no n, n-interpeter i, and assigner in Ui a such that Int ia (f) = O. Hence,

Theorem 3. The set of all nonzero formulas is a proper subset of the set of all 2-valid formulas.

For, if f is a formula and f is nonzero, then, for any 2-interpeter i and assigner in Ui a, Int ia (f) $\neq$ O and so Int ia (f) = 1; that is, f is 2-valid. On the other hand, if f is an atomic formula in which neither I nor ɿ occurs, then IIf is 2-valid, but not nonzero.

Theorem 4. If 2 is smaller than n, then there is a nonzero formula which is not n-valid.

For example, fv$\sim$f where f is an atomic formula in which neither I nor ɿ occur.

Notice also that

Theorem 5. If f is a formula, i is an n-interpeter, and Ui is empty, then f is i-true just in case, for any n and n-interpreter i, if Ui is empty, then f is i-true.

This follows principally from the fact that atomic formulas are always assigned O by an n-interpreter with an empty universe.

We say that a formula f is an n-tautology just in case, for any v, v(f) = 1 when v is a function, the domain of v is the set of all formulas, the range of v is included in Tn, and, for any formulas f and g,

(1) v(IIf) = the z such that either v(f) is in $\{O1\}$ and z = 1 or not and z = O;
(2) v($\vdash$f) = the z such that either v(f) = 1 and z = 1 or not and z = O;
(3) v($\sim$f) = 1—v(f);
(4) v(f$\rightarrow$g) = the smallest member of $\{$1, (1—v(f))+v(g)$\}$;
(5) v(f$\wedge$g) = the smallest member of $\{$v(f) v(g)$\}$;
(6) v(f$\vee$g) = the greatest member of $\{$v(f) v(g)$\}$; and
(7) v(f$\leftrightarrow$g) = (1—the greatest member of $\{$v(f) v(g)$\}$) + the smallest member of $\{$v(f) v(g)$\}$.

A formula is a tautology just in case it is an n-tautology for any n.

Given terms t and u and a term or formula f, we understand Af (the atomic subformula assertion of f), freedom, and PStuf (the result of properly substituting t for u in f) as follows:

(1) if u = f, then u is free in f and PStuf = t;

(2) if u $\neq$ f, then

(a) if f is a variable or an individual constant, then Af = f, u is not free in f, and PStuf = f;

(b) for any positive integer m, m-place predicate p, and m-term sequence of terms v, if f = $((\langle v(1)\rangle ^\frown \langle p\rangle) ^\frown v_{-1}$, then Af = $\vdash((\langle Av(1)\rangle ^\frown \langle p\rangle) ^\frown$ (the m-term sequence w such that w(k) = Av(k) for any k in the domain of $w)_{-1}$, u is free in f just in case u is free in some member of the range of v, and PStuf = $((\langle$Pstu $v(\langle 1\rangle)^\frown\langle p\rangle)]^\frown$ (the m-term sequence w such that w(k) = PStuv(k) for any k in the domain of $w)_{-1}$;

(c) for any sentential connective c and formulas g and h,

(i) if f = cg, then Af = cAg, u is free in f just in case u is free in g, and PStuf = cPStug and

(ii) if f = gch, then Af = AgcAh, u is free in f just in case u is free in g or h, and PStuf = PStugcPStuh; and

(d) for any variable binder b, variable v, and formula g, if f = bvg, then

(i) Af = bvAg;

(ii) u is free in f just in case u is free in g and v is not free in u; and

(iii) PStuf = the z such that

(a) if u is not free in f, then z = f;

(b) if u is free in f and v is not free in t, then z = bvPStug;

(c) if u is free in f, v is free in t, and w = the first variable not occurring in either f or t, then z = bwPStuPSwvg.

A sentence is, of course, a formula in which no variable is free.

Obviously,

Theorem 6. If f is a formula, then f is 2-valid just in case Af is n-valid. This follows principally from the fact that, for any atomic formula f, n-interpreter i, and assigner in Ui a, Int ia (Af) is in $\{O1\}$.

Theorem 6 is of great importance; it establishes that, when 2 is smaller than n, the 2-valid formulas have near duplicates among the n-valid formulas even if the n-valid formulas make up just a fragment of the 2-valid ones. Hence, a 2-valid argument of any kind has a near duplicate among the n-valid arguments which is obtainable from it by just asserting its atomic formulas.

We now turn to the task of listing the most important of the valid formulas.

Theorem 7. If t and u are terms, then IItIu is valid.
This follows from the way in which I is interpreted.

Theorem 8. If m is a positive integer, p is an m-place predicate, t is an m-term sequence of terms, k is in the domain of t, and v is a variable not free in t(k), then $((\langle t(1)\rangle \frown \langle p\rangle) \frown t_{-1} \rightarrow \bigvee$ v vIt(k) is valid.
This follows from the way in which Int ia is defined for atomic formulas.

Theorem 9. If v is a variable, t is a term, and v is not free in t, then $\bigvee$v vIt$\rightarrow$tIt is valid.
This follows from the way in which I is interpreted.

Theorem 10. If v, w, and x are variables, w $\neq$ x, and f is a formula, then $\bigvee$vf$\rightarrow\bigvee$w wIx is valid.
This follows from the way in which assigners are defined.

Theorem 11. If f is a formula and f is an n-tautology, then f is n-valid.
This follows from the fact that, given any n-interpreter i and assigner in Ui a, the function which assigns Int ia (f) to any formula f is one of the kind which assigns 1 to all n-tautologies.

