THE DEDUCTION THEOREM IN THE COMBINATORY THEORY OF RESTRICTED GENERALITY

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1. *Introduction.* This paper contains a formulation of a system of combinatory logic containing a notion of restricted generality, and a detailed proof of one of its basic epitheorems. The paper is based upon [CLg] (1), and acquaintance with that work is necessary in order to follow all the details; but the explanations made in §§ 1-2 are intended to make the main ideas clear without requiring such acquaintance.

As explained in the introduction to [CLg], combinatory logic is divided into two main parts, called *pure* and *illative* combinatory logic respectively. The first of these deals with the combinators by themselves; the main features are summarized below in § 2. Illative combinatory logic, on the other hand, is concerned with combinators in association with the more usual logical notions such as implication and quantification; it is formed by adjoining to pure combinatory logic one or more "illative primitives" expressing these notions. Three stages of illative combinatory logic, called $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ were distinguished in [CLg]. The system $\mathcal{F}_1$ was based on an illative primitive F, in terms of which the fact that a notion is a function from one category into another can be expressed; and consequently this theory is called the *theory of functionality*. In $\mathcal{F}_2$ the illative primitive, Ξ, can express the fact that a property holds for all members of a given category; hence the theory is called *the theory of restricted generality*. Finally, in $\mathcal{F}_3$ there are two illative primitives, Π and P, expressing absolute generality and implication respectively; the system is therefore called the theory of *universal generality*. On the assumption that there is a universal category E, the three systems appear to be of increasing strength in the order named. We shall use '$\mathcal{F}_0$' to indicate the underlying pure combinatory logic on which all of these are based.

The theories $\mathcal{F}_0$ and $\mathcal{F}_1$ were studied in detail in [CLg]. The purpose of this paper is to record a new result concerning $\mathcal{F}_2$. This system $\mathcal{F}_2$ was not taken up in [CLg] for the reasons stated there on p. 274; but it was the subject of rather extensive research in the

(1) For the explanation of the letters in brackets see the Bibliography at the end of the paper.

early 1940s. The results of that research were never published in full; but under the stress of war conditions only brief announcements were made in [CFM] and [ACT]. The result stated here is new in that it goes a little beyond those announced in those papers; but, as compared with the whole sweep of problems considered there, it is rather special in scope. It is presented here with detailed proof; and it is intended to be understandable, without reference to [ACT], provided the reader is acquainted with [CLg] in the sense explained above.

The plan of the paper is as follows. In § 2 there is a summary of the main features of the underlying theory $\mathscr{F}_0$. Then the theory $\mathscr{F}_2$, and the result here proved concerning it, are discussed informally in § 3. The formal treatment begins with § 4, which is concerned with definitions and their immediate consequences. The next two sections (§§ 5-6) are concerned with side issues; viz., the relation of $\mathscr{F}_2$ to implication and to $\mathscr{F}_1$. The formulation of the axiom schemes for $\mathscr{F}_2$ is taken up in § 7. This is followed, in § 8, by a discussion of the notion of canonicalness, which is necessary, as shown in § 3, in order to avoid inconsistency. The final formulation and proof of the deduction theorem occupies § 9.

2. *Resumé of underlying theory.* All illative systems are based upon pure combinatory logic as underlying system. This system will be called here $\mathscr{F}_0$. It is treated in detail in [CLg] Chapters **5-7**. However, since the details of the formulation and proof are irrelevant for the illative theories, the principal features and notations will be recapitulated here as follows:

a. The theory deals with certain objects called *obs*. Nothing is said about the nature of these obs. They may, if one likes, be taken as the wefs (well-formed expressions) of a suitable object language [2]; but no such object language is specified, nor do any symbols of it appear. In interpretation the obs may stand for concepts of an unrestricted nature, including functions of any kind; some of them may be, intuitively speaking, meaningless. Block letters are used exclusively for particular obs; italic capitals and some other letters as variables ranging over obs.

b. The obs are generated from certain primitive ones, the atoms,

(2) For a way of doing this see the next footnote.

by a single binary operation, called *application*. This is indicated by juxtaposition. In interpretation, if X is an n-variate function, XY is the ob formed by giving the first argument the value Y; this will be a n-1-variate function, or, if $n = 1$, a constant; otherwise it is meaningless. However it is understood that any combination of the atoms formed by application is an ob; and that obs formed in distinct ways are distinct as obs. Parentheses are thus necessary, and are used according to the usual conventions, with association to the left (3)

c. There is a single one-place predicate, symbolized by the Frege prefix '$\vdash$'. The elementary statements are thus of the form

$$\vdash X,$$

where X is an ob. We read this as saying 'X is asserted'. The notation (4)

$$X_1,\ X_2,\ \ldots,\ X_m \vdash Y$$

is to mean that from the premises

(3) In view of the common insistence that the obs be explicitly represented as wefs, it is pertinent to suggest the following way of doing this. Let the object-alphabet consist of two letters 'c' and '$*$'. Let the atomic wefs be as many as desired, perhaps all, of the following expressions

$*cc,$
$*c{*}cc,$
$*c{*}c{*}cc,$

i.e. those formed by prefixing '$*c$' any number of times to a single 'c'. If X and Y are wefs, then XY is the wef obtained by writing first '$*$' then X (i.e. the wef named by the U-expression for which 'X' is a temporary abbreviation), then Y; e.g., if X and Y are the second and third atoms in the above list respectively then XY is

$**c{*}cc{*}c{*}c{*}cc.$

A composite wef always starts with two stars; the first constituent (the X) begins with the second star, and the second constituent (the Y) with the first star following the end of the first constituent. It is then effectively decidable whether an expression in the alphabet consisting of '$*$' and 'c' is a wef, and if so how it is to be constructed from the atoms by application; i.e. the system of wefs is tectonic in the sense of [CFS]. A calculus (in the sense of [CFS]) for generating the wefs for the case of infinitely many atoms is as follows:

$\mathfrak{K}_0$ (generates the atoms):
$\vdash *cc$
$\vdash *cx^0,$
$\mathfrak{K}_1$ (generates the wefs):
$x^0 \vdash x^1$
$x^1,\ y^1 \vdash^1 *\,xy.$

This representation is suggested by the notation of Chwistek. A similar suggestion, but with '$*$' in the U-language, was made in [CLg] § 1E3, (q.v. for references).

(4) Due to Rosser [MLV].

$$\vdash X_1, \ \vdash X_2, \ \ldots, \ \vdash X_m$$

we can derive

$$\vdash Y$$

by the rules of the system.

d. There is defined an equality relation, symbolized by infixed '$=$', such that for any obs X and Y

$$X = Y$$

is defined to be the same as

$$\vdash Q\, X\, Y,$$

where Q is a particular ob. This equality has the usual properties; in particular we have

Rule Eq. *If* $X = Y$, *then* $X \vdash Y$.

e. The system is combinatorially complete in the following sense. Let $\mathscr{F}_0(x)_m$ be formed by adjoining to $\mathscr{F}_0$ the indeterminates $x_1, \ldots, x_m$; i.e. these are not already obs of $\mathscr{F}_0$, and they are adjoined as distinct obs without any further properties. Let $\mathfrak{X}$ be an ob of $\mathscr{F}_0(x)_m$ (in general we shall use German letters as variables ranging over obs of such extensions). Then there is an ob X of $\mathscr{F}_0$, such that

$$X x_1 \ldots x_m = \mathfrak{X}$$

is derivable in $\mathscr{F}_0(x)_m$. A particular such X will be indicated by the notation

$$[x_1, \ldots, x_m]\, \mathfrak{X}.$$

This is defined by an induction on m, so that

$$[x, y_1, \ldots, y_n]\, \mathfrak{X} \equiv [x]\, (\, [y_1, \ldots, y_n]\, \mathfrak{X}),$$

where the ob $[y_1, \ldots, y_n]\, \mathfrak{X}$ is defined relative to an extension in which x is treated as a constant.

f. Equality is such that an extensionality principle, called (ζ), holds. This means that if X and Y are obs of $\mathscr{F}_0$, such that in $\mathscr{F}_0(x)_m$, as above,

$$X x_1 \ldots x_m = Y x_1 \ldots x_m,$$

then in $\mathscr{F}_0$ itself we have

$$X = Y.$$

This applies also for any extension of $\mathscr{F}_0$, so that it would be sufficient to state the property for $m=1$.

g. The $[x_1, \ldots, x_m]\mathfrak{X}$ of (e) is a combination of the constants in

$\mathfrak{X}$ and the primitive combinators, I, K, S, where

(1) $\quad \mathsf{I} = [x]\, x,$

(2) $\quad \mathsf{K} = [x, y]\, x,$

(3) $\quad \mathsf{S} = [x, y, z]\,.\, xz(yz).$

Other combinators of some importance are

(4) $\quad \mathsf{B} = [x, y, z].\, x(yz)\,,$

(5) $\quad \mathsf{C} = [x, y, z].\, xzy\,,$

(6) $\quad \mathsf{W} = [x, y].\, xyy\,,$

(7) $\quad \Phi = [x, y, z, u].\, x(yu)(zu)\,.$

For further special combinators and notations reference must be made to [CLg].