Theorem 12. If t is a term or a formula, v is a variable not free in t, i is an n-interpeter, and both a and a$({}^{v}{}_{x})$ are assigners in Ui, then Int ia$({}^{v}{}_{x})$ (t) = Int ia (t).

Theorem 13. If t is a term, v is a variable, f is a term or a formula, i is an n-interpeter, a is an assigner in Ui, and Int ia (t) = $\{m\}$, then Int ia (PStvf) = Int ia $({}^{v}\{{}_{m}\})$ (f).

Theorem 14. If t and u are terms, f is a term or a formula, i is an n-interpeter, a is an assigner in Ui, and Int ia (t) = Int ia (u), then Int ia (PStuf) = Int ia (f).

The proofs of theorems 12 through 14 are by inductions among the members of TF.

Theorem 15. If v and w are variables, f is a formula, t is a term, and w is not free in t, then $\bigwedge$vf $\wedge$ $\bigvee$w wIt$\rightarrow$PStvf is valid.

Assume the antecedent, that i is an n-interpeter, and that a is an assigner in Ui. If Int ia $(\bigvee \mathrm{w}\ \mathrm{wIt}) = \mathrm{O}$, then Int ia $(\bigwedge \mathrm{vf} \wedge \bigvee \mathrm{w}\ \mathrm{wIt} \rightarrow \mathrm{PStvf}) = 1$. Assume then that Int ia $(\bigvee \mathrm{w}\ \mathrm{wIt}) \neq \mathrm{O}$; hence, Int ia $(\bigvee \mathrm{w}\ \mathrm{wIt}) = 1$ and so, for some m in Ui, Int ia $(\mathrm{t}) = \{\mathrm{m}\}$. By theorem 13, Int ia (PStvf) = Int ia $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (f). Also, Int ia $(\bigwedge \mathrm{vf})$ is not greater than Int ia $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (f) and the same as Int ia $(\bigwedge \mathrm{vf} \wedge \bigvee \mathrm{w}\ \mathrm{wIt})$. Hence, Int ia $(\bigwedge \mathrm{vf} \wedge \bigvee \mathrm{w}\ \mathrm{wIt} \rightarrow \mathrm{PStvf}) = 1$ again.

Theorem 16. If v is a variable, f and g are formulas, and v is not free in f, then $\bigwedge \mathrm{v} \langle \mathrm{f} \rightarrow \mathrm{g} \rangle \rightarrow \langle \mathrm{f} \rightarrow \bigwedge \mathrm{vg} \rangle$ is valid.

Assume the antecedent, that i is an n-interpeter, and that a is an assigner in Ui. Let r = Int ia $(\bigwedge \mathrm{v} \langle \mathrm{f} \rightarrow \mathrm{g} \rangle)$ and s = Int ia $(\mathrm{f} \rightarrow \bigwedge \mathrm{vg})$. To establish the theorem, it is sufficient to show that r is not greater than s. If Ui is empty, this is obvious. Assume then that Ui is not empty and, for some real number greater than O, p, $\mathrm{r} = \mathrm{s} + \mathrm{p}$. If $\mathrm{r} = 1$, then, for any m in Ui, Int ia $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (f) is not greater than Int ia $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (g) and so, since Int ia (f) = Int ia $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (f) by theorem 12, Int ia (f) is not greater than Int ia $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (g). But then $\mathrm{s} = 1 = \mathrm{s} + \mathrm{p}$. This is impossible and so r is smaller than 1. It follows that, for some m in Ui, (1—Int ia $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (f)) + Int ia $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (g) $= \mathrm{s} + \mathrm{p}$ and there is no k in Ui such that (1—Int ia $({}^{\mathrm{v}}\{_{\mathrm{k}}\})$ (f)) + Int ia$({}^{\mathrm{v}}\{_{\mathrm{k}}\}$ (g) is smaller than $\mathrm{s}+\mathrm{p}$. Also, it follows that (1—Int ia (f)) + Int ia $(\bigwedge \mathrm{vg}) = \mathrm{s}$. By theorem 12, Int ia $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (f) = Int ia (f) and so p = Int ia $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (g) — Int ia $(\bigwedge \mathrm{vg})$; but then, for some k in Ui, Int ia $(\bigwedge \mathrm{vg})$ = Int ia $({}^{\mathrm{v}}\{_{\mathrm{k}}\})$ (g) and is smaller than Int ia $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (g). But, by theorem 12, Int ia $({}^{\mathrm{v}}\{_{\mathrm{k}}\})$ (f) = Int ia (f) and so (1—Int ia $({}^{\mathrm{v}}\{_{\mathrm{k}}\})$ (f)) + Int ia $({}^{\mathrm{v}}\{_{\mathrm{k}}\})$ (g) is smaller than $\mathrm{s}+\mathrm{p}$. This is impossible and so r is again not greater than s.

Theorem 17. If 2 is smaller than n, then there are a variable v and formulas f and g such that v is not free in f and $\bigwedge \mathrm{v} \langle \mathrm{f} \rightarrow \mathrm{g} \rangle \wedge \mathrm{f} \rightarrow \bigwedge \mathrm{vg}$ is not n-valid.

Assume the antecedent and let v be a variable, c be an individual constant, p and q be distinct 1-place predicates, and i be an n-interpreter such that Ui is i(c) and not empty, Int ia (cp) = 1 divided by (n—1), and Int ia (vq) = O for any assigner in Ui a. Obviously, $\bigwedge \mathrm{v} \langle \mathrm{cp} \rightarrow \mathrm{vq} \rangle \wedge \mathrm{cp} \rightarrow \bigwedge \mathrm{vvq}$ is not i-true.

Theorem 18. If v is a variable and f is a formula, then $\bigvee \mathrm{vf} \leftrightarrow \sim \bigwedge \mathrm{v} \sim \mathrm{f}$ is valid.

Assume the antecedent, that i is an n-interpreter, and that a is an assigner in Ui. To establish the theorem, it is sufficient to show that Int ia $(\bigvee \mathrm{vf})$ = Int ia $(\sim \bigwedge \mathrm{v} \sim \mathrm{f})$. If Ui is empty, this is obviously so.

Assume then that Ui is not empty. By our definitions, Int ia $(\bigvee \mathrm{vf})$ = the greatest member of the set of all r such that, for some m in Ui, Int ia $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (f) = r and so 1—Int ia $(\bigvee \mathrm{vf})$ = (the smallest member of the set of all r such that, for some m in Ui, 1—Int ia$({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (f) = r) = Int ia $(\bigwedge \mathrm{v} \sim \mathrm{f})$. Hence, Int ia $(\sim \bigwedge \mathrm{v} \sim \mathrm{f})$ = 1—Int ia $(\bigwedge \mathrm{v} \sim \mathrm{f})$ = 1—(1—Int ia $(\bigvee \mathrm{vf})$) = Int ia $(\bigvee \mathrm{vf})$ and the theorem holds.

Theorem 19. If t and u are terms and f is a formula, then tIu $\wedge$ PStuf$\rightarrow$f is valid.

Assume the antecedent, that i is an n-interpreter, and that a is an assigner in Ui. If Int ia (tIu) = O, then Int ia (tIu $\wedge$ PStuf$\rightarrow$f) = 1. Assume then that Int ia (tIu) $\neq$ O. Hence, Int ia (tIu) = 1 and so Int ia (tIu $\wedge$ PStuf) = Int ia (PStuf) and Int ia (t) = Int ia (u); but then Int ia (PStuf) = Int ia (f) by theorem 14 and so Int ia (tIu $\wedge$ PStuf$\rightarrow$f) is again 1.

Theorem 20. If v and w are distinct variables, f is a formula, and w is not free in f, then $\bigvee$w wI ɿvf$\leftrightarrow\bigwedge$v IIf $\wedge$ $\bigvee$w$\bigwedge$v$\langle$f$\leftrightarrow$vIw$\rangle$ is valid.

Assume the antecedent, that i is an n-interpreter, and that a is an assigner in Ui. It is sufficient to show that Int ia ($\bigvee$w wI ɿvf) = Int ia ($\bigwedge$vIIf $\wedge$ $\bigvee$w$\bigwedge$v$\langle$f$\leftrightarrow$vIw$\rangle$). If Ui is empty, this is obviously so. Assume then that Ui is not empty. Assume in addition that Int ia ($\bigvee$w wI ɿvf) = O. If Int ia ($\bigwedge$v IIf) = O, then there is no problem. On the other hand, if Int ia ($\bigwedge$vIIf) $\neq$ O and so is 1, then, for any m in Ui, Int ia $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (f) is in {O1}. Also, by theorem 12, there is no k in Ui such that, for any m in Ui, Int i$(\mathrm{a}({}^{\mathrm{w}}\{_{\mathrm{k}}\}))$ $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (f$\leftrightarrow$vIw) = 1; that is, since Int i$(\mathrm{a}({}^{\mathrm{w}}\{_{\mathrm{k}}\}))$ $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (f) and Int i$(\mathrm{a}({}^{\mathrm{w}}\{_{\mathrm{k}}\}))$ $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (vIw) are both in {O1} for any k and m in Ui, Int ia ($\bigvee$w$\bigwedge$v$\langle$f$\leftrightarrow$vIw$\rangle$) = O = Int ia ($\bigwedge$ v IIf $\wedge$ $\bigvee$w$\bigwedge$v$\langle$f$\leftrightarrow$vIw$\rangle$) = Int ia ($\bigvee$w wI ɿvf). Assume finally that Int ia ($\bigvee$w wI ɿvf) $\neq$ O and so is 1. Then, for any m in Ui, Int ia $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (IIf) = 1 and, by theorem 12, there is a k in Ui such that, for any m in Ui, Int i $(\mathrm{a}({}^{\mathrm{w}}\{_{\mathrm{k}}\}))$ $({}^{\mathrm{v}}\{_{\mathrm{m}}\})$ (f$\leftrightarrow$vIw) = 1; that is, Int ia ($\bigvee$w$\bigwedge$v$\langle$f$\leftrightarrow$vIw$\rangle$) = 1 = Int ia ($\bigwedge$vIIf $\wedge$ $\bigvee$w$\bigwedge$v$\langle$f$\leftrightarrow$ vIw$\rangle$ = Int ia ($\bigvee$w wI ɿvf) again.

Theorem 21. If v and w are distinct variables, f is a formula, and w is not free in f, then $\bigwedge$vIIf $\wedge$ $\bigvee$w$\bigwedge$v$\langle$f$\leftrightarrow$vIw$\rangle$ $\rightarrow$ PS$\langle$ɿvf$\rangle$vf is valid.