3. *Informal Discussion.* The ob Ξ, whose adjunction to the theory of combinators forms $\mathscr{F}_2$, is associated with a rule, viz.

Rule Ξ. $\Xi XY,\ XU \vdash YU.$

According to this rule, Ξ can be interpreted as a relation of inclusion, or formal implication, between arbitrary obs. If we interpret X and Y respectively as properties φ, ψ, then the ob ΞXY can be understood as the proposition (expressed in the notation of the Principia Mathematica) $\varphi x \supset_x \psi x$. Here, of course, X and Y are not restricted to belong to specific types; and ΞXY may not be interpretable as a proposition; but the assertion of ΞXY means that YU is asserted for every ob for which XU is asserted, so that ΞX expresses generality over the range X.

As stated in [CLg] § **8**D, the other illative concepts P (implication), Π (universality), and F (functionality) can be defined in terms of Ξ thus:

$$\mathsf{P} \equiv [x,y].\, \Xi(\mathsf{K}x)(\mathsf{K}y) \qquad (= \Psi\Xi\mathsf{K}),$$
$$\Pi \equiv \Xi\mathsf{E},$$
$$\mathsf{F} \equiv [x,y,z].\ \Xi x(\mathsf{B}yz).$$

From these definitions the associated rules follow, viz:

Rule P. $\quad \mathsf{P}XY,\ X \vdash Y,$

Rule Π. $\quad \Pi X,\ \mathsf{E}Y \vdash XY.$

Rule F. $\quad \mathsf{F}XYZ,\ XU \vdash X(ZU).$

Here of course Rule P is the ordinary rule of modus ponens.

The motivation for introducing Ξ is the fact that most generalizations — some say, indeed, all generalizations — are valid only over a restricted range; one can make this range explicit by the use of Ξ. In ordinary predicate calculus one has a fundamental, supposedly very restricted, range of 'individuals', and what is there called 'universal quantification' is really generality with respect to this restricted range. In higher order functional calculuses many different ranges enter; and a means of making them explicit is then useful.

When we wish to consider generalization with respect to two or more variables, one thinks at first of the analog of the Principia's $\varphi xy \supset_{xy} \psi xy$. This cannot be represented directly by Ξ; one would need an analog for Ξ for functions of degree two. But if we regard x as fixed for the moment, we can regard the y as varying over a range which depends on the value of x; then we allow x to vary over a certain range [5]. This leads to a statement which, in Principia notation, would be

$$\varphi_1 x \supset_x . \varphi_2 xy \supset_y \psi xy. \quad \text{[6]}$$

This can be expressed by Ξ. In fact let X_1, X_2, Y correspond to φ_1, φ_2, ψ respectively; then the relation just written becomes

$$\Xi X_1([x]\Xi(X_2x)(Yx)).$$

If we bring in the formalizing combinator Φ from [CLg] Chapter **5**, (especially § 5E), this becomes

$$\Xi X_1(\Phi\Xi X_2 Y).$$

In this way we can define a Ξ_2, viz.

$$\Xi_2 \equiv [x_1, x_2, y].\Xi x_1(\Phi\Xi x_2 y),$$

which expresses a restricted generality relation for functions of two variables. Continuing in this way, we can define by induction a Ξ_m, expressing generality for m-variate functions, for any value of m. The formal definitions will concern us in § **4**.

The deduction theorem for restricted generality is now the following principle. Let $\mathscr{F}_2(x)_m$ be the system formed by adjoining x_1, ..., x_m as indeterminates [7] to $\mathscr{F}_2$. Let ξ_1, ..., ξ_m be obs of $\mathscr{F}_2$, and let $\mathscr{F}_2(\xi; x)_m$ be the extension formed by adjoining to $\mathscr{F}_2(x)_m$ the following axioms

$$(1) \qquad \begin{array}{l} \vdash \xi_1 x_1 \\ \vdash \xi_2 x_1 x_2, \\ \quad \cdots \\ \vdash \xi_m x_1 x_2 \ldots x_m. \end{array}$$

Let $\mathfrak{X}$ be a combination of constants and x_1, ..., x_m such that

(5) Compare the situation in the theory of iterated integration when one integrates over a plane region; the limits of the second integration depend on the first variable.

(6) If φ_1 is universal this reduces to the case previously considered.

(7) When $m=0$, this $\mathscr{F}_2(x)_m$ is to be the same as $\mathscr{F}_2$.

(2) $\vdash \mathfrak{X}$

holds in $\mathscr{F}_2(\xi;x)_m$. Then we have in $\mathscr{F}_2$

(3) $\vdash \Xi_m \xi_1 \ldots \xi_m([x_1, \ldots, x_m]\, \mathfrak{X})$.

This principle cannot be accepted for unrestricted $\xi_1, \ldots, \xi_m$. For, as we shall see in § **4**, this would entail the ordinary deduction theorem for P (as implication). Hence, since modus ponens holds for P, we could infer by the Gentzen method that (PW), viz.

(PW) $\vdash \mathsf{P}(\mathsf{P}X(\mathsf{P}XY))(\mathsf{P}XY)$,

or in ordinary notation

$$\vdash X \supset . X \supset Y : \supset . X \supset Y,$$

would hold for all obs X, Y. This, however, is incompatible with the combinatory completeness which characterizes the theory of combinators. For let Y be an arbitrary ob. By the use of the paradoxical combinator we can find an ob X (viz. $\mathsf{Y}(\mathsf{SP}(\mathsf{CP}Y))$) such that

(4) $X = X \supset . X \supset Y.$

Then we can argue as follows:

	$\vdash X \supset . X \supset Y : \supset . X \supset Y$	(by (PW)).
	$\vdash X \supset . X \supset Y$	(by (4)).
(5)	$\vdash X$	(Rule P).
	$\vdash X \supset Y$	(by (4)).
	$\vdash Y$	(by (5), Rule P).

Since Y is arbitrary, the system would then be inconsistent (8).

In order to avoid this contradiction we must impose restrictions on the $\xi_1, \ldots, \xi_m$. Here we define a class of obs, called *canonical obs*, and restrict $\xi_1, \ldots, \xi_m$ to be a sequence related (9) to that class. The formulation of this class, which is here rather different from that in the theory of functionality, will concern us in § 8. We shall use lower case Greek letters for obs which are assumed to be canonical, or to be in some way restricted in relation to other canonical obs.

Our objective will be to formulate axioms sufficient to give the deduction theorem for the case where $\xi_1, \ldots, \xi_m$ are so related to canonical obs. Taking canonicalness informally, as we do, we shall arrive at an infinite number of these axioms. The question of whether canonicalness can be formalized so as to have a finite set of axioms

(8)This contradiction differs from that in [CLg] in that it follows from (PW) alone. The possibility of such a contradiction was indicated in [CLg] p. 258, footnote 1.

(9) The ξ's in this paper are actually canonical, but certain generalizations seem possible. These are left for later study.

is not investigated. This and some related questions are left for later study.

4. *Formal preliminaries.* We shall now make formal definitions of the sequence of obs Ξ_n and some related notions, and shall derive some properties which are immediate consequences of the definitions and Rule Ξ. The work has thus a notational, technical character. It may be considered an addition to [CLg] § **8**E.

The combinator $\Phi^m{}_n$, which appears here, was defined in [CLg] § **5**E. Its properties may, however, be obtained by (ζ) (§ **2**F) from the "reduction rule".

(1) $$\Phi^m{}_n x y_1 \ldots y_n z_1 \ldots z_m = x(y_1 z_1 z_2 \ldots z_m)(y_2 z_1 \ldots z_m) \ldots (y_n z_1 \ldots z_m).$$

When m is not given it is supposed to be 1; when n is not given it is supposed to be 2.

Definition 1.

(2) $$\Xi_0 \equiv \mathrm{I},$$

(3) $$\Xi_{n+1} \equiv [x, y_1, \ldots, y_n, z].\Xi x(\Phi_{n+1}\Xi_n y_1 \ldots y_n z).$$ (10)

Theorem 1. $\Xi_1 = \Xi$.

Proof. Putting $n = 0$ in (3), we have

$$\vdash \Xi_1 \equiv [x, z]\,\Xi x\,(\Phi_1 \Xi_0 z).$$

By [CLg] § **5**E8,

$$\vdash \Xi_1 \equiv [x, z]\Xi x(\mathsf{B}\mathrm{I}z) = [x, z]\Xi xz = \Xi, \text{ q.e.d.}$$

Theorem 2. *If* X, ξ_1, ..., ξ_n, u_1, ..., u_n *are such that*

(4) $$\vdash \Xi_n \xi_1 \ldots \xi_n X,$$

(5) $$\vdash \xi_k u_1 u_2 \ldots u_k \qquad k = 1, 2, \ldots, n;$$

then

(6) $$\vdash X u_1 \ldots u_n\ .$$

Proof. For $n = 0$, (6) follows from (4) by Rule Eq.