Assume the antecedent, that is is an n-interpreter, and that a is an assigner in Ui. If Int ia ($\bigwedge$vIIf) = O, then the theorem obviously holds. Assume then that Int ia ($\bigwedge$vIIf) $\neq$ O and so is 1. Hence, Int ia ($\bigvee w \bigwedge v \langle f \leftrightarrow vIw \rangle$) is in $\{O1\}$. If it is O, then the theorem again holds; hence, assume that it is 1 and so, by theorem 12, that there is a k in Ui such that, for any m in Ui, Int ia ($^{v}\{_{m}\}$) (f) $= 1$ just in case $m = k$; hence, Int ia (ιvf) $= \{k\}$ and so, by theorem 13, Int ia ($PS\langle \iota vf \rangle vf$) $= 1 =$ Int ia ($\bigwedge vIIf \wedge \bigvee w \bigwedge v \langle f \leftrightarrow vIw \rangle$) and the theorem holds.

Theorem 22. If f and g are formulas, i is an n-interpreter, and both f and $f \rightarrow g$ are i-true, then g is i-true.

Assume the antecedent. It follows that, if a is an assigner in Ui, then Int ia (f) $=$ Int ia ($f \rightarrow g$) $= 1 =$ the smallest member of $\{1, (1 - \text{Int ia}(f)) + \text{Int ia}(g)\}$; hence, Int ia (g) $= 1$ and the theorem holds.

Theorem 23. If f is a formula, i is an n-interpreter, and f is i-true, then $\vdash f$ is i-true.

This is obvious.

Theorem 24. If v is a variable, f is a formula, i is an n-interpreter, and f is i-true, then $\bigwedge vf$ is i-true.

Assume the antecedent. If Ui is empty, then Int ia ($\bigwedge vf$) $= 1$ for any assigner in Ui a and so $\bigwedge vf$ is i-true. On the other hand, if Ui is not empty, then, for any assigner in Ui a, there is an m in Ui such that Int ia ($^{v}\{_{m}\}$) (f) $= 1$ and there is no k in Ui such that Int ia ($^{v}\{_{k}\}$) (f) is smaller than 1 by our assumption. Hence, $\bigwedge vf$ is again i-true.

Among the valid formulas, the tautologies are of particular interest. Some of them are listed in the next theorem.

Theorem 25. If f, g, and h are formulas, then the following formulas are tautologies and so valid [3]:

(1) $f \rightarrow \langle g \rightarrow f \rangle$	(10) $\sim f \rightarrow \sim \langle f \wedge g \rangle$
(2) $\langle f \rightarrow g \rangle \rightarrow \langle \langle g \rightarrow h \rangle \rightarrow \langle f \rightarrow h \rangle \rangle$	(11) $\sim g \rightarrow \sim \langle f \wedge g \rangle$
(3) $\langle \sim f \rightarrow \sim g \rangle \rightarrow \langle g \rightarrow f \rangle$	(12) $\langle f \rightarrow g \rangle \rightarrow \langle \langle g \rightarrow f \rangle \rightarrow \langle f \leftrightarrow g \rangle \rangle$
(4) $\langle \langle f \rightarrow g \rangle \rightarrow g \rangle \rightarrow \langle \langle g \rightarrow f \rangle \rightarrow f \rangle$	(13) $\sim \langle f \rightarrow g \rangle \rightarrow \sim \langle f \leftrightarrow g \rangle$
(5) $\langle \langle f \rightarrow g \rangle \rightarrow \langle g \rightarrow f \rangle \rangle \rightarrow \langle g \rightarrow f \rangle$	(14) $\sim \langle g \rightarrow f \rangle \rightarrow \sim \langle f \leftrightarrow g \rangle$
(6) $f \rightarrow f \vee g$	(15) $\vdash f \rightarrow \langle \vdash \langle f \rightarrow g \rangle \rightarrow \vdash g \rangle$
(7) $g \rightarrow f \vee g$	(16) $\vdash f \rightarrow f$
(8) $\sim f \rightarrow \langle \sim g \rightarrow \sim \langle f \vee g \rangle \rangle$	(17) $\vdash f \rightarrow \vdash \vdash f$
(9) $f \rightarrow \langle g \rightarrow f \wedge g \rangle$	(18) $\vdash f \vee \sim \vdash f$

(19) $\mathrm{II}f \leftrightarrow \vdash \langle f \vee \sim f \rangle$

(20) $\langle f \rightarrow \langle g \rightarrow h\rangle\rangle \rightarrow \langle g \rightarrow \langle f \rightarrow h\rangle\rangle$

(21) $f \rightarrow \langle\langle f\rightarrow g\rangle \rightarrow g\rangle$

(22) $f\rightarrow \sim\sim f$

(23) $\sim\sim f\rightarrow f$

(24) $f\rightarrow f$

(25) $\langle f\rightarrow g\rangle \rightarrow \langle \sim g\rightarrow \sim f\rangle$

(26) $f\rightarrow\langle \sim g \rightarrow \sim\langle f\rightarrow g\rangle\rangle$

(27) $\sim f\rightarrow\langle f\rightarrow g\rangle$

(28) $f\vee g\rightarrow\langle\langle f\rightarrow h\rangle\rightarrow\langle\langle g\rightarrow h\rangle\rightarrow h\rangle\rangle$

(29) $f\vee g\rightarrow\langle\sim f\rightarrow g\rangle$

(30) $f\vee g\leftrightarrow\langle\langle f\rightarrow g\rangle\rightarrow g\rangle$

(31) $\langle f\wedge g\rightarrow h\rangle\rightarrow\langle f\rightarrow\langle g\rightarrow h\rangle\rangle$

(32) $\langle f\rightarrow g\rangle\wedge\langle g\rightarrow h\rangle\rightarrow\langle f\rightarrow h\rangle$

(33) $\langle f\rightarrow g\rangle\rightarrow\langle\langle f\rightarrow h\rangle\rightarrow\langle f\rightarrow g\wedge h\rangle\rangle$

(34) $f\wedge g\leftrightarrow\sim\langle\sim f\vee\sim g\rangle$

(35) $\langle f\leftrightarrow g\rangle\leftrightarrow\langle f\rightarrow g\rangle\wedge\langle g\rightarrow f\rangle$

(36) $f\vee\sim f\leftrightarrow\sim\langle f\wedge\sim f\rangle$

(37) $f\wedge\sim f\rightarrow\langle f\leftrightarrow\sim f\rangle$

(38) $\vdash\langle f\vee\sim f\rangle\leftrightarrow\vdash f\vee\vdash\sim f$

(39) $\vdash\sim\langle f\wedge\sim f\rangle\leftrightarrow\langle\vdash f\leftrightarrow\sim\vdash\sim f\rangle$

(40) $\sim\langle\vdash f\wedge\vdash\sim f\rangle$

(41) $\sim\vdash\langle f\wedge\sim f\rangle$

(42) $\vdash f\wedge\vdash g\leftrightarrow\vdash\langle f\wedge g\rangle$

(43) $\mathrm{II}f\leftrightarrow\vdash\sim\langle f\wedge\sim f\rangle$

(44) $\mathrm{II}f\leftrightarrow\vdash\langle\vdash f\leftrightarrow f\rangle$

(45) $\mathrm{II}f\rightarrow\langle\vdash\langle f\leftrightarrow g\rangle\rightarrow\mathrm{II}g\rangle$

(46) $\mathrm{II}f\rightarrow\langle\langle f\rightarrow\langle f\rightarrow h\rangle\rangle\rightarrow\langle f\rightarrow h\rangle\rangle$

(47) $\mathrm{II}f\rightarrow\langle\langle f\rightarrow\langle g\rightarrow h\rangle\rangle\rightarrow\langle f\wedge g\rightarrow h\rangle\rangle$

(48) $\mathrm{II}f\rightarrow\langle\langle f\rightarrow g\rangle\rightarrow\langle\langle f\rightarrow\langle g\rightarrow h\rangle\rangle\rightarrow\langle f\rightarrow h\rangle\rangle\rangle$

(49) $\mathrm{II}f\rightarrow\langle\langle\sim f\rightarrow g\rangle\rightarrow f\vee g\rangle$

(50) $\mathrm{II}\vdash f$

(51) $\mathrm{II}\sim f\leftrightarrow\mathrm{II}f$

(52) $\mathrm{II}\,\mathrm{II}f$.

The proofs are by cases with the aid of the fact that, for any real numbers r and s, either r = s or r is smaller than s or s is smaller than r.

Certain of the formulas which are not tautologies are also of interest.

Theorem 26. If 2 is smaller than n, then there are formulas f, g, and h such that the following formulas are 2-tautologies and so 2-valid, but neither n-tautologies nor n-valid:

(1) $\langle f\rightarrow\langle f\rightarrow g\rangle\rangle\rightarrow\langle f\rightarrow g\rangle$

(2) $\langle f\rightarrow g\rangle\rightarrow\langle\langle f\rightarrow\langle g\rightarrow h\rangle\rangle\rightarrow\langle f\rightarrow h\rangle\rangle$

(3) $\langle\sim f\rightarrow g\rangle\rightarrow f\vee g$

(4) $f\vee\sim f$

(5) $f\wedge\langle f\rightarrow g\rangle\rightarrow g$

(6) $\langle f\rightarrow\langle g\rightarrow h\rangle\rangle\rightarrow\langle f\wedge g\rightarrow h\rangle$

(7) $\sim\langle f\wedge\sim f\rangle$

(8) $\langle\sim f\rightarrow g\wedge\sim g\rangle\rightarrow f$

(9) $\langle\sim f\rightarrow f\rangle\rightarrow f$

(10) $\langle f\leftrightarrow\sim f\rangle\rightarrow f\wedge\sim f$.

(3) Of these formulas, (1) through (5) are Łukasiewicz's axioms of the denumerably many-valued logic of $\rightarrow$ and $\sim$ from the paper cited in note 2. Also, (6) through (11), (22), (26), and (27) are formulas which Tarski noted held in all of Łukasiewicz's many-valued sentential logics as well as in A. Heyting's intuitionistic one in 'On extensions of incomplete systems of the sentential calculus' (in the book mentioned in note 1).

Assume that 2 is smaller than n and that f, g, and h are atomic sentences whose predicates are distinct and in which neither I nor ℩ occurs. If i is an n-interpreter, a is an assigner in Ui, and v is the function which assigns Int ia (e) to any formula e, then v assigns 1 to all formulas which are either n-tautologies or n-valid; but, for any e of (1) through (10), v(f), v(g), and v(h) can be such that v(e) is smaller than 1. Hence, none of (1) through (10) is either an n-tautology or n-valid whereas each one is obviously a 2-tautology and so 2-valid.