To complete an induction on n it suffices to show that (6) follows from (4), (5) on the supposition that the theorem has been proved

(10) Note that (3) is the case $m=1$ of Theorem 3. We could equally well have taken the case $n=1$. The latter was, essentially, what was done in defining the F-sequence in [CLg] § **8**E. It now seems better to state all such definitions so the functional part ([CLg] § **5**C) of the definiens in the induction step is simple, the complication being in the argument component. This requires a change in the definition of the F-sequence, but this is a pure technicality.

with $n - 1$ in the place of n. By (3), if $n > 1$, we conclude from (4) that

$$\vdash \Xi\xi_1(\Phi_n\Xi_{n-1}\xi_2\ldots\xi_n X).$$

By (5), for $k=1$, and Rule Ξ we conclude from this that

$$\begin{aligned} &\vdash \Phi_n\Xi_{n-1}\xi_2\ldots\xi_n X u_1 \\ =&\vdash \Xi_{n-1}(\xi_2 u_1)\ldots(\xi_n u_1)(Xu_1) \qquad \text{(by (1))}. \end{aligned}$$

From this we have (6) by the theorem with $n—1$ for n, and $\xi_2 u_1$, ..., $\xi_n u_1$, Xu_1, respectively for ξ_1, ..., ξ_{n-1}, X.

THEOREM 3. *For all* m, $n = 0, 1, 2, \ldots,$

(7) $$\Xi_{m+n} = [x_1,\ldots,x_m,y_1,\ldots,y_n,z].\Xi_m x_1\ldots x_m(\Phi^m{}_{n+1}\Xi_n y_1\ldots y_n z).$$

Proof. For $m = 0$ this follows from (2), in view of the fact $X^0 = I$ for any X. To complete an induction on m it suffices to prove, on the assumption that (7) holds, an analogous statement with $m + 1$ in the place of m.

By (3) we have

(8) $$\Xi_{m+n+1} = [u, x_1,\ldots, x_m, y_1, \ldots, y_n]. \Xi u U,$$

where

$$\begin{aligned} U &\equiv \Phi_{m+n+1}\Xi_{m+n}x_1\ldots x_m y_1\ldots y_n z \\ &= [u]\,\Xi_{m+n}(x_1 u)\ldots(x_m u)(y_1 u)\ldots(y_n u)(zu) \qquad \text{(by (1))}. \end{aligned}$$

By the hypothesis of the induction we have

$$\begin{aligned} U &= [u]\,\Xi_m(x_1 u)\ldots(x_m u)(\Phi^m{}_{n+1}\Xi_n(y_1 u)\ldots(y_n u)(zu)) \\ &= \Phi_{m+1}\,\Xi_m x_1\ldots x_m V \qquad \text{(by 1))}, \end{aligned} \tag{9}$$

where

$$\begin{aligned} V &\equiv [u]\Phi^m{}_{n+1}\Xi_n(y_1 u)\ldots(y_n u)(zu) \\ &= \Phi_{n+1}(\Phi^m{}_{n+1}\Xi_n)\, y_1\ldots y_n z \\ &= \Phi^{m+1}{}_{n+1}\Xi_n y_1\ldots y_n z. \end{aligned}$$

Returning to (9) we have

$$\begin{aligned} \Xi u U &= \Xi u(\Phi_{m+1}\Xi_m x_1\ldots x_m V) \\ &= \Xi_{m+1}\, u x_1\ldots x_m V \\ &= \Xi_{m+1}\, u x_1\ldots x_m(\Phi^{m+1}{}_{n+1}\Xi_n y_1\ldots y_n z). \end{aligned}$$

If we put this (8) we have the desired analog of (7). This completes the proof.

THEOREM 4. *If* ξ_1, ..., ξ_m, η_1, ..., η_n, X, u_1, ...u_m *are such that*

(10) $$\vdash \Xi_{m+n}\xi_1\ldots\xi_m\eta_1\ \ldots\ \eta_n X,$$

(11) $$\vdash \xi_k u_1\ldots u_k \qquad k = 1, 2, \quad , m;$$

then

$$\vdash \Xi_n(\eta_1 u_1\ldots u_m)\ldots(\eta_n u_1\ldots u_m)(Xu_1\ldots u_m).$$

Proof. This, in view of (7), is a combination of Theorems 2 and 3. The next definition gives an adaptation for our purposes of the formal implication notation of the Principia mathematica (already used in § 3).

DEFINITION 2. If $\mathfrak{X}$ and $\mathfrak{Y}$ are combinations of constants and the variable x, then

$$\mathfrak{X} \supset_x \mathfrak{Y}. \equiv . \Xi\,([x]\mathfrak{X})([x]\mathfrak{Y}).$$

THEOREM 5. *For each $k = 1, 2, \ldots, n$, let $\mathfrak{X}_k$ be a combination of constants and the variables $x_1, \ldots, x_k$; and let* Y *be a combination of constants and $x_1, \ldots, x_n$. Let*

$$X_k \equiv [x_1, \ldots, x_k]\mathfrak{X}_k, \qquad Y \equiv [x_1, \ldots, x_k]\,\mathfrak{Y}.$$

Then

$$\mathfrak{X}_1 \supset_{x_1} . \mathfrak{X}_2 \supset \ldots \supset_{x_{n-1}} . \mathfrak{X}_n \supset_{x_n} \mathfrak{Y}. = .\Xi_n X_1 \ldots X_n Y. \tag{12}$$

Proof. Let the $\mathfrak{Z}_k$ be defined inductively from $\mathfrak{Z}_n$ backward thus:

$$\mathfrak{Z}_n \equiv \mathfrak{X}_n \supset_{x_n} \mathfrak{Y} = \Xi([x_n]\mathfrak{X}_n)([x_n]\mathfrak{Y}), \tag{13}$$

$$\mathfrak{Z}_k \equiv \mathfrak{X}_k \supset_{x_k} \mathfrak{Z}_{k+1} = \Xi([x_k]\mathfrak{X}_k)([x_k]\mathfrak{Z}_{k+1}). \tag{14}$$

Then we have from (13)

$$\mathfrak{Z}_n = \Xi(X_n x_1 \ldots x_{n-1})(Y x_1 \ldots x_{n-1}). \tag{15}$$

Suppose that, for a definite k,

$$\mathfrak{Z}_{k+1} = \Xi_{n-k}(X_{k+1} x_1 \ldots x_k) \ldots (X_n x_1 \ldots x_k)(Y x_1 \ldots x_k).$$

Then

$$[x_k]\mathfrak{Z}_{k+1} = \Phi_{n-k+1}\Xi_{n-k}(X_{k+1} x_1 \ldots x_{k-1}) \ldots$$
$$(X_n x_1 \ldots x_{k-1})\ (Y x_1 \ldots x_{n-1}).$$

From (14) we should then have, by Definition 1,

$$\mathfrak{Z}_k = \Xi_{n-k+1}(X_k x_1 \ldots x_{k-1}) \ldots (X_n x_1 \ldots x_{k-1})(Y x_1 \ldots x_{k-1}) \tag{16}$$

Since (16) holds for $k = n$ by (15), we have an inductive proof that (16) holds for all $k = 1, 2, \ldots, n$. For $k = 1$ we have (12), q.e.d.

THEOREM 6. *Under the hypotheses of Theorem 1, let*

$$\vdash \mathfrak{X}_1 \supset_{x_1} . \mathfrak{X}_2 \supset_{x_2} . \cdots \supset_{x_{n-1}} . \mathfrak{X}_n \supset_{x_n} \mathfrak{Y}. \tag{17}$$

Let $a_1, \ldots, a_k$ be constants, and let $\mathfrak{X}'_j$ ($\mathfrak{Y}'$) be obtained from $\mathfrak{X}_j$ ($\mathfrak{Y}$) by substitution of $a_1, \ldots, a_k$ respectively for $x_1, \ldots, x_k$; the remaining variables $x_{k+1}, \ldots, x_n$, if any, being left unchanged. Let the $a_1, \ldots, a_k$ be such that

$$\vdash \mathfrak{X}_j' \qquad j = 1, 2, \ldots, k. \tag{18}$$

Then

$$\mathfrak{X}'_{k+1} \supset_{x_{k+1}} . \mathfrak{X}'_{k+2} \supset_{x_{k+2}} . \ \ldots \supset_{x_{n-1}} . \mathfrak{X}'_n \supset_{x_n} \mathfrak{Y}'$$

Proof. This follows from Theorem 5 and Theorem 4.

For purposes where extreme abbreviation is necessary the following is useful.

DEFINITION 3. Let $\xi_1, \ldots, \xi_m$ be obs; and let $\mathfrak{X}$ be a combination of constants and the variables $x_1, \ldots, x_m$. Then

$$(\xi; x)_m \mathfrak{X} \equiv \xi_1 x_1 \supset_{x_1} \ldots \supset_{x_{m-1}} \ldots x_m \supset_{x_m} \mathfrak{X}.$$

5. *Relations to implication.* An ob P representing implication was mentioned in § 3. We consider here some formal matters connected with the relations of this P to Ξ. To give the matter organization, the study is directed toward substantiating the statement, made in § 3, that the ordinary deduction theorem, using P, follows from that stated in § 3; but the intermediate theorems are useful for other purposes, and may be more important than the end result.

DEFINITION 4. $X \supset Y \equiv \mathsf{P}XY$.

This definition allows us to use the ordinary implication infix '$\supset$', and to translate ordinary formulas about implication into combinatory notation and vice versa. Note that in view of Definition 2 of § 4 and the definition of P in § 3 we have

(1) $\qquad X \supset Y = X \supset_x Y,$

where X is any indeterminate not appearing in X or Y. Thus the notation is not in conflict with Definition 2 of § 4.

To represent chain implications of the form

$$X_1 \supset .X_2 \supset . \ \ldots \supset .X_m \supset Y,$$

we need an ob P_m defined as follows [11].

DEFINITION 2.

$$\mathsf{P}_0 \equiv \mathsf{I},$$
$$\mathsf{P}_{n+1} \equiv [x, y_1, \ldots, y_n, z]\mathsf{P}x(\mathsf{P}_n y_1 \ldots y_n z).$$

It is then clear that

(2) $\qquad X_1 \supset .X_2 \supset . \ \ldots \ \supset . \ X_m \supset Y = \mathsf{P}_m X_1 \ldots X_m Y.$

The deduction theorem for P is the following: If

(3) $\qquad X_1, \ldots, X_m \vdash Y,$

then

(4) $\qquad \vdash \mathsf{P}_m X_1 \ldots X_m Y.$

[11] Essentially the same as in [CLg] § **9**E (7) and (8), p. 315.

Assuming the deduction theorem for Ξ, we shall establish this. The proof will show that the theorem holds for canonical $X_1, \ldots, X_m$, provided that for such X_i the $\xi_1, \ldots, \xi_m$, where

$$(5) \qquad \xi_r \equiv \mathsf{K}^r X^r \qquad\qquad r = 1, 2, \ldots, m$$

form a sequence satisfying the canonicalness restrictions of the deduction theorem for Ξ.

THEOREM 7. $\mathsf{K}(\Xi_m Y_1 \ldots Y_m Z) = \Phi_{m+1}\Xi_m(\mathsf{K}Y_1)\ldots(\mathsf{K}Y_m)(\mathsf{K}Z)$.

Proof. By (ζ) (§ 2f) and § 2g(2),

$$\begin{aligned}\mathsf{K}(\Xi_m Y_1 \ldots Y_m Z) &= [x]\Xi_m Y_1 \ldots Y_m Z\\ &= [x]\Xi_m(\mathsf{K}Y_1 x)\ldots(\mathsf{K}Y_m x)(\mathsf{K}Zx)\\ &= \Phi_{m+1}\Xi_m(\mathsf{K}Y_1)\ldots(\mathsf{K}Y_m)(\mathsf{K}Z), \text{ q.e.d.}\end{aligned}$$

THEOREM 8. $\mathsf{P}X(\Xi_m Y_1 \ldots Y_m Z) = \Xi_{m+1}(\mathsf{K}X)(\mathsf{K}Y_1)\ldots(\mathsf{K}Y_m)(\mathsf{K}Z)$.

Proof. By the definition in § 3,

$$\begin{aligned}\mathsf{P}X(\Xi_m Y_1 \ldots Y_m Z) &= \Xi(\mathsf{K}X_1)(\mathsf{K}(\Xi_m Y_1 \ldots Y_m Z))\\ &= \Xi(\mathsf{K}X_1)(\Phi_{m+1}\Xi_m(\mathsf{K}Y_1)\ldots(\mathsf{K}Y_m)(\mathsf{K}Z)) \quad \text{by Th. 7}\\ &= \Xi_{m+1}(\mathsf{K}X_1)(\mathsf{K}Y_1)\ldots(\mathsf{K}Y_m)(\mathsf{K}Z) \quad \text{by Df. 1.}\end{aligned}$$

THEOREM 9. $\mathsf{P}_m = [x_1, \ldots, x_m, y]\Xi_m(\mathsf{K}x_1)(\mathsf{K}^2x_2)\ldots(\mathsf{K}^m x_m)(\mathsf{K}^m y)$.

Proof. This is clear by definition when $m = 1$ (also for $m = 0$). Suppose it is true for a given m, then

$$\begin{aligned}\mathsf{P}_{m+1}xy_1\ldots y_m z &= \mathsf{P}x(\mathsf{P}_m y_1 \ldots y_m z)\\ &= \mathsf{P}x(\Xi_m(\mathsf{K}y_1)(\mathsf{K}^2y_2)\ldots(\mathsf{K}^m y_m)(\mathsf{K}^m z)) &&\text{(Hp. ind.)}\\ &= \Xi_{m+1}(\mathsf{K}x)(\mathsf{K}^2y_1)\ldots(\mathsf{K}^{m+1}y_m)(\mathsf{K}^{m+1}z) &&\text{(Th. 8)}\end{aligned}$$

By (ζ) (§ 2f), this completes a proof by induction on m.

THEOREM 10. *Let* $X_1, \ldots, X_m, Y$ *be such that* (3) *holds, and let the* $\xi_1, \ldots, \xi_m$ *defined by* (5) *be such that the deduction theorem as formulated in § 3 holds for them. Then* (4) *holds.*

Proof.

Suppose that $\vdash \xi_r x_1 \ldots x_r$. $\qquad r = 1, 2, \ldots, m$.
Then by Rule Eq (§ 2d) $\vdash X_r$. $\qquad r = 1, 2, \ldots, m$.
Hence by (3) $\vdash Y$.
Then by Rule Eq, $\vdash \mathsf{K}^m Y x_1 \ldots x_m$.
Therefore, by the deduction theorem of § 3,

$$\vdash \Xi_m \xi_1 \ldots \xi_m \,(\mathsf{K}^m Y)$$

In view of (3) and Theorem 9, this is (4), q.e.d.

6. *Relation to* $\mathscr{F}_1$ ([12]). It was shown in [CLg] § **10** E that, if F is taken as primitive, then there are two obs Ξ' and Ξ'', viz.

$$\Xi' \equiv [x, y]\, \mathsf{F}xy\mathsf{I},$$
$$\Xi'' \equiv [x, y]\, \mathsf{F}x\mathsf{I}y,$$

either of which satisfies Rule Ξ. Furthermore these two obs cannot be proved equal in the system $\mathscr{F}_1$; on the other hand if F is itself defined in terms of Ξ as in § 3, then

(1) $\qquad \Xi = \Xi' = \Xi''.$

Thus (1) holds in $\mathscr{F}_2$, but not in $\mathscr{F}_1$.

Suppose now that we consider in $\mathscr{F}_1$ the equation

(2) $\qquad [x,y,z,u]\mathsf{F}x(\mathsf{B}yz)u = [x,y,z,u]\mathsf{F}xy(\mathsf{B}zu).$

This equation is certainly valid in $\mathscr{F}_2$. In fact, by the definition of F in $\mathscr{F}_2$ we have

$$\mathsf{F}x(\mathsf{B}yz)u = \Xi x(\mathsf{B}(\mathsf{B}yz)u),$$
$$\mathsf{F}xy(\mathsf{B}zu) = \Xi x(\mathsf{B}y(\mathsf{B}zu));$$

by § 2 (4) and (ζ), (cf. [CLg] Theorem 5D3)

$$\mathsf{B}(\mathsf{B}yz)u = \mathsf{B}y(\mathsf{B}zu),$$

and hence (2) follows by (ζ). On the other hand if we take F as primitive and postulate (2), then we have

$$\begin{aligned} \Xi'xy &= \mathsf{F}xy\mathsf{I} && \text{(by df.)},\\ &= \mathsf{F}x(\mathsf{BI}y)\mathsf{I} && \text{(by [CLg] Th. 5D2)},\\ &= \mathsf{F}x\mathsf{I}(\mathsf{B}y\mathsf{I}) && \text{(by (2))},\\ &= \mathsf{F}x\mathsf{I}y && \text{(by [CLg] Th. 5D2)},\\ &= \Xi''xx && \text{(by df.)} \end{aligned}$$

Thus by (ζ),

$$\Xi' = \Xi''.$$

If we define Ξ, e.g., to be Ξ', then (1) will hold; furthermore

$$\begin{aligned} \mathsf{F}xyz &= \mathsf{F}xy(\mathsf{B}z\mathsf{I}) && \text{(by [CLg] Th. 5D2)},\\ &= \mathsf{F}x(\mathsf{B}yz)\mathsf{I} && \text{(by (2))},\\ &= \Xi x(\mathsf{B}yz) && \text{(by df.)}. \end{aligned}$$

Thus postulating F *as primitive with* (2) *as axiom and* Ξ *defined as* Ξ' *(or* Ξ''*) is equivalent to postulating* Ξ *as primitive and defining* F *as in* § 3.

This suggests that one way of formulating $\mathscr{F}_2$ is to adjoin (2) to $\mathscr{F}_1$ as additional axiom. The system so formulated we shall call

([12]) The considerations of this section are not needed for what follows, but are inserted for their intrinsic interest. There is more dependence on [CLg] than in the rest of the paper.

$\mathscr{F}_{12}$. We may use superscripts etc. to distinguish different species of it, just as we did with $\mathscr{F}_1$. The following are some properties which are valid in it.