2. n-VALUED LOGICS

We say that

(1) r is an inference rule just in case r is a function such that
  (a) there is a positive integer m such that the domain of r = the set of all m-term sequences of formulas and
  (b) the range of r is included in the set of all sets of formulas;
(2) d is a deductive system just in case, for some s and r, d = ⟨sr⟩, s is a set of formulas, and r is a set of inference rules; and
(3) if f is a formula and d is a deductive system, then f is d-provable just in case f is in every set k such that d(1) is included in k and, for any r in d(2) and s in the domain of r, if the range of s is included in k, then r(s) is included in k.

By an n-valued logic, we mean a deductive system d such that the set of all d-provable formulas = the set of all n-valid formulas. Obviously, ⟨the set of all n-valid formulas the empty set⟩ is an n-valued logic. A higher-valued logic is an n-valued logic for some n greater than 2 and a finitely many-valued logic is an n-valued logic for some n.

We understand MP, AS, UG, and Ln as follows:

(1) MP, AS, and UG are inference rules and, for any s,
  (a) if s is in the domain of MP, then s is a 2-term sequence and either s(2) = s(1)→g and MP(s) = {g} for some formula g or not and MP(s) is empty;
  (b) if s is in the domain of either AS or UG, then s is a 1-term sequence, AS(s) = {⊢s(1)}, and UG(s) = the set of all u such that, for some variable v, u = ⋀vs(1); and
(2) Ln is the deductive system d such that d(2) = {MP AS UG} and d(1) = the set of all e such that, for some distinct variables v and w, terms t and u such that w is not free in t, positive integer m, m-place predicate p, m-term sequence of terms s, k in the domain

of s such that w is not free in s(k), and formulas f and g such that w is not free in f, e is one of the following (4):

(1) $IItIu$
(2) $\langle\langle s(1)\rangle^\frown\langle p\rangle\rangle^\frown s_{-1}' \to \bigvee w\; wIs(k)$
(3) $\bigvee w\; wIt \to tIt$
(4) $\bigvee vf \to \bigvee w\; wIv$
(5) an n-tautology
(6) $\bigvee vf \wedge \bigvee w\; wIt \to PStuf$
(7) $\bigwedge w\langle f \to g\rangle \to \langle f \to \bigwedge wg\rangle$
(8) $\bigvee vf \leftrightarrow \sim \bigwedge v \sim f$
(9) $tIu \wedge PStuf \to f$
(10) $\bigvee w\; wI\iota vf \leftrightarrow \bigwedge vIIf \wedge \bigvee w \bigwedge v\langle f \leftrightarrow vIw\rangle$
(11) $\bigwedge v\; IIf \wedge \bigvee w \bigwedge v\langle f \leftrightarrow vIw\rangle \to PS\langle \iota vf\rangle vf$.

We say that a formula f is n-provable just in case f is Ln-provable and that f is provable just in case f is n-provable for any n. From theorems 7 through 11, 15, 16, and 18 though 24, it follows that

Theorem 27. If f is a formula, then f is n-provable only if f is n-valid for any n and so f is provable only if f is valid.

It is not difficult to prove the following theorems:

Theorem 28. If v and w are distinct variables and t is a term in which w is not free, then the following formulas are provable and so valid:

(1) $\bigvee w\; wIt \leftrightarrow tIt$
(2) $\bigvee v\; vIv \leftrightarrow vIv$
(3) $II\bigvee v\; vIv$.

Theorem 29. If v and w are distinct variables, t is a term in which w is not free, and f is a formula, then the following formulas are provable and so valid:

(1) $\bigvee w\; wIt \wedge PStvf \to \bigvee vf$
(2) $\bigvee v\; vIv \wedge \bigwedge vf \to f$
(3) $\bigvee v\; vIv \wedge f \to \bigvee vf$
(4) $\langle \bigvee v\; vIv \to \bigwedge vf\rangle \to \bigwedge vf$.

Theorem 30. If v is a variable, f and g are formulas, v is not free in f, and $f \to g$ is n-provable, then $f \to \bigwedge vg$ is n-provable.

Theorem 31. If f, g, h, and i are formulas, $f \to \langle g \to h\rangle$ is n-provable, and $f \to \langle h \to i\rangle$ is n-provable, then $f \to \langle g \to i\rangle$ is n-provable.

(4) Actually, (3) follows from (9). We include it here since, given any variable w, term t, and existence predicate e, if we replace the occurrences of $\bigvee wwIt$ in (1) through (11) with te, then the resulting (3) seems to be needed.

Theorem 32. If f is a formula and v is a variable not free in f, then $f \rightarrow \bigwedge vf$ and $\bigvee vf \rightarrow f$ are provable and so valid.

A slighly more difficult theorem is

Theorem 33. If v is a variable and f and g are formulas, then $\bigwedge v \langle f \rightarrow g \rangle \rightarrow \langle \bigwedge vf \rightarrow \bigwedge vg \rangle$ is provable and so valid.