THEOREM 11. $\mathsf{F}_m X_1 \ldots X_m Y(\mathsf{B}^m Z U) = \mathsf{F}_m X_1 \ldots X_m (\mathsf{B} Y Z) U$.

Proof. For $m = 1$, this is true by (2). To get an induction on m, we proceed thus:

$$
\begin{aligned}
& \mathsf{F}_{m+1} X_1 \ldots X_{m+1} Y(\mathsf{B}^{m+1} Z U) \\
& \quad = \mathsf{F}_m X_1 \ldots X_m (\mathsf{F} X_{m+1} Y)(\mathsf{B}^m (\mathsf{B} Z) U) && \text{(by df. of } \mathsf{F}_m \text{ and } \mathsf{B}_m\text{)}, \\
& \quad = \mathsf{F}_m X_1 \ldots X_m (\mathsf{B}(\mathsf{F} X_{m+1} Y)(\mathsf{B} Z)) U && \text{(by Hp. induction)}, \\
& \quad = \mathsf{F}_m X_1 \ldots X_m ([v] \mathsf{F} X_{m+1} Y(\mathsf{B} Z v)) U \\
& \quad = \mathsf{F}_m X_1 \ldots X_m ([v] \mathsf{F} X_{m+1} (\mathsf{B} Y Z) v) U && \text{(by (2))}, \\
& \quad = \mathsf{F}_m X_1 \ldots X_m (\mathsf{F} X_{m+1} (\mathsf{B} Y Z)) U && \text{(by } (\zeta)\text{)}, \\
& \quad = \mathsf{F}_{m+1} X_1 \ldots X_m X_{m+1} Y(\mathsf{B} Y Z) U && \text{(by df. } \mathsf{F}_m\text{)}
\end{aligned}
$$

This completes the induction.

The next theorem concerns the notion

$$(\forall f_1 x_1) \ldots (\forall f_m x_m). \mathfrak{X}$$

defined in [CLg] § **10**E1. Since (2) holds, we can drop the accents. The $f_1, \ldots, f_m$ are arbitrary obs.

THEOREM 12. $(\forall f_1 x_1) \ldots (\forall f_m x_m) \mathfrak{X} = \mathsf{F}_m f_1 \ldots f_m \mathsf{I} X$, *where* $X \equiv [x_1, \ldots, x_m] \mathfrak{X}$.

Proof. For $m = 1$ this follows by the definition ([CLg] § **10**E1, (4)). For induction on m we have, since

$$
\begin{aligned}
& [x_2, \ldots, x_{m+1}] \mathfrak{X} = X x_1, \\
& (\forall f_1 x_1) \ldots (\forall f_m x_m)(\forall f_{m+1} x_{m+1}) \mathfrak{X} \\
& \quad = (\forall f_1 x_1)((\forall f_2 x_2) \ldots (\forall f_{m+1} x_{m+1}) \mathfrak{X}) && \text{(by df.)}, \\
& \quad = (\forall f_1 x_1)(\mathsf{F}_m f_2 \ldots f_{m+1} \mathsf{I}(X x_1)) && \text{(Hp. induction)} \\
& \quad = \mathsf{F} f_1 \mathsf{I}([x_1] \mathsf{F}_m f_2 \ldots f_{m+1} \mathsf{I}(X x_1)) && \text{(by df.)}, \\
& \quad = \mathsf{F} f_1 \mathsf{I}(\mathsf{B}(\mathsf{F}_m f_2 \ldots f_{m+1} \mathsf{I}) X) && \text{(by § 2 (4))}, \\
& \quad = \mathsf{F} f_1 (\mathsf{B} \mathsf{I}(\mathsf{F}_m f_2 \ldots f_{m+1} \mathsf{I})) X && \text{(by (2))}, \\
& \quad = \mathsf{F} f_1 (\mathsf{F}_m f_2 \ldots f_{m+1} \mathsf{I}) X. && \text{(by [CLg] Th. 5D2)}, \\
& \quad = \mathsf{F}_{m+1} f_1 \ldots f_{m+1} \mathsf{I} X. && \text{(by df. of } \mathsf{F}_m\text{)}.
\end{aligned}
$$

This completes the induction.

THEOREM 13. *Let $\mathfrak{X}$ be an ob in an extension formed by $x_1, \ldots, x_m$ such that*

(3) $\quad \vdash (\forall \xi_1 x_1) \ldots (\forall \xi_m x_m) \ \mathfrak{X}$.

Let $\mathfrak{Y}_i$ $(i = 1, 2, \ldots, m)$ be an ob in the extension formed by $y_1, \ldots, y_n$ such that

(4) $\vdash \mathsf{F}_n \eta_1 \eta_2 \ldots \eta_n \xi_i Y_i,$

where

$$Y_i \equiv [y_1, \ldots, y_n]\mathfrak{Y}_i \qquad i = 1, 2, \ldots, m.$$

Let $\mathfrak{Z}$ *be formed by substituting* $\mathfrak{Y}_1, \mathfrak{Y}_2, \ldots, \mathfrak{Y}_m$ *for* $x_1, \ldots, x_m$ *in* $\mathfrak{X}$. *Then*

(5) $\vdash (\forall \eta_1 y_1) \ldots (\forall \eta_n y_n)\, \mathfrak{Z}.$

Proof. Let

$$X \equiv [x_1, \ldots, x_m]\, \mathfrak{X}.$$

Then by (3) and Theorem 12,

(6) $\vdash \mathsf{F}_m \xi_1 \ldots \xi_m\, \mathsf{I} X.$

Also,

$$\mathfrak{Z} = X(Y_1 y_1 \ldots y_n) \ldots (Y_m y_1 \ldots y_n);$$

and hence, by (ζ),

$$Z \equiv [y_1, \ldots, y_n]\mathfrak{Z} = \Phi^n{}_m\, X Y_1 \ldots Y_n.$$

Now $\Phi^n{}_m$ is a stratified combinator; hence, by the stratification theorem [13] one can deduce in the basic theory of functionality that it has the functional character $\mathsf{F}_{m+1}(\mathsf{F}_m \xi_1 \ldots \xi_m \zeta)(\mathsf{F}_n \eta_1 \ldots \eta_n \xi_1) \ldots \mathsf{F}_n \eta_1 \ldots \eta_n \xi_m)(\mathsf{F}_n \eta_1 \ldots \eta_n \zeta)$. From this, with $\zeta \equiv \mathsf{I}$, and (4), (6) we have

$$\vdash \mathsf{F}_n \eta_1 \ldots \eta_n \mathsf{I} Z.$$

By Theorem 2, this is the same as (4).

Remark. We could not apply directly the substitution theorem of [CLg] Theorem 9D3, because of special assumptions made in that theorem. The detour via the functional character of Φ_m was necessary in order to get the theorem, and might well have been used to get the theorem in [CLg]; but there the interest was in getting an F-deduction.

If $\mathfrak{B}$ in [CLg] § 9D3 consisted of (4) and (6) only, then there might be a difficulty because X is not a variable. This does not cause trouble in functional character of $\Phi^n{}_m$.

Note. Theorem 13 is the only theorem considered here which involves any axioms of $\mathcal{F}_1$. As given it is valid in $\mathcal{F}_{12}$, since it used I as an F-simple.

7. *Formulation of* $\mathcal{F}_2$. As a formulation of $\mathcal{F}_2$ system $\mathcal{F}_{12}$ is not fully satisfactory because the axiom schemas for $\mathcal{F}_1$ are based on

[13] [CLg] Th. 9D1.

generalization with respect to univariate premises only. For bivariate functions we get only theorems of the form

$$\alpha x \supset_x . \beta y \supset_y . \gamma(fxy),$$

whereas in $\mathcal{F}_2$ we may need those of the form

$$\alpha x \supset_x . \beta xy \supset_y . \gamma xy.$$

The situation is similar to the predicament we should be in if we could evaluate double integrals over rectangles only.

Nevertheless the formulation suggests axioms to choose for a more adequate formulation. Thus the scheme (FK) becomes in our present notation

$$\vdash \alpha x \supset_x . \beta y \supset . \alpha(\mathsf{K}xy)$$
$$= \vdash \alpha x \supset_x . \beta y \supset . \alpha x$$

This suggests the axiom scheme

(1) $\qquad \vdash \alpha x \supset_x . \beta xy \supset . \alpha x,$

i.e.

(2) $\qquad \vdash \Xi_2 \alpha\beta(\mathsf{BK}\alpha).$

The axiom scheme (FS) becomes similarly

$$\vdash \alpha u \supset_u . \beta v \supset_v \gamma(xuv) . \supset_x : \alpha u \supset_u \beta(yu) . \supset \alpha z \supset_z \gamma(xz(yz)).$$

Here, if we put βu for β, we have

$$\vdash \alpha u \supset_u . \beta uv \supset_v \gamma(xuv) . \supset_x : \alpha u \supset_u \beta u(yu) . \supset \alpha z \supset_z \gamma(xz(yz)).$$

This in turn suggests the scheme

(3) $\qquad \vdash \alpha u \supset_u . \beta uv \supset_v . \gamma xuv . \supset_x : \alpha u \supset_u \beta u(yu) . \supset_y . \alpha z \supset_z \gamma xz(yz).$

This is the axiom chosen for (ΞS) in [ACT]. Note it can be written

(4) $\qquad \vdash \Xi_2 \alpha\beta(\gamma x) \supset_x . \Xi\alpha(\mathsf{S}\beta y) \supset_y \Xi\alpha(\mathsf{S}(\gamma x)y).$

A possible alternative would be the scheme

(5) $\qquad \vdash \alpha u \supset_u . \beta uv . \supset \gamma uv . \supset_x : \alpha u \supset_u \beta u(yu) . \supset . \alpha z \supset_z . \gamma z(yz),$

i.e.

(6) $\qquad \vdash \Xi_2 \alpha\beta\gamma . \supset_x \Xi\alpha(\mathsf{S}\beta y) \supset_y . \Xi\alpha(\mathsf{S}\gamma y).$

This scheme is sufficient for the purposes of this paper. It follows from (3) by substitution of $\mathsf{K}\gamma$ for γ. The converse deduction of (3) from (5), requires substitution of γx for γ and generalization with respect to x. Whether this circumstance introduces problems later is not yet entirely clear. Accordingly (3) is retained as the axiom scheme [ΞS] for the time being; it may be possible later to replace it by (5).

We adopt the notation

$$[\Xi\mathsf{K}] \equiv [x,y]\Xi_2 xy(\mathsf{BK}x)$$
$$[\Xi\mathsf{S}] \equiv [x,y,z]\Xi_2 xy(zu) \supset_u \Xi x(\mathsf{S}yv) \supset_v \Xi_x(\mathsf{S}(zu)v) .$$

then the schemes (2), (4) are respectively

$$\vdash [\Xi \mathsf{K}]\ \alpha\beta,$$
$$\vdash [\Xi \mathsf{S}]\ \alpha\beta\gamma;$$

while the scheme (5), (6) is

$$\vdash [\Xi \mathsf{S}]\ \alpha\beta(\mathsf{K}\gamma).$$

8. *Canonicalness.* We turn to the formulation of the restrictions to be imposed on the obs denoted by Greek letters. To this end we formulate a class of canonical obs. The method of doing this is rather different from that used in the theory of functionality; for here a canonical ob need not be, in interpretation, a category, but may be thought of as a predicate of any number of arguments, or, in other terms, as a concept of any degree. It might be possible to distinguish canonicalness of different degrees; but it seems likely that we shall get a simpler formalism by taking a single concept of canonicalness which applies to all degrees.

The definition of canonicalness is an inductive one, and it is not asserted that the question of whether or not an ob is canonical is decidable. But the definition will be such that if $X = Y$ and there is a proof that X is canonical, then there will be a proof that Y is also. Also any consequence of canonical premises will be canonical if the theory is Q-consistent.

DEFINITION 5. X is canonical just when its being so can be derived by the following rules:

(a) Certain atoms ϑ_i are canonical. These are atoms of the underlying theory, and so are not available as adjoined indeterminates.

(b) If we define Q by

$$\mathsf{Q} \equiv [x,y]\Xi(\mathsf{SI}(\mathsf{K}x))(\mathsf{SI}(\mathsf{K}y)),$$

then

$$\Xi(\mathsf{SI}(\mathsf{K}U))(\mathsf{SI}(\mathsf{K}V))$$

is canonical for any U, V. We abbreviate this as $[\mathsf{Q}UV]$.

(c) ΞUV is canonical if U and V are canonical.

(d) If U is canonical, then UV is canonical for any V.

(e) If $\mathfrak{U}$ is canonical and x is an adjoined indeterminate, then $[x]\ \mathfrak{U}$ is also canonical.

(f) If U is canonical and $V \succ\!\!- U$ [14], then V is canonical.

THEOREM 14. *If $\mathfrak{X}$ is canonical and x is an adjoined indeterminate, and $X' \equiv [Y/x]\mathfrak{X}$, then X' is canonical.*

[14] Here $\succ\!\!-$ is the strong reduction defined in [CLg] § 6F.

Proof. We use an induction on the number of applications of (a) — (f). We divide into cases according to the last rule applied.

(a) If $\mathfrak{X} \equiv \vartheta_i$, $X' \equiv \mathfrak{X}$.

(b) If $\mathfrak{X} \equiv [\mathrm{Q}\mathfrak{U}\mathfrak{V}]$, then $X' \equiv [QU'V']$, where U', V' are, related to $\mathfrak{U}\mathfrak{V}$ respectively as X' to $\mathfrak{X}$.

(c) If $\mathfrak{X} \equiv \Xi\,\mathfrak{U}\mathfrak{V}$, then $X' \equiv \Xi U'V'$. By the hypothesis of the induction, U' and V' are canonical; hence X' is.

(d) Let $\mathfrak{X} \equiv \mathfrak{U}\mathfrak{V}$, where $\mathfrak{U}$ is canonical. Then $X' \equiv U'V'$ and again U' is canonical.

(e) Let $\mathrm{X} \equiv [u]\mathfrak{U}$. Since the case where x does not occur in $\mathfrak{X}$ is trivial, we have $u \neq x$. Since u is an adjoined indeterminate relative to the extension in which Y occurs, u does not occur in Y. Supposing that the algorithm used is (abcf), it follows by [CLg] Theorem **6**D4 that

$$X' \equiv [u]\ U'.$$

By the hypothesis of the induction U' is canonical, hence again X' is.

Remark. There are typographical errors in [CLg] Theorem **6**D4, in that $=$ and $\equiv$ are confused. If the algorithm is (abcf), identity signs can appear throughout.

(f) Let $\mathfrak{X} \succ\!\!- \mathfrak{Y}$, $\mathfrak{Y}$ canonical. Then $X' \succ\!\!- Y'$, and Y' is canonical.

Remark. The validity is not obvious in regard to strong reduction, unless Y is constant, because of the H-transformation at the end of a strong reduction; but the λ-prefix as used can be with regard to variables not occuring in Y.

THEOREM 15. *A necessary and sufficient condition that* X *be canonical is that there exist indeterminates* $x_1, \ldots, x_m$, *and an ob* $\mathfrak{X}$ *in the extension formed by adjoining them, such that an ob of one of the types* (a), (b), (c) *is a leading element in* $\mathfrak{X}$ *and*

$$\mathrm{X}x_1 \ldots x_m \succ\!\!- \mathfrak{X}.$$

Proof of sufficiency. Such an $\mathfrak{X}$ is canonical by (d) and so $[x_1, \ldots, x_m]\mathfrak{X}$ is by (e). Then X is by (f).

Proof of necessity. We proceed by the same type of induction as in Theorem 14. In cases (a), (b), (c) we can take $\mathfrak{X} \equiv \mathrm{X}$, $m=0$.

(d) Let $\mathrm{X} \equiv UV$, U canonical. Let $\mathfrak{U}$ be associated with U. If $m = 0$ take $\mathfrak{X} \equiv \mathfrak{U}\,V$; if $m > 1$ take $\mathfrak{X} \equiv [V/x_1]\,\mathfrak{U}$. This will begin with an ob of type (a), (b), or (c).

(e) Let $\mathrm{X} \equiv [x]\,\mathfrak{U}$. Let $\mathfrak{U}$ be associated with U; then $\mathfrak{U}$ is also associated with X, m being increased by one.

(f) Let $X \succ\!\!- Y$ and Y be associated with $\mathfrak{Y}$. Then X is associated with the same $\mathfrak{Y}$.

This completes the proof.

THEOREM 16. *Let X be canonical and*

$$X = Y\,;$$

then Y is also canonical.

Proof. Let $x_1, \ldots, x_m$ be adjoined indeterminates occuring in neither X nor Y, and let $\mathfrak{X}$ be associated with X as in Theorem 15. By the Church-Rosser property of strong reduction there is a $\mathfrak{Y}$ such that.

$$Xx_1 \ldots x_m \succ\!\!- \mathfrak{X} \succ\!\!- \mathfrak{Y},$$
$$Yx_1 \ldots x_m \succ\!\!- \mathfrak{Y}.$$

We show that $\mathfrak{Y}$ has the properties described in Theorem 15. There are three cases (a), (b), (c) according to the type of the leading element of $\mathfrak{X}$.

(a) $\mathfrak{X} \equiv \vartheta_i\, \mathfrak{U}_1\, \mathfrak{U}_2 \ldots \mathfrak{U}_n$. Then
$\mathfrak{Y} \equiv \vartheta_i\, \mathfrak{V}_1\, \mathfrak{V}_2 \ldots \mathfrak{V}_n$,

where $\mathfrak{U}_i \succ\!\!- \mathfrak{V}_i$ ([CLg] Th. 6F5).

(b) $\mathfrak{X} \equiv [Q\mathfrak{U}_1\, \mathfrak{U}_2]\, \mathfrak{U}_3 \ldots \mathfrak{U}_n$. Then
$\mathfrak{Y} \equiv [Q\mathfrak{V}_1\, \mathfrak{V}_2]\, \mathfrak{V}_3 \ldots \mathfrak{V}_n$

and this is again of Type (b).