Assume the antecedent. By theorem 29, $\bigvee v\, vIv \wedge \bigwedge v \langle f \rightarrow g \rangle \rightarrow \langle f \rightarrow g \rangle$ and $\bigvee v\, vIv \wedge \bigwedge vf \rightarrow f$ are provable and so $\bigvee v\, vIv \rightarrow \langle \bigwedge vf \rightarrow f \rangle$ is; but $\bigvee v\, vIv \wedge \bigwedge v \langle f \rightarrow g \rangle \rightarrow \bigvee v\, vIv$ is and so, by theorem 31, $\bigvee v\, vIv \wedge \bigwedge v \langle f \rightarrow g \rangle \rightarrow \langle \bigwedge vf \rightarrow g \rangle$ is. Hence, by theorem 30, $\bigvee v\, vIv \vee \bigwedge v \langle f \rightarrow g \rangle \rightarrow \bigwedge v \langle \bigwedge vf \rightarrow g \rangle$ is and so $\bigvee v\, vIv \wedge \bigwedge v \langle f \rightarrow g \rangle \rightarrow \langle \bigwedge vf \rightarrow \bigwedge vg \rangle$ is. Hence, by theorem 29, $\bigwedge v \langle f \rightarrow g \rangle \rightarrow \langle \bigwedge vf \rightarrow \bigwedge vg \rangle$ is provable and so valid.

From theorem 33, it follows that

Theorem 34. If v is a variable and f and g are formulas, then the following formulas are provable and so valid:

(1) $\bigwedge v \langle f \leftrightarrow g \rangle \rightarrow \langle \bigwedge vf \leftrightarrow \bigwedge vg \rangle$

(2) $\bigwedge v \langle f \rightarrow g \rangle \rightarrow \langle \bigvee vf \rightarrow \bigvee vg \rangle$

(3) $\bigwedge v \langle f \leftrightarrow g \rangle \rightarrow \langle \bigvee vf \leftrightarrow \bigvee vg \rangle$.

In addition, we have

Theorem 35. If f and g are formulas and v is a variable not free in f, then $\langle f \rightarrow \bigwedge vg \rangle \rightarrow \bigwedge v \langle f \rightarrow g \rangle$ is provable and so valid.

Assume the antecedent. By theorem 29, $\bigvee v\, vIv \wedge \bigwedge vg \rightarrow g$ is provable and so $\bigvee v\, vIv \rightarrow \langle \bigwedge vg \rightarrow g \rangle$ is; hence, by theorem 31, $\bigvee v\, vIv \wedge \langle f \rightarrow \bigwedge vg \rangle \rightarrow \langle f \rightarrow g \rangle$ is and so, by theorem 30, $\bigvee v\, vIv \wedge \langle f \rightarrow \bigwedge vg \rangle \rightarrow \bigwedge v \langle f \rightarrow g \rangle$ is. Hence, by theorem 29, $\langle f \rightarrow \bigwedge vg \rangle \rightarrow \bigwedge v \langle f \rightarrow g \rangle$ is provable and so valid.

Theorem 36. If v and w are variables, f is a formula, and w is not free in f, then $\bigwedge vf \leftrightarrow \bigwedge wPSwvf$ and $\bigvee vf \leftrightarrow \bigvee wPSwvf$ are provable and so valid.

Theorem 37. If t, u, and v are terms, then $tIu \rightarrow uIt$ and $tIu \wedge uIv \rightarrow tIv$ are provable and so valid.

Theorem 38. If w is a variable, t and u are terms in which w is not free, and f is a formula, then $\sim \bigvee w\, wIt \wedge \sim \bigvee w\, wIu \wedge PStuf \rightarrow f$ is provable and so valid.

The proof is by an induction among the members of TF. Hence,

Theorem 39. If w is a variable, t and u are terms in which w is not free, and f is a formula, then $\langle \sim \bigvee w\, wIt \wedge \sim \bigvee w\, wIu \rangle \vee tIu \rightarrow \langle PStuf \leftrightarrow f \rangle$ is provable and so valid.

Hence,

Theorem 40. If v and w are distinct variables, f and g are formulas, and w is not free in f or g, then the following formulas are provable and so valid:

(1) $\sim \bigvee w\, wI\iota vf \rightarrow \langle PS\langle \iota vf\rangle vg \leftrightarrow PS\,\langle \iota v \sim vIv\rangle vg\rangle$

(2) $wI\iota vf \rightarrow \langle PS\langle \iota vf\rangle vg \leftrightarrow PSwvg\rangle$

(3) $\bigvee w\, wI\iota vf \rightarrow \langle PS\langle \iota vf\rangle vg \leftrightarrow \bigvee w\langle wI\iota vf \wedge PSwvg\rangle\rangle$

(4) $PS\langle \iota vf\rangle vg \leftrightarrow \bigvee w\langle wI\iota vf \wedge PSwvg\rangle \vee \langle \sim \bigvee w\, wI\iota vf \wedge PS\langle \iota v \sim vIv\rangle vg\rangle$.

If x is a set of formulas, then c is a conjunction from x just in case either x is empty and c = $II\bigvee v\, vIv$ or x is not empty and c is in every set k such that x is included in k and, for any f and g in k, $f \wedge g$ is in k . Also,

(1) if f is a formula, then x n-implies f just in case there is a conjunction from x c such that $c \rightarrow f$ is n-provable and (2) x is n-consistent just in case there is a formula f such that x does not n-imply f.

From these definitions, we have

Theorem 41. If f is a formula, then f is n-provable just in case every set of formulas n-implies f.

Theorem 42. If x is a set of formulas, f and g are formulas, and the union of x and $\{f\}$ n-implies g, then x n-implies $f \rightarrow g$.

Theorem 43. If x is a set of formulas, f is a formula, and f is n-provable, then x is n-consistent just in case x does not n-imply $\sim f$.

On the other hand, we also have

Theorem 44. If 2 is smaller than n, then there are a set of formulas x and formulas f and g such that x n-implies f and $f \rightarrow g$, but not g.

Assume the antecedent. By (7) of theorem 26, there is a formula h such that $\sim\langle h \wedge \sim h\rangle$ is not n-provable. Let $x = \{h \wedge \sim h\}$, $f = h$, and $g = \sim\langle h \rightarrow h\rangle$. Obviously, x n-implies both f and $f \rightarrow g$. Also, if x n-implies g, then x n-implies $\sim\langle h \rightarrow h\rangle$ and so $\sim\langle h \wedge \sim h\rangle$ is n-provable, then x is n-consistent just in case x does not n-imply $\sim f$.

Theorem 45. If 2 is smaller than n, then there are a set of formulas x and formulas f and g such that x n-implies $f \rightarrow g$, but the union of x and $\{f\}$ does not n-imply g.

This can be shown by letting h be as for theorem 44, letting $x = \{\sim h\}$, letting $f = h$, and letting $g = \sim\langle h \rightarrow h\rangle$.

Theorem 46. If 2 is smaller than n, then there are a set of formulas x and a formula f such that x n-implies f, x n-implies $\sim f$, and x is n-consistent.

Assume the antecedent. By (7) of theorem 26, there is a formula f such that $\sim\langle f \wedge \sim f\rangle$ is not n-provable. Let $x = \{f \wedge \sim f\}$. Obviously, x n-implies both f and $\sim f$. Also, if x is not n-consistent, then x n-implies $\sim\langle f \rightarrow f\rangle$ by theorems 25 and 43 and so $\sim\langle f \wedge \sim f\rangle$ is n-provable. Hence, x is also n-consistent.

Because of theorems 32 through 40, it seems likely that every n-valid formula is n-provable and so that Ln is an n-valued logic. Nevertheless, unless $n = 2$, this is not provable in quite the usual way ([5]) because, among other things,

Theorem 47. If 2 is smaller than n, then there is a 1-membered set of sentences x such that x is n-consistent and there is no n-interpreter i such that, for any s in x, s is i-true.

Assume the antecedent and let f be as for (7) of theorem (26); this f is, of course, a sentence. Also, let $s = f \wedge \sim f$ and let $x = \{s\}$. Obviously, x is n-consistent; yet, there is no n-interpreter i such that s is i-true.

3. HIGHER-VALUED LOGICS VERSUS 2-VALUED LOGICS

We now turn to the question of the relative adequacies of higher-valued and 2-valued logics.

It is difficult to find any sense in which higher-valued logics are more adequate than 2-valued ones. The higher-valued ones do preserve truth with respect to interpreters which determine more than 2 truth values whereas the 2-valued ones do not. This could be understood as meaning that higher-valued logics allow us to express ourselves in a more carefree manner than 2-valued logics do; if, for instance, we use a 3-valued logic, then we can interpret our sentences in a way which changes some of our falsehoods into mere half-truths. But then the negations of these sentences plus a vast array of logical principles sink to the level of half-truths. Moreover, if we want to speak only truths, we can speak half-truths no more than we can falsehoods and so have not gained much and lost a great deal by admitting half-truths. Besides, how many truth values should we settle for ? The more the merrier ? Or just 3 ? Finally, of what importance is such a reinterpretation to our deductions anyhow ? For, by theorem 2, any n-valid argument is 2-valid and, by theorem 6, any

([5]) For an instance of this way, the reader is referred to the author's 'Contributions to syntax, semantics, and the philosophy of science' (in the *Notre Dame Journal of Formal Logic*, vol. 4, 1963).

2-valid argument can be translated into an n-valid one by prefixing each of the atomic formulas occurring in it with $\vdash$.

Although it is hard to find any sense in which higher-valued logics are more adequate than 2-valued ones, there are many in which 2-valued logics are more adequate than higher-valued ones. For example (6):

(1) There is no finitely many-valued logic which is less than 2-valued; thus, 2-valued logics are in a sense the most economical of the finitely many-valued logics.

(2) Every higher-valued logic is just a fragment of every 2-valued logic in the sense of theorem 2.

(3) Every nonzero formula is provable in a 2-valued logic by theorem 3; on the other hand, no higher-valued logic has this kind of universality by theorem 4.

(4) By theorem 26, many of the most plausible and important principles of reasoning are not provable in higher-valued logics although they are in 2-valued logics. In particular, many forms of indirect reasoning, the principle of excluded middle, and even the principle of non-contradiction are not provable in any higher-valued logic.

(5) The 2-valued logics are the only finitely many-valued logics whose sentential portions are syntactically complete in the sense that they have no extensions by addition of purely sentential axiom schemata which add some, but not all formulas to their theorems (7).

(6) The metamathematics of any higher-valued logic is artificial and weak in the sense of theorems 44 through 47 (8).

Thus, higher-valued logics do not seem to have enough to offer to be good alternatives to 2-valued ones. Nevertheless, they are of great philosophical interest and their study gives us a better understanding of both the logical constants and of 2-valued logics themselves.

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(6) It should be noted that analogues to most of the clauses listed here are applicable to intuitionistic logics.

(7) We content ourselves with just making this vague assertion here since its elaboration would lead us far afield. Related results which are in certain respects stronger were given by Tarski in the second of his papers mentioned in note 3.

(8) This inadequacy is not just an accidental result of the syntactic and semantic definitions given here since no plausible substitutes for them can lead to the validity of such sentences as $\sim (f \wedge \sim f)$ for all sentences f. Notice, however, that theorems 44 through 47 have to do with the systems Ln (although analogues to these theorems for n-valued logics can be established by the same methods).