(c) $\mathfrak{X} \equiv \Xi\, \mathfrak{U}_1\, \mathfrak{U}_2\, \mathfrak{U}_3 \ldots \mathfrak{U}_n$

where $\mathfrak{U}_1\, \mathfrak{U}_2$ are canonical. Then

$$\mathfrak{Y} \equiv \Xi\, \mathfrak{V}_1\, \mathfrak{V}_2 \ldots \mathfrak{V}_n$$

where $\mathfrak{U}_i \succ\!\!- \mathfrak{V}_i$. By the hypothesis of an induction on the number of applications of (a)—(f) in the construction of X, $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are canonical. Then $\mathfrak{Y}$ is of type (c).

Note that the number of applications of (c) in the construction of X is mirrored exactly in the construction of $\mathfrak{X}$. I.e., we start a construction of X and at every step construct the corresponding $\mathfrak{X}$. Then $\mathfrak{U}_1$ and $\mathfrak{U}_2$ will have preceded $\mathfrak{X}$ in the construction. Hence if we have an induction on the number of steps in the proof that X is canonical, this induction will be sufficient to complete the proof.

THEOREM 17. *If the theory is Q-consistent, then every step in a proof by Rules Ξ and Eq from canonical premises valid in the theory will be canonical.*

Proof. The proof is by deductive induction. The premises are canonical by hypothesis. The case of Rule Eq in the induction is taken care of by Theorem 16.

Suppose, then, we have an inference by Rule Ξ, viz.

$$\frac{\vdash \Xi XY \qquad\qquad \vdash XZ}{\vdash YZ}$$

There are two possibilities.

1) The left premise comes under (c). Then X and Y are both canonical, and so the conclusion is.

2) The left premise comes under (b). Then by the Q-consistency, $X = Y$. Hence the case comes under Rule Eq.

Just as in $\mathcal{F}_1$, it is desirable to have in $\mathcal{F}_2$ all instances of the reflexive property of Ξ, i.e.

$$\vdash \Xi XX$$

for any ob X. To include this possibility we can add to Definition 1 the case

(g) ΞUV is canonical if $U = V$. Evidently this addition will not disturb the proofs of any of Theorems 14-17 ([15]).

If we adjoin clause (g) to Definition 1, and at the same time cancel clause (b), then Theorems 14-17 remain valid. Now however it is not necessary to postulate Q-consistency, since ΞXY is now canonical only then either $X = Y$ or X and Y are both canonical. Moreover Q-consistency is a consequence of Theorem 17; ([16]) for since $[z]zU$ is never canonical, $\mathsf{Q}UV$ is never canonical unless $U = V$.

This proves the following theorem: ([17])

THEOREM 18. *If clause* (g) *is added to Definition 1 with or without clause* (b), *then Theorems 13-17 remain valid. Moreover the theory is Q-consistent if all the axioms are canonical according to the modified definition with clause* (b) *deleted.*

A theory lacking clause (b) is of course extraordinarily weak; in fact such an ob as

$$\mathsf{E}x \supset_x . \; x = y \supset_y . \; y = z \supset_z . \; x = z,$$

which expresses the transitive property of equality, is not canonical in it. Nevertheless, in view of the fact that inconsistencies do arise

([15]) We must add (g) to the types (a), (b), (c) in Theorem 15.

([16]) Under the assumption, of course, that all the axioms are canonical in the new sense.

([17]) The role of Rule E, viz. $\mathsf{E}X, \mathsf{E}Y \vdash \mathsf{E}(XY)$, and of axioms of the form $\vdash \mathsf{E}X$ has been ignored in the foregoing. For this we may either take E to be one of the ϑ_i, or define E as WQ, so that $\mathsf{E}X$ reduces to $[\mathsf{Q}XX]$. In either case $\mathsf{E}X$ is always canonical, so that ignoring the E-rules and E-axioms causes no disturbance.

in what appear to be very weak systems, its Q-consistency is of some significance. Stronger definitions of canonicalness are under consideration for future study.

9. *The deduction theorem.* A preliminary statement of the deduction theorem was given in § 3. In order to give a more adequate formulation we must first formulate certain conditions upon the system $\mathcal{F}_2$ and its extensions.

The axioms of $\mathcal{F}_2$ we suppose given in at most three ways, viz.

1°) By substitution of canonical obs for certain parameters, called the *canonical parameters,* in certain axiom schemes. We call these axiom schemes, and the axioms obtained from them, *regular.*

2°) By substitution of arbitrary obs for certain parameters, called *free parameters,* in certain axiom schemes, possibly with further substitution of canonical obs for additional parameters which are here also called canonical parameters. Such axiom schemes, and the axioms obtained from them, we call *special.*

3°) By listing of individual axioms not containing any parameters for which substitutions can be made. The obs asserted in such axioms are specific obs [18] of $\mathcal{F}_2$. We call these axioms the *fixed axioms.*

In contradistinction to the fixed axioms we call the axioms of types 1° and 2° *schematic.*

An axiom of $\mathcal{F}_2(x)_m$ which actually contains x_m we call a proper axiom of $\mathcal{F}_2(x)_m$; it is not an axiom of $\mathcal{F}_2(x)_{m-1}$. Since the $x_1, \ldots, x_m$ are indeterminates, any such axiom must be schematic; and at least one of the obs substituted for a parameter must contain x_m.

This established we make the following assumptions concerning $\mathcal{F}_2$.

(A) The regular axiom schemes include the following (the canonical parameters being such of α, β, γ as actually appear):

(A1) $[\Xi\mathsf{K}]\,\alpha\beta$,

(A2) $[\Xi\mathsf{S}]\,\alpha\beta\gamma$,

(B) The special axiom schemes include only

(B1) $\vdash \mathsf{E}u$

where u is the free parameter. With this goes the fixed axiom

(B2) $\vdash \Xi\mathsf{E}(\mathsf{W}\Xi)$.

This with (B1) gives the reflexiveness of Ξ over arbitrary obs.

[18] This is true of any axiom of $\mathcal{F}_2$. But the point is that proper axioms of $\mathcal{F}_2(x)_m$ for any $m>0$ cannot be obtained from fixed axioms.

(C) All axioms are canonical [19]. (This is true for the axioms and axiom schemes explicitly listed; but has to be assumed for any additional ones there may be.)

(D) If

$$\vdash \mathfrak{A}$$

is a proper axiom of $\mathscr{F}_2(x)_m$, then

$$\Xi(\xi_m x_1 \ldots x_{m-1})([x_m]\mathfrak{A})$$

is derivable in $\mathscr{F}_2(\xi; x)_{m-1}$.

On this basis we proceed to state and prove the deduction theorem.

THEOREM 19. *Let $\mathfrak{X}$ be an ob of* $\mathscr{F}_2(x)_m$ such that

(1) $\vdash \mathfrak{X}$

is derivable by Rules Ξ and Eq from axioms of $\mathscr{F}_2(x)_m$ together with

(2) $\vdash \xi_q x_1 \ldots x_q \qquad q = 1, 2, \ldots, m.$

Let the conditions (A) - (D) *be fulfilled. Then*

(3) $\vdash (\xi; x)_m \mathfrak{X}$

is derivable in $\mathscr{F}_2$.

Proof. We use a double induction. The primary induction is with respect to m; the secondary induction is a deductive induction on the proof of (1).

The theorem is trivial for $m = 0$. Hence it suffices to assume $m > 0$, and to suppose the theorem proved for $\mathscr{F}_2(\xi; x)_{m-1}$. This is the hypothesis of the primary induction.

Let $\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_n$ ($\equiv \mathfrak{X}$) be the steps in the proof of (1). Let

$$X_k \equiv [x_1, \ldots, x_m]\, \mathfrak{X}_k.$$

Then what we have to show is that

(4) $\vdash \Xi_m \xi_1 \ldots \xi_m X_k$

holds for $k = n$. We do this under the assumption that (4) holds for all $k < n$ (if any). This will complete the secondary (deductive) induction. We distinguish five cases, as follows:

Case 1. $\mathfrak{X}_n$ is an axiom of $\mathscr{F}_2(\xi; x)_m$ which does not contain x_m; hence an axiom of $\mathscr{F}_2(\xi; x)_{m-1}$ and so either an axiom of $\mathscr{F}_2(x)_{m-1}$ or one of the first m - 1 of the statements (2). By (A) we have in $\mathscr{F}_2(x)_{m-1}$, if $\mathfrak{A}$ and $\mathfrak{B}$ are canonical obs of $\mathscr{F}_2(x)_{m-1}$,

(5) $\vdash \mathfrak{A}y \supset_y . \mathfrak{B}yz \supset_z \mathfrak{A}y.$

[19] With respect to some definition of canonicalness for which the conclusion of Theorem 17 is valid. If Theorem 17 requires Q-consistency, then Q-consistency must be added to the premises of Theorem 19.

Here take [20] $\mathfrak{A} \equiv \mathsf{K}\,\mathfrak{X}_n$, $\mathfrak{B} \equiv \mathsf{K}(\xi_m x_1,\ldots x_{m-1})$. Now, since $\mathfrak{A}Y = \mathfrak{X}_n$,

$$\vdash \mathfrak{A}\, Y$$

is true in $\mathcal{F}(\xi\, x)_{m-1}$ for any Y; hence we conclude from (5) and Rule Ξ that

(6) $\qquad \vdash \xi_m x_1\ldots x_{m-1} z \supset_z \mathfrak{X}_n$

is valid in $\mathcal{F}(\xi;x)_{m-1}$. By the hypothesis of the primary induction it follows from (6) (replacing z by x_m) that

$$\vdash (\xi;x)_{m-1}.\xi_m x_1\ldots x_m \supset_{x_m} \mathfrak{X}_n$$

This is the same as (4) by Definition 3 and Theorem 5.

Case 2. $\mathfrak{X}_n$ is $\xi_m x_1\ldots x_m, X_n = \xi_m$. In this case we use (B) to conclude that, in $\mathcal{F}(x)_{m-1}$ and hence a fortiori in $\mathcal{F}(\xi;x)_{m-1}$,

$$\begin{aligned}\vdash \Xi(\xi_m x_1\ldots x_{m-1})(\xi_m x_1\ldots x_{m-1})\\ = \xi_m x_1\ldots x_m \supset_{x_m} \xi_m x_1,\ldots,x_m.\end{aligned}$$

Applying the hypothesis of the primary induction, and reasoning as in the last step of Case 1, we conclude that

$$\vdash \Xi_m \xi_1\cdots\xi_m \xi_m$$

holds in $\mathcal{F}_2$. This is (4) for this case.

Case 3. $\mathfrak{X}_n$ is a proper axiom of $\mathcal{F}(x)_m$. Here we use (D) to conclude that (6) holds in $\mathcal{F}(\xi;x)_{m-1}$. From that point we proceed as in Case 1.

Case 4. $\mathfrak{X}_n$ is derived by Rule Eq. Then $\mathfrak{X}_n = \mathfrak{X}_k$ for some $k > n$. Hence $X_n = X_k$, and (3) follows from (4) by Rule Eq.

Case 5. $\mathfrak{X}_n$ is derived by Rule Ξ. Let the premises be $\mathfrak{X}_i$ and $\mathfrak{X}_j$, $i<n$, $j<n$. Then for some $\mathfrak{U}$, $\mathfrak{B}$, $\mathfrak{Z}$ we have

$$\mathfrak{X}_i \equiv \Xi\,\mathfrak{U}\,\mathfrak{B}, \quad \mathfrak{X}_j \equiv \mathfrak{U}\,\mathfrak{Z}, \quad \mathfrak{X}_n \equiv \mathfrak{B}\,\mathfrak{Z}.$$

We may suppose that $\mathfrak{U}$ and $\mathfrak{B}$ are canonical; for $\mathfrak{X}_i$ is canonical by (C) and § 8 [21], and the case where $\mathfrak{U} = \mathfrak{B}$ reduces to Case 4.

Let

$$U \equiv [x_1,\ldots,x_m]\mathfrak{U}, \qquad V \equiv [x_1,\ldots x_m]\,\mathfrak{B},$$
$$Z \equiv [x_1,\ldots,x_m]\,\mathfrak{Z}$$

Then [22]

$$X_i = \Phi^m \Xi UV, \qquad X_j = \mathsf{S}^{(m)}UZ, \qquad X_n = \mathsf{S}^{(m)}VZ$$

From (4) for $k = i$ we have

[20] $\mathfrak{A}$ is canonical by (C) and § 8.

[21] Cf. footnote to (C).

[22] For definition of $\mathsf{S}^{(m)}$ see [CLg] § 5E.

$$\vdash \Xi_m \xi_1 \ldots \xi_m (\Phi^m \Xi U V)$$
$$= \vdash \Xi_{m+1} \xi_1 \ldots \xi_m U V \qquad \text{(by § 4).}$$

Hence, using the first $m - 1$ of the postulates (2) and Theorem 4, we have in $\mathcal{F}_2(\xi;x)_{m-1}$

(7) $\vdash \Xi_2 \mathfrak{A}' \mathfrak{U}' \mathfrak{B}'$,

where $\mathfrak{A}' \equiv \xi_m x_1 \ldots x_{m-1}$, $\mathfrak{U}' \equiv U x_1 \ldots x_{m-1} = [x_m] \mathfrak{U}$,

$$\mathfrak{B}' \equiv V x_1 \ldots x_{m-1} = [x_m] \mathfrak{B}.$$

By the axiom scheme [ΞS] with $\mathfrak{A}'$, $\mathfrak{U}'$, $K\mathfrak{B}'$ for α, β, γ, we have in $\mathcal{F}_2(x)_{m-1}$

$$\vdash \Xi_2 \mathfrak{A}' \mathfrak{U}' \mathfrak{B}' \supset_y . \Xi \mathfrak{A}' (\mathsf{S} \mathfrak{U}' z) \supset_z \Xi \mathfrak{A}' (\mathsf{S} \mathfrak{B}' z).$$

From this and (7) we conclude by Rule Ξ that

(8) $\vdash \Xi \mathfrak{A}' (\mathsf{S} \mathfrak{U}' z) \supset_z \Xi \mathfrak{A}' (\mathsf{S} \mathfrak{B}' z)$.

holds in $\mathcal{F}_2(\xi;x)_{m-1}$.

From (4) for $k=j$ we have in $\mathcal{F}_2$

$$\Xi_m \xi_1 \ldots \xi_m (\mathsf{S}^{(m)} U Z).$$

Here we use the first $m - 1$ relations (2) and Theorem 4 to derive a result in $\mathcal{F}_2(\xi;x)_{m-1}$. Since

$$\mathsf{S}^{(m)} U Z X_1 \ldots x_{m-1} = \mathsf{S} \mathfrak{U}' \mathfrak{Z}',$$

this result is

$$\vdash \Xi \mathfrak{A}' (\mathsf{S} \mathfrak{U}' \mathfrak{Z}').$$

Then by (8) and Rule Ξ, we have

$$\vdash \Xi \mathfrak{A}' (\mathsf{S} \mathfrak{B}' \mathfrak{Z})$$
$$= \vdash \Xi_m x_1 \ldots x_m \supset_{x_m} \mathfrak{B} \, \mathfrak{Z}$$

This is (6) for this case. Then (4) follows as in Case 1.

This completes the proof of Theorem 19.

Remark 1. The troublesome assumption needed for the proof is (D). This seems not to be derivable from the axiom schemes listed under (A) and (B). A sufficient condition for its derivability may be written as follows. Consider an axiom scheme

(8) $\vdash \mathfrak{A}$.

where $\mathfrak{A}$ has the canonical parameters $\alpha_1, \ldots, \alpha_p$. Let $y_1, \ldots, y_q$ be indeterminates not appearing in $\mathfrak{A}$, and let $\mathfrak{A}'$ be obtained by substituting $\alpha_j y_1 \ldots y_q$ for α_j ($j = 1, \ldots, p$) in $\mathfrak{A}$. Let $\beta_1, \ldots, \beta_q$ be additional canonical parameters, and let

$$\mathfrak{B} \equiv \Xi_q \beta_1 \ldots \beta_q ([y_1, \ldots y_q] \mathfrak{A}').$$

Under just these circumstances we shall say that $\mathfrak{B}$ is derived from $\mathfrak{A}$ by *canonical generalization of order q*. Then a sufficient condition for (D) is that the axiom schemes be closed with respect to canonical generalization of order m.

That this is indeed sufficient may be seen as follows. Let an axiom of $\mathcal{F}(x)_m$ assert an ob $\mathfrak{A}^*$ obtained from the $\mathfrak{A}$ of (8) by substituting obs $\mathfrak{U}_1, \ldots, \mathfrak{U}_p$, which may contain $x_1, \ldots, x_m$, for $\alpha_1, \ldots, \alpha_p$ respectively. Let $q = m$, and identify y_i with x_i. Let

$$U_j \equiv [x_1, \ldots, x_m] \mathfrak{U}_j.$$

Then $\mathfrak{A}^*$ can be obtained from $\mathfrak{A}'$ by substituting $U_1, \ldots, U_p$ respectively for $\alpha_1, \ldots, \alpha_p$ in $\mathfrak{A}'$ and applying *Eq*. Let $\mathfrak{B}^*$ be obtained from $\mathfrak{B}$ by substituting ξ_i for β_i and U_j for α_j ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, p$) in $\mathfrak{B}$. Then $\mathfrak{B}^*$ is an axiom of $\mathcal{F}_2$. From this and the first m—1 equations (2) we have (by Theorem 4)

$$\vdash \Xi(\xi_m x_1 \ldots x_{m-1})([x_m]\mathfrak{A}^*).$$

The situation is thus, in a way, analogous to that in the ordinary predicate calculus. There we have two rules, modus ponens and generalization. To get a system with modus ponens only we need a set of axioms closed under generalization.

Remark 2. If we postulate under (A) the scheme

$$\vdash \Xi\alpha\mathsf{E}$$

in which α is a canonical parameter, then I suspect we could avoid using (D) for special axioms. However the proof is not complete, and the question is left open.

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